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Domination Numbers of Graphs Containing Vertex-Disjoint Cycles in Graphs

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Abstract Domination in graphs has been an extensively researched branch of graph theory. In a graph, a domination set is a subset S of the vertices such that every vertices of $V-S$ is adjacent to a vertex of S. The main object of this article is to study the domination numbers of graph containing vertex disjoint cycles with some identities of domination set and independent set. Further, we present some identities related to domination number, upper domination number, Independence number and independent domination number of graphs containing vertex-disjoint cycles.

Keywords: *independent numbers, domination numbers, graph*

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1. Introduction

In the last five decades graph theory has been an explosive growth due to its interaction with and application in several areas like Physical Sciences, Life Sciences, Engineering, Operation Research etc. The fastest growing area within graph theory is the study of domination, the reason being its many and varied applications in such fields as social sciences, communications networks, algorithmic designs etc. The topic of domination was given formal Mathematical definition by C. Berge in 1958 and O. Ore [9] in 1962. Berge called the domination as external stability and domination number of coefficient of external stability. Ore introduced the world domination in his famous book [9]. This concept lived in hibernation until 1975 when a paper [6] published in 1977. This paper brought to light new ideas and potentiality of being applied in variety of areas. The research in domination theory has been broadly classified in [11,12].

In this paper we present some identities of domination set and independent set of graphs containing vertex-disjoint cycles.

2. Preliminaries and Notations

Let $G=(V,E)$ be a simple graph (i.e., undirected, without loops and multi edges). The number of vertices namely the cardinality of V is called the order of G and is denoted by $|G|$. The number of edges of a graph namely the cardinality of E is called the size of G and is denoted by $|E|$. We write $e=v_iv_j \in E(G)$ to mean the pair $v_i, v_j \in E(G)$ and if $e=v_iv_j \in E(G)$ we say that v_i and v_j are adjacent and e and v_i, e and v_j re incident.

The open neighbourhood $N(v)$ of the vertex v consists of the set of vertices adjacent to v. That is $N(v)=\{w \in V : vw \in E\}$. The closed neighbourhood of v is $N[v]=N(v)\cup\{v\}$. For a set $S \subset V$, the open neighbourhood N(S) is defined by $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighbourhood $N[S]$ by $N[S]=N(S)\cup S$. A vertex $v \in S$ is called "an enclave of S", if $N[S]\subset S$. A vertex $v \in S$ is called "an isolate of of S", if $N(S)\subset V-S$.

The degree of a point v is denoted by $\deg(v)$ is defined as the number of edges incident with v. That is $\deg(v)=|N(v)|$. The maximum and minimum of the degree of vertices of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. If $\Delta(G)=\delta(G)=r$, then G is said to be a regular graph of degree r or simply r-regular.

A walk of length k is an alternative sequence of vertices and edges with $e_i=u_{i-1}u_i$, i.e., $W=u_0e_1u_1e_2u_2e_3 \ldots u_{k-1}e_ku_k$. If all k edges are distinct in a walk, then it is called a Trial. If all the k+1 vertices are distinct in a walk, then it is called a Path. If $u_0=u_k$ and if $u_1,u_2,\ldots u_{k-1}$ are distinct in a walk then it is called a cycle. If $u_0=u$ and $u_k=v$, then W is said to be a $u-v$ walk of a length k.

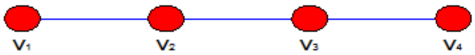
We now explain independent set and dominating set of graphs [11,12].

3. Independent Set and Dominating Set

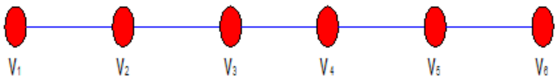
Counting independent sets in graphs is one of several combinatorial counting problems which have received recent attention. Independent sets were introduced into the communication theory on noisy channels [7]. From application point of view, domination problems appear in numerous practical setting, ranging from strategic decision such as locating radar stations or emergency services through computational biology to voting system. Variations of dominations such as multiple domination, even/odd domination, distance domination, directed domination, independent domination and connected domination have found numerous application and significant theoretical interest recently.

Independent set

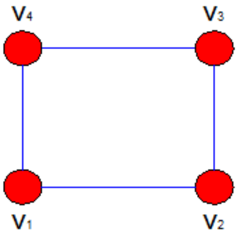
A sub set $S \subseteq V(G)$ is an independent set, if no two vertices of S are adjacent. Moreover, the subset containing only one vertex and the empty set also are independent. The number of all independent sets in G is denoted by $NI(G)$. For a graph G on $V(G) = \phi$, we put $NI(G) = 1$. Independence number $\beta(G)$ of graph G is the maximal cardinality of an independent set of vertices. Following are the examples of independent set:



Independent set $\{v_1, v_3\}$



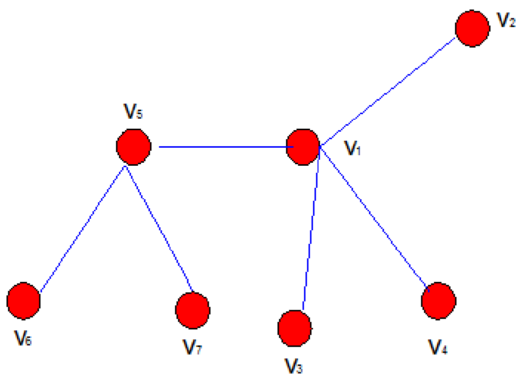
Independent set $\{v_2, v_4, v_6\}$



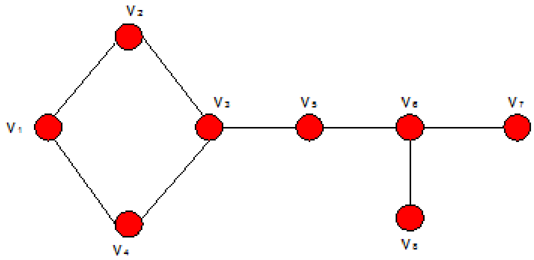
Independent set $\{v_1, v_3\}$

Dominating Set

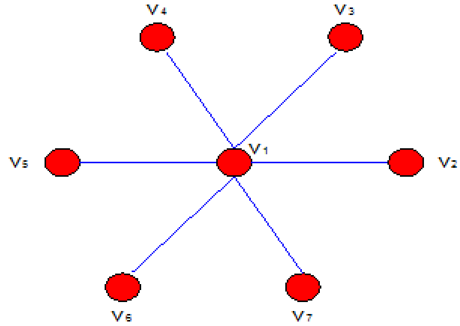
A subset $D \subseteq V(G)$ is called a dominating set, if every vertices of $V - D$ is adjacent to a member of D. A dominating set of G with minimum cardinality is called a minimum dominating set and the cardinality of a minimum dominating set is called the domination number and denoted by $\gamma(G)$. The upper domination number of a graph G denoted by $\Gamma(G)$ is defined as the maximum cardinality of a minimum dominating set of G. Following are the examples of dominating set:



Dominating set $\{v_1, v_5\}$



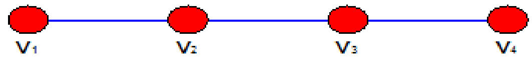
Dominating set $\{v_3, v_4, v_7, v_8\}$



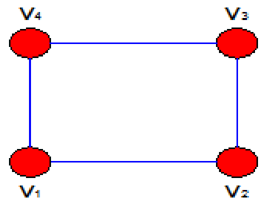
Dominating set $\{v_1\}$

Independent Dominating set

A set D of vertices in a graph G is called an independent dominating set of G if D is both an independent and a dominating set of G. This set is also called a Stable set or a Kernel of the graph. Independent dominating sets were introduced into the theory of games by Neumann and Margenstern in 1944 [8]. The independent domination number $i(G)$ is the cardinality of the smallest independent domination set. Following are the examples of independent dominating set:



Independent dominating set $\{v_1, v_3\}$



Independent dominating set $\{v_1, v_3\}$

4. Main Results

We now present identities of domination set and independent set of graphs containing vertex-disjoint cycles. Also present identities related to domination number, upper domination number, Independence number, independent domination number.

Theorem (4.1). Let G be a graph of order n and containing two vertex-disjoint cycles. A dominating set $D \subseteq V$ is a minimal dominating set if and only if (i) if $N(u) \subset V - D$, i.e. u is an isolate of D. (ii) $\exists$ a vertex $v \in V - D$ for which $N(v) \cap D = \{u, w\}$.

Proof. Let D be a minimal dominating set of a graph G having two vertex-disjoint cycles. Suppose $N(u) \not\subset V-D$, $\exists$ a vertex $v \in D$ such that

$$uv \in E(G). \qquad (4.1)$$

Suppose for every $v \in V-D$, $N(u) \cap D \neq \{u, w\}$, then $\exists$ a vertex $t_v \in D$ such that

$$t_v \neq u, t_v \neq w \text{ and } t_v \in N(v) \cap D. \qquad (4.2)$$

Since D is a dominating set, we get $N(u) \cap D \neq \phi$

Consider $D-\{u, w\}$, then

$$V-(D-\{u, w\}) = V-D \cup \{u, w\}.$$

Let $v \in V-D \cup \{u, w\}$, if

$$v \neq u, v \neq w, \textit{then } N(v) \cap (D-\{u, w\}) = \phi.$$

Also by 4.1, $N(u) \cap (D-\{u, w\}) \neq \phi.$

Thus $D-\{u, w\}$ is a dominating set which is contradiction, since D is minimal dominating set. Hence $N(u) \subset V-D.$

Conversely, Let D be a dominating set and $u, w \in D.$

Since $N(u) \subset V-D,$ then $D-\{u, w\}$ is not a dominating set.

Suppose (ii) holds, then $N(v) \cap (D-\{u, w\}) = \phi.$ for some $v \subset V-D.$

Hence $D-\{u, w\}$ is not a dominating set. Thus D is a minimal dominating set.

Theorem (4.2). Let G be a graph of order n and containing two vertex-disjoint cycles. If $D \subseteq V$ is a dominating set $D \subseteq V$, then $\gamma(G) \leq \frac{n}{2} - 1,\ n \geq 6.$

Proof: Let D be a minimum dominating set of a graph G containing two vertex-disjoint cycles, then $|D| = \gamma.$

Since D is a dominating set and if $u \in D$, then by theorem 4.2, either u is an isolate of S, i.e., $N(u) \subset V-D$ or $\exists$ a vertex $v \in V-D$ for which $N(v) \cap D = \{u, w\}.$

Since $v \in V-D$ such that v is adjacent to u, thus $V-D$ is also a dominating set and hence

$$|V-D| \geq \gamma(G) + 2 \textit{ or } |V| - |D| \geq \gamma + 2$$

$$|V| \geq 2\gamma + 2 = 2(\gamma+1) \textit{ or } \gamma(G) \leq \frac{n}{2} - 1, n \geq 6.$$

Theorem (4.3). Let G be a graph of order n and containing two vertex-disjoint cycles, then every maximal independent set is a minimal dominating set of G.

Proof. Let G be a graph containing two vertex-disjoint cycles and D be a maximal dominating set in G, then clearly D is a dominating set.

If we assume D is not a minimal dominating set, then $\exists$ a vertex $v \in D$ for which $D-\{v\}$ is a dominating set. Thus $\exists$ a vertex $u \in D-\{v\}$ such that u and v are adjacent, which is contradiction. Hence D is a minimal dominating set of G.

Now we are in a position to state the following theorems without proof because it can be easily proved from the theorems given above.

Theorem (4.4). Let G be a graph of order n containing two vertex-disjoint cycles. Then

(a) $\beta(G) = \Gamma(G) = n-4,\ n \geq 6,$

(b) $\gamma(G) = i(G),\ n \geq 6$ and

(c) $\Delta(G) > \delta(G),\ n \geq 6.$

Theorem (4.5). Let G be a graph of order n and containing three vertex-disjoint cycles. A dominating set $D \subseteq V$ is a minimal dominating set if and only if (i) if $N(u) \subset V-D$, i.e. u is an isolate of D. (ii) $\exists$ a vertex $v \in V-D$ for which $N(v) \cap D = \{u, w, z\}.$

Theorem (4.6). Let G be a graph of order n and containing three vertex-disjoint cycles. If $D \subseteq V$ is a dominating set $D \subseteq V$, then $\gamma(G) \leq \frac{n-3}{2},\ n \geq 9.$

Theorem (4.7). Let G be a graph of order n and containing three vertex-disjoint cycles, then every maximal independent set is a minimal dominating set of G.

Next we state identity related to domination number, independence number of a graph contains three vertex-disjoint cycles.

Theorem (4.8). Let G be a graph of order n and containing three vertex-disjoint cycles. Then

(a) $\beta(G) = \Gamma(G) = n-6,\ n \geq 9,$

(b) $\gamma(G) = i(G),\ n \geq 9$

and (c) $\Delta(G) > \delta(G),\ n \geq 9.$

Theorem (4.9). Let G be a graph of order n and containing four vertex-disjoint cycles. A dominating set $D \subseteq V$ is a minimal dominating set if and only if

(i) if $N(u) \subset V-D$, i.e. u is an isolate of D.

(ii) $\exists$ a vertex $v \in V-D$ for which

$$N(v) \cap D = \{u, w, z, l\}.$$

Theorem (4.10). Let G be a graph of order n and containing four vertex-disjoint cycles. If $D \subseteq V$ is a dominating set $D \subseteq V$, then $\gamma(G) \leq \frac{n-4}{2},\ n \geq 12.$

Theorem (4.11). Let G be a graph of order n and containing four vertex-disjoint cycles, then every maximal independent set is a minimal dominating set of G.

Next we state identity related to domination number, independence number of a graph contains four vertex-disjoint cycles.

Theorem (4.12). Let G be a graph of order n and containing four vertex-disjoint cycles, then

(a) $\beta(G) = \Gamma(G) = n-8,\ n \geq 12,$

(b) $\gamma(G) = i(G),\ n \geq 12$ and

(c) $\Delta(G) > \delta(G),\ n \geq 12.$

5. Conclusion

In this paper, we have presented some identities of domination set and independent set of graphs containing vertex-disjoint cycles. Also presented some identities related to domination number, upper domination number, Independence number and independent domination number of graphs containing vertex-disjoint cycles. This consideration given here would be interesting to apply to other types of graphs.

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