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Some New Wilker and Generalized Lazarević Type Inequalities for Modified Bessel Functions

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Abstract In this paper our aim is to deduce some new Wilker types inequalities for modied Bessel function of the first kind. In addition, a generalized Lazarević's inequality is established.

Keywords: *The modified Bessel functions, Wilker type inequalities, Lazarević inequality*

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1. Introduction

In the literature there are many inequalities satisfied by Bessel and modified Bessel functions of the first kind. Different types of inequalities for this functions are proved, like Turán types inequalities [4,15], Jordan's type Inequalities [5], Redheffer's type inequalities [6,8,24], Huygens types inequalities [10,11] and Frame's types inequalities [9], ...etc. This paper is a continuation of some inequalities for this functions. Wilker [19] proposed two open problems:

a. Prove that if $x \in \left(0, \frac{\pi}{2}\right)$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan(x)}{x} > 2, \tag{1}$$

b. Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{2}$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

In [16], inequality (1) was proved, and the following inequality:

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x \quad \text{for } 0 < x < \frac{\pi}{2},$$

where the constants $(2/\pi)^4$ and $\frac{8}{45}$ are best possible, was also established.

Wilker-type inequalities (1) and (2) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [3,23] and the references cited therein).

In this paper, some new Wilker-type inequalities involving modified Bessel functions of the first kind are established. Moreover, we present a new proof of generalization of the Lazarević and Wilker-type inequalities proved by Baricz [2].

Let $\nu > -1$, let us consider the function $\mathcal{I}_\nu : \mathbb{R} \to [1, \infty)$ defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+1)} x^{2n} \tag{3}$$

and I_ν is the modified Bessel function of the first kind defined by [[18], p. 77]

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{2^{2n+p}\Gamma(n+1)\Gamma(\nu+n+1)} x^{2n+p}, x \in \mathbb{R}. \tag{4}$$

It is worth mentioning that in particular we have,

$$\mathcal{I}_{-\frac{1}{2}}(x) = \cosh(x) \tag{5}$$

$$\mathcal{I}_{\frac{1}{2}}(x) = \frac{\sinh(x)}{x} \tag{6}$$

$$\mathcal{I}_{\frac{3}{2}}(x) = -3\left(\frac{\sinh(x)}{x^3} - \frac{\cosh(x)}{x^2}\right). \tag{7}$$

2. Lemmas

In order to establish our main results, we need several lemmas, which we present in this section.

Lemma 1. [14] *Let a_n and $b_n (n = 0,1,2,...)$ be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$*

and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ *be convergent for* $|x|<R$. *If* $b_n>0$ *for* $n=0,1,..,$ *and if* $\frac{a_n}{b_n}$ *is strictly increasing (or decreasing) for* $n=0,1,2,...,$ *then the function* $\frac{A(x)}{B(x)}$ *is strictly increasing (or decreasing) on* $(0,R)$.

Lemma 2. [1,8,13] *Let* $f,g:[a,b]\to\mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g'\neq 0$ *on* (a,b). *If* $\frac{f'}{g'}$ *is increasing (or decreasing) on* (a,b), *then the functions* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *and* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *are also increasing (or decreasing) on* (a,b).

3. Wilker and Lazarević Type Inequalities for Modified Bessel Functions

The first aim of this paper is to prove the following inequalities.

Theorem 1. *Let* $\nu>-1$, *the following inequalities*

$$\mathcal{I}_{\nu+1}(x)>\left(\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}\right)^{\nu+1} \qquad (8)$$

and

$$\mathcal{I}_{\nu+1}^2(x)+\left(\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)}\right)^{\nu+1}>2. \qquad (9)$$

holds for all $x\in(0,\infty)$.

Proof. Let $\nu>-1$, we define the function F_ν on $(0,\infty)$ by

$$F_\nu(x)=\frac{\log\left(\mathcal{I}_{\nu+1}(x)\right)}{\log\left(\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}\right)}=\frac{f(x)}{g(x)}$$

where $f(x)=\log\left(\mathcal{I}_{\nu+1}(x)\right)$ and $g(x)=\log\left(\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}\right)$.

By using the differentiation formula [[18], p. 79]

$$\mathcal{I}_\nu'(x)=\frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x) \qquad (10)$$

we can easily show that

$$f'(x)=\frac{x\mathcal{I}_{\nu+2}(x)}{2(\nu+2)\mathcal{I}_{\nu+1}(x)}, \qquad (11)$$

and

$$g'(x)=\frac{x\left(\frac{1}{\nu+1}\mathcal{I}_{\nu+1}^2(x)-\frac{1}{\nu+2}\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)\right)}{2\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}. \qquad (12)$$

Thus

$$\frac{f'(x)}{g'(x)}=\frac{\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}{\frac{\nu+2}{\nu+1}\mathcal{I}_{\nu+1}^2(x)-\mathcal{I}_{\nu+2}(x)\mathcal{I}_\nu(x)} \qquad (13)$$

Using the Cauchy product

$$\mathcal{I}_\mu(x)\mathcal{I}_\nu(x)=\sum_{n\geq 0}\frac{\Gamma(\nu+1)\Gamma(\mu+1)\Gamma(\nu+\mu+2n+1)x^{2n}}{\left[\begin{array}{l}2^{2n}\Gamma(n+1)\Gamma(\nu+\mu+n+1)\\ \Gamma(\mu+n+1)\Gamma(\nu+n+1)\end{array}\right]} \qquad (14)$$

we get

$$\frac{f'(x)}{g'(x)}=\frac{\sum_{n=0}^{\infty}a_n(\nu)x^{2n}}{\sum_{n=0}^{\infty}b_n(\nu)x^{2n}}$$

where

$$a_n(\nu)=\frac{\Gamma(\nu+1)\Gamma(\nu+3)\Gamma(2\nu+2n+3)}{\left[\begin{array}{l}2^{2n}\Gamma(n+1)\Gamma(2\nu+n+3)\\ \Gamma(\mu+n+1)\Gamma(\nu+n+3)\end{array}\right]}, \qquad (15)$$

and

$$b_n(\nu)=\frac{\Gamma(\nu+1)\Gamma(\nu+3)\Gamma(2\nu+2n+3)}{\left[\begin{array}{l}2^{2n}\Gamma(n+1)\Gamma(2\nu+n+3)\\ \Gamma(\mu+n+2)\Gamma(\nu+n+3)\end{array}\right]}. \qquad (16)$$

Now, we define the sequence $c_n=\frac{a_n}{b_n}$ for $n=0,1,...,$ thus

$$C_n(\nu)=\nu+n+1.$$

We conclude that c_n is increasing for $n=0,1,2,...$ and $\frac{f'(x)}{g'(x)}$ is also increasing on $(0,\infty)$ by Lemma 1. Thus

$$F_\nu(x)=\frac{f(x)}{g(x)}=\frac{f(x)-f(0)}{g(x)-g(0)}, \qquad (17)$$

is increasing on $(0,\infty)$ by Lemma 2. Furthermore,

$$\lim_{x\to 0^+}F_\nu(x)=c_0(\nu)=\nu+1,$$

from which follows the inequality (8) holds for all $\nu>-1$ and $x>0$. Finally, using the the arithmeticgeometric mean inequality and equality (9), we get

$$\frac{1}{2}\left[\mathcal{I}_{\nu+1}(x)+\left(\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)}\right)^{\nu+1}\right]\geq\sqrt{\frac{\mathcal{I}_{\nu+1}^2(x)}{\exp(\nu+1)\mathcal{I}_\nu(x)}}\geq 1. \qquad (18)$$

Since $\mathcal{I}_\nu(x)\geq 1$, the inequality (9) holds for all $\nu>-1$ and $x>0$. So, the proof of Theorem 1 is complete.

In this theorem, we establish new inequalities of the Wilker type for modified Bessel function of the first kind.

Theorem 2. *Let* $\nu>-1$ *and* $x>0$, *the following Wilker type inequality*

$$\left(\mathcal{I}_{\nu+1}(x)\right)^{\frac{1}{\nu+1}}+\frac{\mathcal{I}_{\nu}(x)}{\mathcal{I}_{\nu+1}(x)}\geq 2, \tag{19}$$

is valid. In particular, the following inequality

$$\left(\frac{\sinh x}{x}\right)^{2}+\frac{x}{\tanh x}\geq 2 \tag{20}$$

hold for all $x>0$.

Proof. We define the function $H_{\nu}(x)$ on $(0,\infty)$ by

$$H_{\nu}(x)=\left(\mathcal{I}_{\nu+1}(x)\right)^{\frac{1}{\nu+1}}-\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_{\nu}(x)}.$$

The Mittag-Leffler expansion for the modified Bessel functions of first kind, which becomes [[7], Eq. 7.9.3]

$$\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_{\nu}(x)}=\sum_{n=1}^{\infty}\frac{2x}{j_{\nu,n}^{2}+x^{2}}, \tag{21}$$

where $0<j_{\nu,1}<j_{\nu,2}<\ldots<j_{\nu,n}<\ldots,$ are the positive zeros of the Bessel function J_{ν}, and the differentiation formula (10) we have

$$H_{\nu}'(x)=\frac{x}{2(\nu+1)(\nu+2)}\mathcal{I}_{\nu+2}(x)\left(\mathcal{I}_{\nu+1}(x)\right)^{-\frac{\nu}{\nu+1}}$$
$$+2\sum_{n\geq 1}\frac{x}{\left(x^{2}+j_{\nu,n}^{2}\right)^{2}}>0.$$

Thus implies that the function $H_{\nu}(x)$ is increasing on $(0,\infty)$ for all $\nu>-1,$ and consequently $H_{\nu}(x)>0$ for all $x>0.$ So, the following inequality

$$\mathcal{I}_{\nu}(x)\geq\left(\mathcal{I}_{\nu+1}(x)\right)^{\frac{\nu}{\nu+1}}, \tag{22}$$

holds for all $\nu>-1$ and $x\in(0,\infty)$. By using the arithmetic-geometric mean inequality we obtain

$$\frac{1}{2}\left[\left(\mathcal{I}_{\nu+1}(x)\right)^{\frac{1}{\nu+1}}+\frac{\mathcal{I}_{\nu}(x)}{\mathcal{I}_{\nu+1}(x)}\right]\geq\sqrt{\frac{\mathcal{I}_{\nu}(x)}{\left(\mathcal{I}_{\nu+1}(x)\right)^{\frac{\nu}{\nu+1}}}}\geq 1.$$

Finally, Observe that using (5) and (6) in particular for $\nu=-\frac{1}{2}$ we obtain the inequality (20).

In the next theorem we present a generalization of the Lazarević and Wilker-type inequalities to modified Bessel functions of the first kind. The next result exist in [3]. We give an elementary proof.

Theorem 3. *let $\nu>-1$ and $x>0$. Then, the following inequalities*

$$\left(\mathcal{I}_{\nu+1}(x)\right)^{q}\geq\mathcal{I}_{\nu}(x) \tag{23}$$

and

$$\left(\mathcal{I}_{\nu+1}(x)\right)^{q-1}+\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_{\nu}(x)}\geq 2, \tag{24}$$

holds if and only if $q\geq\frac{\nu+2}{\nu+1}$.

Proof. Let $G_{\nu}(x)=\frac{\log\left(\mathcal{I}_{\nu}(x)\right)}{\log\left(\mathcal{I}_{\nu+1}(x)\right)}=\frac{f_1(x)}{g_1(x)}$ where $f_1(x)=\log\left(\mathcal{I}_{\nu}(x)\right)$ and $g_1(x)=\log\left(\mathcal{I}_{\nu+1}(x)\right).$ From the differentiation formula (10) we get

$$\frac{f_1'(x)}{g_1'(x)}=\left[\frac{\nu+2}{\nu+1}\right]\frac{\mathcal{I}_{\nu+2}^{2}(x)}{\mathcal{I}_{\nu}(x)\mathcal{I}_{\nu+2}(x)}=\frac{\sum_{n=0}^{\infty}a_n(\nu)x^{2n}}{\sum_{n=0}^{\infty}b_n(\nu)x^{2n}}$$

where

$$a_n(\nu)=\left[\frac{\nu+2}{\nu+1}\right]\frac{\Gamma^{2}(\nu+2)\Gamma(2\nu+2n+3)}{2^{2n}\Gamma(n+1)\Gamma(2\nu+n+3)\Gamma^{2}(\nu+n+2)}$$

and

$$b_n(\nu)=\frac{\Gamma(\nu+3)\Gamma(\nu+1)\Gamma(2\nu+2n+3)}{2^{2n}\Gamma(n+1)\Gamma(2\nu+n+3)\Gamma(\nu+n+3)\Gamma(\nu+n+1)}.$$

We define the sequence $(c_n)_{n\geq 0}$ by $c_n=\frac{a_n}{b_n}$ for $n=0,1,\ldots,$ thus implies that

$$\frac{c_{n+1}}{c_n}=\frac{(\nu+n+3)(\nu+n+1)}{(\nu+n+2)^{2}}<1$$

and consequently c_n is decreasing for $n=0,1,\ldots,$ so the function $\frac{f'(x)}{g'(x)}$ is decreasing on $(0,\infty)$ by Lemma 1. Thus $G_{\nu}(x)=\frac{f_1(x)-f_1(0^{+})}{g_1(x)-g_1(0^{+})}$ is decreasing on $(0,\infty)$ by Lemma 2. Now,

$$\lim_{x\to 0^{+}}G_{\nu}(x)=c_0=\frac{\nu+2}{\nu+1}.$$

Therefore, (23) holds. From arithmetic-geometric mean inequality and inequality (23), we have

$$\mathcal{I}_{\nu+1}^{q-1}(x)+\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_{\nu}(x)}\geq 2\sqrt{\frac{\mathcal{I}_{\nu+1}^{q}(x)}{\mathcal{I}_{\nu}(x)}}>2,$$

and the proof of theorem is complete.

4. Concluding Remarks

1. In proof of Theorem 1, we can see that the following Turán type inequality

$$\mathcal{I}_{\nu}(x)\mathcal{I}_{\nu+2}(x)<\frac{\nu+2}{\nu+1}\mathcal{I}_{\nu+2}^{2}(x), \tag{25}$$

holds for all $\nu>-1$ and $x\in\mathbb{R}$. This inequality is not new and it is actually equivalent to a very well-known Turán type inequality for the modified Bessel function of the first kind, firstly discovered by Thiruvenkatachar and Nanjundiah, see [17]

2. On the other hand, by using (5), (6) and (7) in particular for $\nu=\frac{-1}{2},$ the Turán type inequality (25) becomes

$$\cosh(x)\left(\cosh(x) - x\sinh(x)\right) < \sinh^2(x)$$

3. Inequality (23) is a natural generalization of the Lazarević inequality [[12], p. 207]

$$\cosh(x) < \left(\frac{\sinh(x)}{x}\right)^3 \quad (26)$$

where $x \neq 0$. Recently, Zhu gives a new proof of the inequality (26) in [20] and extends the inequality (26) to the following result in [21] by: for $p > 0$ and $x \in (0, \infty)$, then

$$\frac{p}{2}\cosh(x) + \frac{2-p}{2}\frac{\sinh(x)}{x} \leq \left(\frac{\sinh(x)}{x}\right)^q \quad (27)$$

holds if and only if $p \geq q+1$. Moreover, in [22], Zhu gives another generalization of the inequality (26) as follows: for $p > 1$ or $p < \frac{8}{15}$, then

$$p + (1-p)\cosh(x) \leq \left(\frac{\sinh(x)}{x}\right)^q \quad (28)$$

holds if and only if $q \geq 3(1-p)$.

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