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Generalized *a*-Browder's and Generalized *a*-Weyl's Theorems for Banach Space

M. H. M. RASHID*

Department of Mathematics& Statistics, Faculty of Science P.O.Box(7), Mu'tah University, Al-Karak, Jordan
*Corresponding author: malik_okasha@yahoo.com

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Abstract In this paper, we give necessary and sufficient conditions for a Banach space T to satisfy the generalized a-Browder's theorem. We also prove that the spectral mapping theorem holds for the left Drazin invertible and for analytic functions on a neighborhood of $\sigma(T)$. As applications, we show that if T^* is algebraically $\omega F(p,r,q)$ for each $p,r>0$ and $q\geq 1$, or if T^* is algebraically quasi-class A, then the generalized a-Weyl's theorem hold for $f(T)$, where $f\in Hol(\sigma(T))$, the space of functions analytic on an open neighborhoods of $\sigma(T)$.

Keywords: *Weyl's theorem, a-Weyl's theorem, generalized a-Weyl's theorem, a-Browder's theorem, generalized a-Browder's theorem, reduction a-isoloid, reduced subspace*

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1. Introduction

Throughout this paper let $\mathbf{B}(\mathbb{X}), \mathbf{F}(\mathbb{X}), \mathbf{K}(\mathbb{X})$, denote, respectively, the algebra of bounded linear operators, the set of finite rank operators and the ideal of compact operators acting on an infinite dimensional Banach space $\mathbb{X}$. If $T\in\mathbf{B}(\mathbb{X})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ (or $ran(T)$) for the null space and range of T, respectively. Also, let $\alpha(T):=\dim\ker(T)$, $\beta(T):=\dim R(T)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T\in\mathbf{B}(\mathbb{X})$ called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T):=\alpha(T)-\beta(T).$$

T is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm of finite ascent and descent.

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p)=\ker(T^{p+1})$. If such integer does not exist we put $a(T)=\infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathcal{R}\left(T^q\right)=\mathcal{R}\left(T^{q+1}\right)$, , and if such integer does not exist we put $d(T)=\infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Fredholm}\}$$

$$\sigma_W(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T)\subseteq\sigma_W(T)\subseteq\sigma_b(T)\subseteq\sigma_F(T)\cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K\subseteq\mathbb{C}$.

Following [12], we say that *Weyl's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T)=E_0(T)$, where $E_0(T)$ is the set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$. And *Browder's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T)=\pi_0(T)$, where $\pi_0(T)$ is the set of all poles of T of finite rank.

Let $SF_+(\mathbb{X})$ be the class of all upper semi-Fredholm operators, $SF_+^-(\mathbb{X})$ be the class of all $T\in SF_+(\mathbb{X})$ with $i(T)\leq 0$, and for any $T\in\mathbf{B}(\mathbb{X})$ let

$$\sigma_{SF_+^-}(T)=\left\{\lambda\in\mathbb{C}:T-\lambda I\notin SF_+^-(\mathbb{X})\right\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [23], we say that T satisfies a*-Weyl's theorem* if $\sigma_{SF_+^-}(T)=\sigma_a(T)\setminus E_0^a(T)$. It follows from [[23], Corollary 2.5]] a-Weyl's theorem implies Weyl's theorem.

In [11] Berkani define the class of *B*-Fredholm operators as follows. For each integer n, define T_n to be the restriction of T to $\mathcal{R}\left(T^n\right)$ viewed as a map from $\mathcal{R}\left(T^n\right)$ into $\mathcal{R}\left(T^n\right)$ (in particular $T_0 = T$). If for some n the range $\mathcal{R}\left(T^n\right)$ is closed and T_n is Fredholm (resp. Semi-Fredholm) operator, then T is called a *B-Fredholm* (resp. *Semi-B-Fredholm*) operator. In this case and from [5] T_m is a Fredholm operator and $i(T_m) = i(T_n)$ for each $m \geq n$. The index of a B -Fredholm operator T is defined as the index of the Fredholm operator T_n, where n is any integer such that the range $\mathcal{R}\left(T^n\right)$ is closed and T_n is Fredholm operator (see [11]).

Let $BF(\mathbb{X})$ be the class of all B-Fredholm operators. In [5] Berkani has studied this class of operators and has proved that an operator $T \in \mathbf{B}(\mathbb{X})$ is a B-Fredholm if and only if $T = T_0 \oplus T_1$, where T_0 is a Fredholm operator and T_1 is a nilpotent operator.

Recall that an operator $T \in \mathbf{B}(\mathbb{H})$ is called a *B-Weyl* operator(see [7]) if it is a B -Fredholm operator of index 0 . The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a } B-\text{Weyl operator}\}.$$

In the case of a normal operator T acting on a Hilbert space $\mathbb{X}$, Berkani [[11], Theorem 4.5] showed that $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where $E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T . This result gives a generalization of the classical Weyl's theorem.

Let $SBF_+(\mathbb{X})$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(\mathbb{X})$ the class of all $T \in SBF_+(\mathbb{X})$ such that $i(T) \leq 0$, and

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathbb{X})\}.$$

Recall that an operator $T \in \mathbf{B}(\mathbb{X})$ satisfies the *generalized a-Weyl's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$, where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. Note that generalized a-Weyl's theorem implies a-Weyl's theorem, (see [10]).

Recall that an operator $T \in \mathbf{B}(\mathbb{X})$ is a *Drazin invertible* if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is nilpotent operator and T_1 is invertible operator (see [[19], Proposition A]). The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of all poles.

An operator $T \in \mathbf{B}(\mathbb{X})$ is called *left Drazin invertible* if $a(T) < \infty$ and $R(T^{a(T)+1})$ is closed (see [[8], Definition 2.4]). The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

Recall [[8], Definition 2.5] that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible operator and $\lambda \in \sigma_a(T)$ is a left pole of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. We will denote $\pi^a(T)$ the set of all left pole of T , and by $\pi_0^a(T)$ the set of all left pole of T of finite rank. We have $\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T)$. Note that if $\lambda \in \pi^a(T)$, then it is easily seen that $T - \lambda$ is an operator of topological uniform descent. Therefore it follows from ([[10], Theorem 2.5]) that λ is isolated in $\sigma_a(T)$. Following [8] if $T \in \mathbf{B}(\mathbb{X})$ and $\lambda \in \mathbb{C}$ is an isolated in $\sigma_a(T)$, then $\lambda \in \pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$ and $\lambda \in \pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$. The quasinilpotent part $H_0(T - \lambda)$ and the analytic core $K(T - \lambda)$ of $T - \lambda$ are defined by

$$H_0(T - \lambda) := \{x \in \mathbb{X} : \lim_{n \longrightarrow \infty} \left\|(T - \lambda)^n x\right\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} K(T - \lambda) = \{x \in \mathbb{X} &: \text{there exists a sequence } \{x_n\} \subset \mathbb{X} \\ &\text{and } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \\ &\text{and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \cdots\}. \end{aligned}$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are generally non-closed hyper-invariant subspaces of $T - \lambda$ such that $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for all $p = 0, 1, \cdots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T - \lambda) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in \mathbb{X}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow \mathbb{X}$ that satisfies $(T - \mu) f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$ (see [14]).

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [16], we say that $T \in \mathbf{B}(\mathbb{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the only analytic function $f : U_\lambda \to \mathbb{X}$ which satisfies the equation $(T - \mu) f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathbb{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathbf{B}(\mathbb{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [[21], Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Proposition 1.1. [20] *Let* $T \in \mathbf{B}(\mathbb{X})$.

(i) *If* T *has the SVEP, then* $i(T-\lambda I)\leq 0$ *for every* $\lambda\in\rho_{SBF}(T)$.

(ii) *If* T^* *has the SVEP, then* $i(T-\lambda I)\geq 0$ *for every* $\lambda\in\rho_{SBF}(T)$.

(iii) *If* T^* *has the SVEP, then*

(a) $\sigma_{SF_+^-}(T)=\sigma_W(T)$ and (b) $\sigma_{SBF_+^-}(T)=\sigma_{BW}(T)$

In [27] H. Weyl examined the spectra of all compact perturbations of a Hermitian operator T on a Hilbert space and proved that their intersection coincides with the isolated point of the spectrum $\sigma(T)$ which are the eigenvalues of finite multiplicity. Weyl's theorem has been extended to several classes of Hilbert space operators including seminormal operators [3,4]. In [6] M. Berkani introduced the concepts of the generalized Weyl's theorem and generalized Browder's theorem, and they showed that T satisfies the generalized Weyl's theorem whenever T is a normal operator on Hilbert space. More recently, [9] extended this result to hyponormal operators. In [26] extended this result to $\log$ -hyponormal operators. Recently, Rashid et al. [25] showed that if T is quasi-class A then generalized Weyl's theorem holds $f(T)$, for every $f\in Hol(\sigma(T))$. More recently, I. J. An and Y. M. Han [2] showed that Weyl's theorem holds for algebraically quasi-class A operators.

In this paper we extend this result to several classes much larger than that of normal operators. we first find necessary and sufficient conditions for a Banach space operator T to satisfy the generalized a -Browder's theorem. We then characterize the smaller class of operators satisfying the generalized a -Weyl's theorem. A long the way we prove that the spectral mapping theorem always holds for the left Drazin spectrum and for analytic functions on an open neighborhood of $\sigma(T)$. we have three main applications of our results: if T is an algebraically $\omega F(p,r,q)$ operators with $p,r>0$ and $q\geq 1$, or if T is an algebraically quasi-class A, then the generalized a -Weyl's theorem holds for $f(T)$, for each $f\in Hol(\sigma(T))$; and if T is reduced by each of its eigenspaces, then generalized a -Browder's theorem holds for $f(T)$, for each $f\in Hol(\sigma(T))$. Throughout of this paper, we will use gaW to signify that an operator $T\in\mathbf{B}(\mathbb{X})$ obeys generalized a-Weyl's theorem, analogous meaning is attached to the abbreviations gaB with respect to Browder's theorem.

2. Structural Properties of Operators in gaB and gaW

Theorem 2.1. *Let* $T\in\mathbf{B}(\mathbb{X})$. *Then the following statements are equivalent*:

(i) $T\in gaB$;

(ii) $\sigma_{SBF_+^-}(T)=\sigma_{lD}(T)$;

(iii) $\sigma_a(T)=\sigma_{SBF_+^-}(T)\cup E^a(T)$;

(iv) $acc(\sigma_a(T))\subseteq\sigma_{SBF_+^-}(T)$;

(v) $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)\subseteq E^a(T)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $T\in gaB$. Then $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)=\pi^a(T)$. Let $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Then $\lambda\in\pi^a(T)$, and so $T-\lambda I$ is left Drazin invertible. Therefore $\lambda\in\sigma_a(T)\setminus\sigma_{LD}(T)$, and hence, $\sigma_{LD}(T)\subseteq\sigma_{SBF_+^-}(T)$. On the other hand, since $\sigma_{SBF_+^-}(T)\subseteq\sigma_{LD}(T)$ is always verified for any operator T [[10], Lemma 2.12.].

(ii) $\Rightarrow$ (i). We assume that $\sigma_{SBF_+^-}(T)=\sigma_{LD}(T)$ and we will establish that $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)=\pi^a(T)$. Suppose first that $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Then $\lambda\in\sigma_a(T)\setminus\sigma_{LD}(T)$, and so $T-\lambda I$ is left Drazin invertible. Therefore $d=a(T)<\infty$ and $ran(T^{d+1})$ is closed. Since $\lambda\in\sigma_a(T)$, we have $\lambda\in\pi^a(T)$. Thus $\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)\subseteq\pi^a(T)$.

Conversely, suppose that $\lambda\in\pi^a(T)$. Then $T-\lambda I$ is left Drazin invertible but not bounded below. Since λ is an isolated point of $\sigma_a(T)$, then $T-\lambda\in SBF_+^-(X)$. Therefore $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Thus $\pi^a(T)\supseteq\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$.

(ii) $\Rightarrow$ (iii). Let $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Then $\lambda\in\sigma_a(T)\setminus\sigma_{LD}(T)$, and so $T-\lambda I$ is left Drazin invertible but not bounded below. Therefore $\lambda\in E^a(T)$. Thus $\sigma_a(T)\subseteq\sigma_{SBF_+^-}(T)\cup E^a(T)$. Since the other inclusion is always true, we must have $\sigma_a(T)=\sigma_{SBF_+^-}(T)\cup E^a(T)$.

(iii) $\Rightarrow$ (ii). Suppose $\sigma_a(T)=\sigma_{SBF_+^-}(T)\cup E^a(T)$. To show that $\sigma_{SBF_+^-}(T)=\sigma_{LD}(T)$. it suffices to show that $\sigma_{SBF_+^-}(T)\subseteq\sigma_{LD}(T)$. Suppose that $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Then $T-\lambda I\in SBF_+^-(X)$ but not invertible. Since $\sigma_a(T)=\sigma_{SBF_+^-}(T)\cup E^a(T)$, we see that $\lambda\in E^a(T)$. In particular, λ is an isolated point of $\sigma_a(T)$. Hence, $T-\lambda I$ is left Drazin invertible, and so $\sigma_{SBF_+^-}(T)=\sigma_{LD}(T)$.

(i) $\Leftrightarrow$ (iv). Suppose $T\in gaB$. Then $\sigma_{SBF_+^-}(T)=\sigma_a(T)\setminus\pi^a(T)$. Let $\lambda\in\sigma_a(T)\setminus\sigma_{SBF_+^-}(T)$. Then $\lambda\in\pi^a(T)$, and so λ is an isolated point of $\sigma_a(T)$. Therefore $\lambda\in\sigma_a(T)\setminus acc(\sigma_a(T))$, and hence, $acc(\sigma_a(T))\subseteq\sigma_{SBF_+^-}(T)$.

Conversely, let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, it follows that $\lambda \in iso(\sigma_a(T))$ and $T-\lambda I \in SBF_+^-(\mathbb{X})$. It follows from [[10], Theorem 2.8.] that $\lambda \in \pi^a(T)$. Therefore $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T)$.

For the converse, suppose $\lambda \in \pi^a(T)$. Then λ is a left pole of the resolvent of T, and so λ is an isolated point of $\sigma_a(T)$. Therefore $\lambda \in \sigma_a(T) \setminus acc(\sigma_a(T))$. It follows from [[10], Theorem 2.11.] that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Thus $\pi^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, and so $T \in gaB$.

(iv) $\Leftrightarrow$ (v). Suppose that $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T-\lambda \in SBF_+^-(\mathbb{X})$ but not bounded below. Since $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, λ is an isolated point of $\sigma_a(T)$. It follows from [[10], Theorem 2.8.] that λ is a left pole of of the resolvent of T. Therefore $\lambda \in \pi^a(T)$, and hence, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$.

Conversely, suppose that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$ and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$. Then $\lambda \in E^a(T)$, and so λ is an isolated point of $\sigma_a(T)$. Therefore $\lambda \in \sigma_a(T) \setminus acc(\sigma_a(T))$, which implies that $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$.

Recall that $gaW \subseteq gaB$, see [10]. However, the reverse inclusion does not hold, as the following example shows.

Example 2.2. Let $\mathbb{X} = \ell^2$, let $T_1, T_2 \in \mathbf{B}(\mathbb{X})$ be given by $T_1(x_1, x_2, \cdots) := (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \cdots)$ and $T_2 := 0$, and let $T := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $\mathbb{X} \oplus \mathbb{X}$.

Then $\sigma_a(T) = \omega(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}$ and $\pi^a(T) = \emptyset$. Therefore $T \in gaB \setminus gaW$.

The next result gives simple necessary and sufficient conditions for an operator $T \in gaB$ to belong to the smaller class gaW.

Theorem 2.3. *Let $T \in gaB$. The following statements are equivalent:*

(i) $T \in gaW$.

(ii) $\sigma_{SBF_+^-}(T) \cap E^a(T) = \emptyset$.

(iii) $\pi^a(T) = E^a(T)$.

Proof. (i) $\Rightarrow$ (ii). Assume $T \in gaW$, that is, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. It then easily that $\sigma_{SBF_+^-}(T) \cap E^a(T) = \emptyset$, as required for (ii).

(ii) $\Rightarrow$ (iii). Let $\lambda \in E^a(T)$. The condition in (ii) implies that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, and since $T \in gaB$, we must have $\lambda \in \pi^a(T)$. It follows that $E^a(T) \subseteq \pi^a(T)$, and since the reverse inclusion always holds, we obtain (iii).

(iii) $\Rightarrow$ (i). Since $T \in gaB$, we know that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$, and since we are assuming $E^a(T) = \pi^a(T)$, it follows that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$, that is, $T \in gaW$.

Theorem 2.4. *Let $T \in \mathbf{B}(\mathbb{X})$. Then*

$$\sigma_{LD}(T) = \sigma_{SBF}(T) \cup acc(\sigma_a(T)).$$

Proof. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$. Then $T-\lambda I$ is left Drazin invertible but not bounded below. Then $\lambda \in \pi^a(T)$. Therefore, $s = a(T-\lambda I) < \infty$ and $ran((T-\lambda I)^{d+1})$ is closed. Hence it follows from [[10], Remark 2.7] that λ is an isolated point of $\sigma_a(T)$. Hence $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF}(T) \cup acc(\sigma_a(T)))$.

Conversely, suppose that $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF}(T) \cup acc(\sigma_a(T)))$. Then $T-\lambda I$ is a semi-B-Fredholm and λ is an isolated point of $\sigma_a(T)$. Since $T-\lambda I$ is semi- B -Fredholm, it follows from [[10], Corollary 2.10] that $T-\lambda I$ can be decompose as $T-\lambda I = T_1 \oplus T_2$, where T_1 is an upper semi-Fredholm operator with $i(T_1) \le 0$ and T_2 is nilpotent. We consider two cases.

Case I. Suppose that T_1 is bounded below. Then $T-\lambda I$ is left Drazin invertible, and so $\lambda \notin \sigma_{LD}(T)$.

Case II. Suppose that T_1 is not bounded below. Then 0 is an isolated point of $\sigma_a(T_1)$. But T_1 is an upper semi-Fredholm operator, hence it follows from the punctured neighborhood theorem that T_1 is a -Browder. Therefore there exists a finite rank operator S_1 such that $T_1 + S_1$ is bounded below and $T_1 S_1 = S_1 T_1$. Put $F := S_1 \oplus 0$. Then F is a finite rank operator, $TF = FT$ and $T - \lambda I + F = T_1 \oplus T_2 + S_1 \oplus 0 = (T_1 + S_1) \oplus T_2$ is left Drazin invertible. Hence $\lambda \notin \sigma_{LD}(T)$.

As shown in [11] that the spectral mapping theorem holds for the Drazin spectrum. We prove here the spectral mapping theorem holds for left Drazin spectrum.

Theorem 2.5. *Let $T \in \mathbf{B}(\mathbb{X})$ and let $f \in Hol(\sigma(T))$. If $f(T) \in gaB$, then $\sigma_{LD}(f(T)) = f(\sigma_{LD}(T))$.*

Proof. Suppose that $\mu \notin f(\sigma_{LD}(T))$ and set $h(\lambda) = f(\lambda) - \mu I$. Then h has no zeros in $\sigma_{LD}(T)$. Since $\sigma_{LD}(T) = \sigma_{SBF}(T) \cup acc(\sigma_a(T))$ by Theorem 2.4, we conclude that h has finitely many zeros in $\sigma_a(T)$. Now we consider two cases.

Case I. Suppose that h has no zeros in $\sigma_a(T)$. Then $h(T) = f(T) - \mu I$ is bounded below, and so $\mu \notin \sigma_{LD}(f(T))$.

Case II. Suppose that h has at least one zero in $\sigma_a(T)$. Then

$$h(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)g(\lambda),$$

where $c, \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ and $g(\lambda)$ is a nonvanishing analytic function on an open neighborhood. Therefore

$$h(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T),$$

where $g(T)$ is invertible. Since $\mu \notin f(\sigma_{LD}(T))$, $\lambda_1, \lambda_2, \cdots, \lambda_n \notin \sigma_{LD}(T)$. Therefore $T - \lambda_j I$ is left Drazin invertible, and hence each $T - \lambda_j I \in SBF(\mathbb{X})$, $j = 1, 2, \cdots, n$. But each λ_j is an isolated point of $\sigma_a(T)$, it follows from [10, Theorem 2.8] that each λ_j is a left pole of the resolvent of T. Therefore $a(T - \lambda_j I) = d < \infty$ and $ran(T - \lambda_j I)^{d+1}$ is closed $(j = 1, 2, \cdots, n)$, so

$$a((T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)) = s < \infty$$

and $ran((T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n))^{s+1}$ is closed. Since $g(T)$ isinvertible, $a(h(T)) = t < \infty$ and $ran((h(T)^{t+1}))$ is closed. Therefore $h(T)$ is left Drazin invertible, and so $0 \notin \sigma_{LD}(h(T))$. Hence, $\mu \notin \sigma_{LD}(f(T))$. It follows from case I and II that $\sigma_{LD}(f(T)) \subseteq f(\sigma_{LD}(T))$.

Conversely, suppose that $\lambda \notin \sigma_{LD}(f(T))$. Then $f(T) - \lambda I$ is left Drazin invertible. We again consider two cases.

Case I. Suppose that $f(T) - \lambda I$ is bounded below. Then $\lambda \notin \sigma_a(f(T)) = f(\sigma_a(T))$, and hence $\lambda \notin f(\sigma_{LD}(T))$.

Case II. Suppose that $\lambda \in \sigma_a(f(T)) \setminus \sigma_{LD}(f(T))$. Write

$$f(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T),$$

where $c, \lambda_1, \lambda_2, \cdots, \lambda_n \in C$ and $g(T)$ is invertible. Since $f(T) - \lambda I$ is left Drazin invertible

$$f(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T)$$

has finite ascent say r and $ran(f(T))^{r+1}$ is closed. Hence, $T - \lambda_j I$ has finite ascent say r_j and $ran(T - \lambda_j)^{r_j+1}$ is closed for every $j = 1, 2, \cdots, n$. Therefore each $T - \lambda_j I$ is left Drazin invertible, and so $\lambda_1, \cdots, \lambda_n \notin \sigma_{LD}(T)$.

We now wish to prove that $\lambda \notin f(\sigma_{LD}(T))$. Assume not; then there exists $\mu \in \sigma_{LD}(T)$ such that $f(\mu) = \lambda$. Since $g(\mu) \neq 0$, we must have $\mu = \mu_j$ for some $j = 1, 2, \cdots, n$, which implies $\mu_j \in \sigma_{LD}(T)$, a contradiction. Hence, $\lambda \notin f(\sigma_{LD}(T))$, and so $f(\sigma_{LD}(T)) \subseteq \sigma_{LD}(f(T))$. This completes the proof.

Corollary 2.6. Let $T \in \mathbf{B}(\mathbb{X})$ and let $f \in Hol(\sigma(T))$. If $f(T) \in gaB$, then

$$\sigma_{SBF_+^-}(f(T) = f(\sigma_{SBF_+^-}(T)).$$

Proof. Since $T \in gaB$, it follows from Theorem 2.1 that $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ and $\sigma_{SBF_+^-}(f(T)) = \sigma_{LD}(f(T))$. By Theorem 2.5 we have

$$\sigma_{LD}(f(T)) = f(\sigma_{LD}(T)) \quad \text{for every } f \in Hol(\sigma(T)).$$

Hence

$$f(\sigma_{SBF_+^-}(T)) = f(\sigma_{LD}(T)) = \sigma_{LD}(f(T)) = \sigma_{SBF_+^-}(f(T)).$$

Theorem 2.7. *Let $S, T \in \mathbf{B}(\mathbb{X})$. If T has SVEP and $S \prec T$, then $f(S) \in gaB$ for every $f \in Hol(\sigma(T))$. In particular, if T has SVEP, then $T \in gaB$.*

Proof. Suppose that T has SVEP. Since $S \prec T$, it follows from the proof of [[13], Theorem 3.2] that S has SVEP. We now show that $S \in gaB$. Let $\lambda \in \sigma_a(S) \setminus \sigma_{SBF_+^-}(S)$; then $S - \lambda I \in SBF_+^-(\mathbb{X})$ but not bounded below. Since $S - \lambda I \in SBF_+^-(\mathbb{X})$, it follows from from [[10], Corollary 2.10] that $S - \lambda I = S_1 \oplus S_2$, where S_1 is an upper semi-Fredholm operator with $i(S_1) \leq 0$, and S_2 is nilpotent. Since S has SVEP, S_1 and S_2 also have SVEP. Therefore a-Browder's theorem holds for S_1, and hence, $\sigma_{ab}(S_1) = \sigma_{SF_+^-}(S_1)$. Since S_1 is semi-Fredholm with $i(S_1) \leq 0$, S_1 is a-Browder's. Hence, λ is an isolated point of $\sigma_a(S)$. It follows from Theorem 2.1 that $S \in gaB$.

Now let $f \in Hol(\sigma(T))$. since the SVEP is stable under the functional calculus, then $f(S)$ has the SVEP. Therefore $f(S) \in gaB$, by the first part of the proof.

We now recall that the generalized a-Weyl's theorem may not hold for quasinilpotent operators, and that it does not necessarily transfer to or from adjoints.

Example 2.8. *Let $T \in \mathbf{B}(\mathbb{X})$ defined on ℓ^2 by*

$$T(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then T is quasinilpotent operator and

$$\sigma(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}.$$

Thus T does not obey generalized a-Weyl's theorem.

Now $\sigma(T^*) = \sigma_{SBF_+^-}(T^*) = \{0\}$ and $E^a(T^*) = \emptyset$.

Therefore $T^* \in gaW$.

3. Operators Reduced by Their Eigenspaces

Let $\mathbf{B}(\mathbb{H})$ denote, the algebra of bounded linear operator acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. Let $T \in \mathbf{B}(\mathbb{H})$. Suppose that T is reduced by each of its eigenspaces. If we let

$$\mathfrak{M} := \bigvee \{\ker(T - \lambda I) : \lambda \in \sigma_p(T)\},$$

it follows that $\mathfrak{M}$ reduces T. Let $T_1 = T|_{\mathfrak{M}}$ and $T_2 = T|_{\mathfrak{M}^{\perp}}$. By [[4], Proposition 4.1] we have:

(i) T_1 is normal operator with pure point spectrum;

(ii) $\sigma_p(T_1) = \sigma_p(T)$;

(iii) $\sigma(T_1) = \overline{\sigma_p(T_1)}$;

(iv) $\sigma_p(T_2) = \emptyset$.

Corollary 3.1. *Suppose that* $T \in \mathbf{B}(\mathbb{H})$ *is reduced by each of its eigenspaces. Then* $f(T) \in gaB$ *for every* $f \in Hol(\sigma(T))$. *In particular,* $T \in gaB$.

Proof. Since T is reduced by each of its eigenspaces, $T - \lambda I$ has finite ascent for each $\lambda \in \mathbb{C}$. Therefore T has SVEP, and since the SVEP is stable under functional calculus, then $f(T)$ has SVEP for every $f \in Hol(\sigma(T))$. It follows that $f(T) \in gaB$.

Corollary 3.2. *Suppose that* $T \in \mathbf{B}(\mathbb{H})$ *is reduced by each of its eigenspaces. Assume that* $\sigma_a(T)$ *has no isolated points. Then* $T, T^* \in gaW$. *Moreover, if* $f \in Hol(\sigma(T))$, $f(T) \in gaW$.

Proof. We first prove that $T \in gaW$. Since T is reduced by each of its eigenspaces, it follows from Corollary 3.1 that $T \in gaB$. By Theorem 2.1, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$.

But $iso\sigma_a(T) = \emptyset$, hence, $E^a(T) = \emptyset$, which implies $\sigma_a(T) = \sigma_{SBF_+^-}(T)$. Therefore $T \in gaW$. On the other hand, observe that $\sigma_a(T^*) = \overline{\sigma_a(T)}$, $\sigma_{SBF_+^-}(T^*) = \overline{\sigma_{SBF_+^-}(T)}$ and $E^a(T^*) = \overline{E^a(T)} = \emptyset$. Hence, $T^* \in gaW$. Let $f \in Hol(\sigma(T))$. Since T is reduced by each of its eigenspaces, T has SVEP, and so $f(T)$ has SVEP. Therefore

$$\sigma_{SBF_+^-}(f(T)) = \sigma_{LD}(f(T)) = f(\sigma_{LD}(T)) = f(\sigma_{SBF_+^-}(T)).$$

Thus $\sigma_{SBF_+^-}(f(T)) = f(\sigma_{SBF_+^-}(T))$. But $\sigma_a(T)$ has no isolated points, hence, T is a-isoloid. It follows from Theorem 2.3 that generalized a-Weyl's theorem holds for $f(T)$.

Lemma 3.3. *Suppose that* $T \in \mathbf{B}(\mathbb{H})$ *is* a*-isoloid. Then for any* $f \in Hol(\sigma(T))$ *we have*

$$\sigma_a(f(T)) \setminus E^a(f(T)) = f(\sigma_a(T)) \setminus E^a(T)).$$

Proof. Let $\lambda \in \sigma_a(f(T)) \setminus E^a(f(T))$ then $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$. We distinguish two cases:

Case I. $\lambda \notin iso(f(\sigma_a(T)))$, then there is an infinite sequence $\{\eta_n\}_{n\in N} \in \sigma_a(T)$ such that $\lambda = f(\eta_0)$ and $\eta_n \to \eta_0$. But $f \in Hol(\sigma(T))$, therefore $f(\eta_n) \to f(\eta_0) = \lambda$ and $\lambda \in f(\sigma_a(T) \setminus E^a(T))$.

Case II. $\lambda \in iso(f(\sigma_a(T)))$, since $\lambda \notin E^a(f(T))$ then λ is not an eigenvalue of $f(T)$. Then

$$f(T) - \lambda I = (T - \eta_1 I)^{t_1}(T - \eta_2 I)^{t_2} \cdots (T - \eta_m I)^{t_m} g(T),$$

where $\eta_1, \cdots, \eta_m$ are scalars and g is invertible. Since λ is not an eigenvalue of $f(T)$, then for each $j \in \{1, \cdots, m\}$, η_j is not an eigenvalue of T. Hence, $\eta_j \in \sigma_a(T) \setminus E^a(T)$ and $\lambda = f(\eta_j) \in f(\sigma_a(T) \setminus E^a(T))$.

Conversely, Let $\lambda \in f(\sigma_a(T) \setminus E^a(T))$ then $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$. Assume that $\lambda \in E^a(f(T))$. Then

$$f(T) - \lambda I = (T - \eta_1 I)^{t_1}(T - \eta_2 I)^{t_2} \cdots (T - \eta_m I)^{t_m} g(T),$$

where $\eta_1, \cdots, \eta_m$ are scalars and g is invertible. if $\eta_j \in \sigma_a(T)$, then $\eta_j \in iso(\sigma_a(T))$. Since T is a-isoloid, η_j is an eigenvalue of T. Hence, $\eta_j \in E^a(T)$. So $\lambda = f(\eta_j)$ this leads a contraction to the fact that $\lambda \in f(\sigma_a(T) \setminus E^a(T))$.

Theorem 3.4. Sup*pose that* $T \in \mathbf{B}(\mathbb{H})$ *is* a*-isoloid and* $T \in gaW$. *Then for any* $f \in Hol(\sigma(T))$.

$$f(T) \in gaW \Leftrightarrow f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T)).$$

Proof. ($\Rightarrow$) Suppose $f(T) \in gaW$. Then

$$\sigma_{SBF_+^-}(f(T)) = \sigma_a(f(T)) \setminus \pi^a(f(T)).$$

Since T is a-isoloid, it follows from Lemma 3.3 that $\sigma_a(f(T)) \setminus E^a(f(T)) = f(\sigma_a(T) - E^a(T))$. But $T \in gaW$, hence, $\sigma_a(T) \setminus E^a(T) = \sigma_{SBF_+^-}(T)$, which implies $f(\sigma_{SBF_+^-}(T)) = f(\sigma_a(T) \setminus E^a(T))$. Therefore

$$f(\sigma_{SBF_+^-}(T)) = f(\sigma_a(T) \setminus E^a(T))$$
$$= \sigma_a(f(T)) \setminus E^a(f(T)) = \sigma_{SBF_+^-}(f(T)).$$

($\Leftarrow$) Suppose that $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$. Since T is a-isoloid, it follows from Lemma 3.3 that

$$f(\sigma_a(T) \setminus E^a(T)) = \sigma_a(f(T)) \setminus E^a(f(T)).$$

Since $T \in gaW$, we have $\sigma_a(T) \setminus E^a(T) = \sigma_{SBF_+^-}(T)$.

Therefore

$$f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$$
$$= f(\sigma_a(T) \setminus E^a(T))$$
$$= \sigma_a(f(T)) \setminus \pi^a(f(T)),$$

and hence, $f(T) \in gaW$.

Definition 3.5. *Let* $T \in \mathbf{B}(\mathbb{H})$. *Then* T *is said to be reduction-* a*-isoloid if the restriction of* T *to every reducing subspace is* a*-isoloid.*

Theorem 3.6. *Suppose that* $T \in \mathbf{B}(\mathbb{H})$ *is both reduction-* a *-isoloid and reduced by each of its eigenspaces. Then* $f(T) \in gaW$ *for every* s

Proof. We first show that $T \in gaW$. In view of Theorem 3.4, it suffices to show that $E^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Suppose that $\lambda \in E^a(T)$. Then

$$\lambda \in E^a(T_1) \cap [iso\sigma_a(T_2) \cup \rho_a(T_2)].$$

If $\lambda \in iso\sigma_a(T_2)$, then since T_2 is a -isoloid we have $\lambda \in \sigma_p(T_2)$. But $\sigma_p(T_2) = \emptyset$, hence, we must have $\lambda \in E^a(T_1) \cap \rho_a(T_2)$. Since T_1 is normal, $T_1 \in gaW$. Hence, $T_1 - \lambda I \in SBF_+^-(\mathbb{X})$ and so is $T - \lambda I$, which implies $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Therefore $E^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, and hence, $T \in gaW$. Now, let $f \in Hol(\sigma(T))$. Since T is reduced by each of its eigenspaces, it follows from the proof of Corollary 3.2 that $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$. Therefore $f(T) \in gaW$ by Theorem 3.4.

4. Applications

In this section we show that the generalized a -Weyl's theorem holds to algebraically $\omega F(p,r,q)$ operators with $p,r > 0$ and $q \geq 1$ and to algebraically quasi-class A operators, using the results in section 2 and 3. We begin with the following definition.

4.1. Algebraically Class *wF*(*p*,*r*,*q*) operators with *p*, *r*>0 and *q*>1

Definition 4.1. *For* $p > 0, r > 0$ *and* $q \geq 1$, *an operator* $T \in \mathbf{B}(\mathbb{H})$ *is called of class* $wF(p,r,q)$ *if*

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$$
$$\text{and } |T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p |T^*|^{2r} |T|^p)^{(1-\frac{1}{q})}.$$

We say that $T \in \mathbf{B}(\mathbb{H})$ is an algebraically class $\omega F(p,r,q)$ for each $p,r > 0$ and $q \geq 1$ if there exists a nonconstant complex polynomial p such that $p(T)$ is of class $wF(p,r,q)$ with $p,r > 0$ and $q \geq 1$.

An operator $T \in \mathbf{B}(\mathbb{H})$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator $T \in \mathbf{B}(\mathbb{H})$ is called normaloid if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T. An operator $T \in \mathbf{B}(\mathbb{H})$ is said to be convexoid if

$$\overline{W(T)} = conv(\sigma(T)),$$

where $conv(\sigma(T))$ means the convex hull of the spectrum $\sigma(T)$ of T. $X \in \mathbf{B}(\mathbb{H})$ is called a quasiaffinity if it has trivial kernel and dense range. $S \in \mathbf{B}(\mathbb{H})$ is said to be a quasiaffine transform of $T \in \mathbf{B}(\mathbb{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in \mathbf{B}(\mathbb{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$ then we say that S and T are quasisimilar.

In general, the following implications hold: class $wF(p,r,q) \Rightarrow$ algebraically class $wF(p,r,q)$ for each $p,r > 0$ and $q \geq 1$.

The following facts follow from the above definition and some well known facts about class $wF(p,r,q)$ for each $p,r > 0$ and $q \geq 1$.

Lemma 4.2. For each $p,r > 0$ and $q \geq 1$.

(i) If $T \in \mathbf{B}(\mathbb{H})$ be an algebraically class $wF(p,r,q)$ and $M \subset H$ is invariant under T, then $T|_M$ is an algebraically class $wF(p,r,q)$.

(ii) If $T \in \mathbf{B}(\mathbb{H})$ is algebraically class $wF(p,r,q)$ then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

Remark. In what follows, we use the notation ωF to denote the class $\omega F(p,r,q)$ operators with $p,r > 0$ and $q \geq 1$.

Lemma 4.3. *Let* $T \in \mathbf{B}(\mathbb{H})$ *belong to class* $wF(p,r,q)$ *with* $p,r > 0$ *and* $q \geq 1$. *Let* $\lambda \in \mathbb{C}$. *Assume that* $\sigma(T) = \{\lambda\}$. *Then* $T = \lambda I$

Proof. We consider two cases:

Case (I). $(\lambda = 0)$: Since T belongs class wF for each $p,r > 0$ and $q \geq 1$, T is normaloid. Therefore $T = 0$.

Case (II). $(\lambda \neq 0)$: Here T is invertible, and since T belongs class wF for each $p,r > 0$ and $q \geq 1$, we see that T^{-1} is also belongs class wF for each $p,r > 0$ and $q \geq 1$. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$.

Proposition 4.4. *Let* T *be a quasinilpotent algebraically* wF *operator. Then* T *is nilpotent.*

Proof. Assume that $p(T)$ is ωF operator for some nonconstant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasinilpotent. Thus Lemma 4.3 would imply that

$$cT^m(T - \lambda_1 I) \cdots (T - \lambda_n I) \equiv p(T) - p(0) = 0,$$

where $m \geq 1$. Since $T - \lambda_j I$ is invertible for every $\lambda_j \neq 0$, we must have $T^m = 0$.

In [28] they showed that every ωF operator is isoloid. We can prove more:

Proposition 4.5. *Let* T *be an algebraically* wF *operator. Then* T *is polaroid.*

Proof. Suppose T is an algebraically ωF operator. Then $p(T)$ is ωF for some nonconstant polynomial p. Let $\lambda \in iso(\sigma(T))$. Using the spectral projection $P := \frac{1}{2i\pi}\int_{\partial D}(\mu - T)^{-1} d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T=\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ and } \sigma(T_1)=\{\lambda\} \text{ and } \sigma(T_2)=\sigma(T)\setminus\{\lambda\}.$$

Since T_1 is algebraically class ωF and $\sigma(T_1)=\{\lambda\}$. But $\sigma(T_1-\lambda I)=\{0\}$ it follows from Proposition 4.4 that $T_1-\lambda I$ is nilpotent. Therefore $T_1-\lambda$ has finite ascent and descent. On the other hand, since $T_2-\lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T-\lambda I$ has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda \in iso(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $iso(\sigma(T)) \subset \pi(T)$. Hence, T is polaroid.

Corollary 4.6. *Let T be an algebraically wF operator. Then T is isoloid.*

Following [17] we say that $T\in \mathbf{B}(\mathbb{H})$ belongs to class A if $|A^2| \geq |A|^2$. Recall [15,18] that $T\in \mathbf{B}(\mathbb{H})$ is called a quasi-class A operator if $T^*|T^2|T \geq T^*|T|^2 T$.

Definition 4.7. [2] An operator $T\in \mathbf{B}(\mathbb{H})$ is called an algebraically quasi-class A operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a quasi-class A operator.

In general, the following implications hold:

class A $\Rightarrow$ quasi-class A $\Rightarrow$ algebraically quasi-class A.

The following facts follow from the above definition and some well known facts about quasi-class A operators.

(i) If $T\in \mathbf{B}(\mathbb{H})$ is algebraically quasi-class A then so is $T-\lambda I$ for each $\lambda\in\mathbb{C}$.

(ii) If $T\in \mathbf{B}(\mathbb{H})$ is algebraically quasi-class A and M is a closed T-invariant subspace of H then $T|_M$ is algebraically quasi-class A.

Lemma 4.8. [18] *Let $T\in \mathbf{B}(\mathbb{H})$ be quasi-class A and T not have dense range. Then*

$$T=\begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix} \text{ on } H=\overline{ranT}\oplus \ker(T^*),$$

where $A=T|_{\overline{ran(T)}}$, the restriction of T to $\overline{ran(T)}$, belongs to class A. Moreover, $\sigma(T)=\sigma(A)\cup\{0\}$.

Lemma 4.9. [[2], Lemma 2.2.] *Suppose T is a quasinilpotent algebraically quasi-class A operator. Then T is nilpotent.*

Proposition 4.10. *Suppose T is an algebraically quasi-class A operator. If $\sigma(T)=\{\mu\}$, then $T-\mu I$ is nilpotent.*

Proof. Assume $p(T)$ is quasi-class A for some non-constant complex polynomial $p(z)$. Since

$$\sigma(p(T))=p(\sigma(T))=p(\{\mu\})=\{p(\mu)\},$$

the operator $p(T)-p(\mu)I$ is nilpotent by Lemma 4.9. Let

$$p(z)-p(\mu)=c(z-\mu)^{k_0}(z-\mu_1)^{k_1}\cdots(z-\mu_n)^{k_n},$$

where $\mu_i\neq\mu_j$ for $i\neq j$, k_j is an integer, $i,j=0,1,\cdots n$. Then

$$\begin{aligned} 0&=(p(T)-p(\mu)I)^t \\ &=c^t(T-\mu I)^{k_0 t}(T-\mu_1 I)^{k_1 t}\cdots(T-\mu_n I)^{k_n t}, \end{aligned}$$

and hence, $(T-\mu I)^{k_0 t}=0$.

Let $\Upsilon(\mathcal{H})$ be the set of all $T\in \mathbf{B}(\mathbb{H})$ such that T is an algebraically quasi-class A or algebraically class $\omega F(p,r,q)$ operator with $p,r>0$ and $q\geq 1$.

Theorem 4.11. *Let T or $T^*\in\Upsilon(\mathcal{H})$. Then $f(\sigma_{SBF_+^-}(T))=\sigma_{SBF_+^-}(f(T))$ for all $f\in Hol(\sigma(T))$.*

Proof. If T or $T^*\in\Upsilon(\mathcal{H})$. Then T or T^* has SVEP. Hence $T\in gaB$. Now the result follows from Corollary 3.4.

Proposition 4.12. *If $T^*\in\Upsilon(H)$. Then T is a-isoloid.*

Proof. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is polaroid. Therefore it is isoloid, and hence $\lambda\in\sigma_p(T)$. Thus T is a-isoloid.

Corollary 4.13. Let $T\in \mathbf{B}(\mathbb{H})$. *If $T^*\in\Upsilon(\mathcal{H})$, then $\sigma_a(f(T))\setminus E^a(f(T))=f(\sigma_a(T)\setminus E^a(T))$ for every $f\in Hol(\sigma(T))$. Consequently, $E^a(f(T))=\pi^a(f(T))$.*

Theorem 2.4 of [29] affirms that if T^* or T has the SVEP and if T is a-isoloid and generalized a-Weyl's holds for T then generalized a-Weyl's theorem holds for $f(T)$, for every $f\in Hol(\sigma(T))$. If $T^*\in\Upsilon(\mathcal{H})$, then we have:

Theorem 4.14. *Let $T^*\in\Upsilon(\mathcal{H})$. Then generalized a-Weyl's holds for $f(T)$, for every $f\in Hol(\sigma(T))$.*

Proof. If $T^*\in\Upsilon(\mathcal{H})$. Then T^* has SVEP then it follows from Corollary 2.45 of [1] that $\sigma(T)=\sigma_a(T)$ and consequently $E(T)=E^a(T)$. Let $\lambda\notin\sigma_{SBF_+^-}(T)$ be given, then $T-\lambda$ is semi-B-Fredholm and $i(T-\lambda)\leq 0$. Then Proposition 1.1 implies that $i(T-\lambda)=0$ and consequently $T-\lambda$ is B-Weyl. Hence, $\lambda\notin\sigma_{BW}(T)$. Hence, it follows from [[29], Theorem 3.1] that $\lambda\in E(T)=E^a(T)$.

For the converse, let $\lambda\in E^a(T)$. Then $\lambda\in iso\sigma_a(T)$. Since T^* has SVEP then it follows from Corollary 2.45 of [1] $\sigma(T)=\sigma_a(T)$. Hence, $\overline{\lambda}\in\sigma(T^*)$. Now we represent T^* as the direct sum $T^*=T_1\oplus T_2$, where $\sigma(T_1)=\{\overline{\lambda}\}$ and $\sigma(T_2)=\sigma(T)\setminus\{\overline{\lambda}\}$. Since $T\in\Upsilon(\mathcal{H})$ then so does T_1, and so we have two cases:

Case I. $(\overline{\lambda}=0)$: then T_1 is quasinilpotent. Hence, it follows that T_1 is nilpotent. Since T_2 is invertible, Then T^* is a B-Weyl's.

Case II. $(\overline{\lambda} \neq 0)$: Since $\sigma(T_1) = \{\overline{\lambda}\}$, then $T_1 - \overline{\lambda}$ is nilpotent and $T_2 - \overline{\lambda}$ is invertible, it follows from [29] that $T^* - \overline{\lambda}$ is B -Weyl's. Thus in any case $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Let $f \in Hol(\sigma(T))$. Since T is a -isoloid, then it follows from Theorem 4.11 that

$$\sigma_{SBF_+^-}(f(T)) = f(\sigma_{SBF_+^-}(T))$$
$$= f(\sigma_a(T) \setminus E^a(T)) = \sigma_a(f(T)) \setminus E^a(f(T)).$$

Thus generalized a -Weyl's theorem holds for $f(T)$.

An operator $T \in \mathbf{B}(\mathbb{H})$ is called a -polaroid if $iso\sigma_a(T) \subset \pi^a(T)$. In general, if T is a -polaroid then it is polaroid. However, the converse is not true. Consider the following example.

Example 4.15. *Let R be the unilateral right shift on $\ell^2(N)$ and define*

$$U(x_1, x_2, \cdots) := (0, x_2, x_3, \cdots) \text{ for all } x_n \in \ell^2(N).$$

Clearly, U is a quasi-nilpotent operator. Let $T := R \oplus U$. We have $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ is the unit disc of C , so $iso(\sigma(T)) = E(T) = \emptyset$ and hence, T is polaroid. Moreover, $\sigma_a(T) = \partial\mathbf{D} \cup \{0\}$. Since $\sigma_a(T)$ does not cluster at 0, then T has the SVEP at 0, as well as at the points $\lambda \notin \sigma_a(T)$. Since T has SVEP at all points $\lambda \in \partial\sigma(T)$ it then follows that T has SVEP. Finally, $\sigma_{SBF_+^-}(T) = \partial\sigma(T)$ so $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\}$. Hence, T is not a -polaroid.

In [2] they showed that every algebraically quasi-class A operator is polaroid. We can prove more:

Theorem 16. *Let $T^* \in \Upsilon(\mathcal{H})$. Then T is a -polaroid.*

Proof. Since T satis_es generalized *a*-Weyl's theorem by Theorem 4.14 and a-isoloid by Proposition 4.12. Then it follows from [[8], Lemma 3.2] that *T* is a-polaroid.

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