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Inequalities of Type Hermite-Hadamard for Fractional Integrals via Differentiable Convex Functions

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Abstract In this paper, we established some new Hermite-Hadamard-type inequalities for differentiable convex functions via Reimann-Liouville fractional integrals. Moreover, our results improve and extend the corresponding ones in the literature.

Keywords: *integral inequalities, Riemann-Liouville fractional integral, Hermite-Hadamard inequality, convex function, Hölder inequality*

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1. Introduction

The following definition is well known in the literature.

Definition 1.1. [7,8] A function: $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to beconvex on the interval I, if for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y).$$

Many significant inequalities have been studied for the class of convex functions, but the most important is the following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \qquad (1)$$

that is known as Hermite-Hadamard inequality [1]. For more systematic information, please refer to the monographs [3-8] and closely related references therein.

In what follows we recall the following definition [9].

Definition 1.2. Let $f \in L_1[a,b]$. The left Riemann-Liouville fractional integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, x > a,$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, x < b,$$

respectively. Here, Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

Tomar et. al. [2] established the following Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals:

Lemma 1.1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on the interior I° of an interval I such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. Then for each $h \in (0,1)$ and $\alpha > 0$ the following equality holds*:

$$I_f(f;\alpha,a,b) = \frac{(b-a)^2}{8}\left[\begin{array}{l}\int_0^1 t^{\alpha+1} f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\ +\int_0^1 t^{\alpha+1} f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\end{array}\right], \qquad (2)$$

where

$$I_f(f;\alpha,a,b) = \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^\alpha}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)\right] -(\alpha+1) f\left(\frac{a+b}{2}\right).$$

Theorem 1.1. *The assupmtions of Lemma 1.1 are satisfied. If $|f''|$ is convex on $[a,b]$, then for each $h \in (0,1)$ the following inequality holds*:

$$\left|I_f(f;\alpha,a,b)\right| \le \frac{(b-a)^2}{8(\alpha+2)}\left(\left|f''(a)\right|+\left|f''(b)\right|\right). \qquad (3)$$

Theorem 1.2. *The assupmtions of Lemma 1.1 are satisfied. If $|f''|^q$ is convex on $[a,b]$ for $q > 1$, then the following inequality holds*:

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{8(\alpha+2)}\left(\frac{2}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left(\left|f''(a)\right|+\left|f''(b)\right|\right). \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Theorem 1.3. *Let the assupmtions of Lemma 1.1 be satisfied. If* $|f''|^q$ *is convex on* $[a,b]$ *for* $q>1$, *then the following inequality holds*:

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{8}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left[\begin{array}{l}\left(\begin{array}{l}\frac{1}{2(\alpha+3)}\left|f''(a)\right|^q \\ +\left(\frac{1}{\alpha+2}-\frac{1}{2(\alpha+3)}\right)\left|f''(b)\right|^q\end{array}\right)^{\frac{1}{q}} \\ +\left(\begin{array}{l}\left(\frac{1}{\alpha+2}-\frac{1}{2(\alpha+3)}\right)\left|f''(a)\right|^q \\ +\frac{1}{2(\alpha+3)}\left|f''(b)\right|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right]. \tag{5}$$

The main objective of this article is to establish some Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. Our results improve and generalize some types of Hermite-Hadamard inequalities in [2].

2. Main Results

We start with the following lemma:

Lemma 2.1. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be twice differentiable mapping on the interior* I° *of an interval* I *such that* $f''\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *Then for each* $h\in(0,1)$ *and* $\alpha>0$ *the following equality holds*:

$$I_f\left(f;\alpha,a,b\right) = \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\int_0^1 t^{\alpha+1}f''\left(tw+(1-t)a\right)dt \\ +h^{\alpha+1}\int_0^1 t^{\alpha+1}f''\left(tw+(1-t)b\right)dt\end{array}\right], \tag{6}$$

where

$$\begin{aligned} I_f\left(f;\alpha,a,b\right) &= \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^\alpha}\left[J^\alpha_{w^+}f(b)+J^\alpha_{w^-}f(a)\right] \\ &+2^{\alpha-1}(b-a)\left[(1-h)^{\alpha+1}-h^{\alpha+1}\right]f'(w) \\ &-2^{\alpha-1}(\alpha+1)f(w), \end{aligned}$$

and $w=ha+(1-h)b.$

Proof. Denoting

$$\begin{aligned} &(1-h)^{\alpha+2}\int_0^1 t^{\alpha+1}f''\left(tw+(1-t)a\right)dt \\ &+h^{\alpha+2}\int_0^1 t^{\alpha+1}f''\left(tw+(1-t)b\right)dt \\ &=(1-h)^{\alpha+2}I_1+h^{\alpha+2}I_2. \end{aligned} \tag{7}$$

Integrating by parts twice and changing variable of definite integral, we have

$$\begin{aligned} I_1 &= \int_0^1 t^{\alpha+1}f''\left(tw+(1-t)a\right)dt \\ &=\frac{1}{w-a}f'(w)-\frac{\alpha+1}{(w-a)^2}f(w) \\ &+\frac{\alpha(\alpha+1)}{(w-a)^2}\int_0^1 t^{\alpha+1}f\left(tw+(1-t)a\right)dt \\ &=\frac{1}{w-a}f'(w)-\frac{\alpha+1}{(w-a)^2}f(w) \\ &+\frac{\alpha(\alpha+1)}{(w-a)^{\alpha+2}}\int_a^w (u-a)^{\alpha-1}f(u)du \\ &=\frac{1}{(1-h)(b-a)}f'(w)-\frac{\alpha+1}{(1-h)^2(b-a)^2}f(w) \\ &+\frac{\Gamma(\alpha+2)}{(1-h)^{\alpha+2}(b-a)^{\alpha+2}}J^\alpha_{w^-}f(a), \end{aligned} \tag{8}$$

and similarly

$$\begin{aligned} I_2 &= \int_0^1 t^{\alpha+1}f''\left(tw+(1-t)b\right)dt \\ &=\frac{1}{w-b}f'(w)-\frac{\alpha+1}{(w-b)^2}f(w) \\ &+\frac{\alpha(\alpha+1)}{(w-b)^2}\int_0^1 t^{\alpha+1}f\left(tw+(1-t)b\right)dt \\ &=\frac{1}{w-b}f'(w)-\frac{\alpha+1}{(w-b)^2}f(w) \\ &+\frac{\alpha(\alpha+1)}{(w-b)^{\alpha+2}}\int_w^b (b-u)^{\alpha-1}f(u)du \\ &=-\frac{1}{h(b-a)}f'(w)-\frac{\alpha+1}{h^2(b-a)^2}f(w) \\ &+\frac{\Gamma(\alpha+2)}{h^{\alpha+2}(b-a)^{\alpha+2}}J^\alpha_{w^+}f(b). \end{aligned} \tag{9}$$

Using equations (8)-(9) in (7) and by the simple calculations, we obtain the desired result.

Remark 2.1. *If we take* $h=\frac{1}{2}$ *in Lemma 2.1, then the identity (6) reduces to the identity (2) which was proved by Tomar et. al. [2].*

Theorem 2.1. *The assupmtions of Lemma 1.1 are satisfied. If* $|f''|$ *is convex on* $[a,b]$, *then for each* $h\in(0,1)$ *the following inequality holds*:

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{2^{1-\alpha}(\alpha+2)(\alpha+3)} \times\left[\begin{array}{l}(\alpha+2)\left\{(1-h)^{\alpha+2}+h^{\alpha+2}\right\}\left|f''(w)\right| \\ +(1-h)^{\alpha+2}\left|f''(a)\right|+h^{\alpha+2}\left|f''(b)\right|\end{array}\right]. \tag{10}$$

Proof. From Lemma 2.1, we have

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)a)\right|dt \\ +h^{\alpha+2}\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)b)\right|dt\end{array}\right].$$

Now, since $|f''|$ is convex, we have

$$\begin{aligned}\left|I_f\left(f;\alpha,a,b\right)\right| &\le \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left\{\begin{array}{l}\left|f''(w)\right|\int_0^1 t^{\alpha+2}dt \\ +\left|f''(a)\right|\int_0^1(1-t)t^{\alpha+1}dt\end{array}\right\} \\ +h^{\alpha+2}\left\{\begin{array}{l}\left|f''(w)\right|\int_0^1 t^{\alpha+2}dt \\ +\left|f''(a)\right|\int_0^1(1-t)t^{\alpha+1}dt\end{array}\right\}\end{array}\right] \\ &= \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left\{\frac{\left|f''(w)\right|}{\alpha+3}+\frac{\left|f''(a)\right|}{(\alpha+2)(\alpha+3)}\right\} \\ +h^{\alpha+2}\left\{\frac{\left|f''(w)\right|}{\alpha+3}+\frac{\left|f''(b)\right|}{(\alpha+2)(\alpha+3)}\right\}\end{array}\right] \\ &= \frac{(b-a)^2}{2^{1-\alpha}(\alpha+2)(\alpha+3)}\left[\begin{array}{l}(\alpha+2)\left\{\begin{array}{l}(1-h)^{\alpha+2} \\ +h^{\alpha+2}\end{array}\right\}\left|f''(w)\right| \\ +(1-h)^{\alpha+2}\left|f''(a)\right|+h^{\alpha+2}\left|f''(b)\right|\end{array}\right],\end{aligned}$$

and this ends the proof.

Remark 2.2. *If we take* $h=\frac{1}{2}$ *in Theorem 2.1, then inequality* (10) *reduces to* (3).

Theorem 2.2. *The assupmtions of Lemma 1.1 are satisfied. If* $|f''|^q$ *is convex on* $[a,b]$ *for* $q>1$, *then the following inequality holds*:

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\frac{\left|f''(w)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}} \\ +h^{\alpha+2}\left(\frac{\left|f''(w)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\end{array}\right] \tag{11}$$

$$\le \frac{(b-a)^2}{2^{2-\alpha}}\left(\frac{2}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\begin{array}{l}\left|f''(w)\right| \\ +\left|f''(a)\right|\end{array}\right) \\ +h^{\alpha+2}\left(\left|f''(w)\right|+\left|f''(b)\right|\right)\end{array}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the well known Hölder integral inequality, we have

$$\begin{aligned}\left|I_f\left(f;\alpha,a,b\right)\right| &\le \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^1\left|f''(tw+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +h^{\alpha+2}\left(\int_0^1 t^{(\alpha+1)p}dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^1\left|f''(tw+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right] \\ &= \frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\int_0^1\left|f''(tw+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +h^{\alpha+2}\left(\int_0^1\left|f''(tw+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right].\end{aligned} \tag{12}$$

Since $|f''|$ is convex, then we have

$$\begin{aligned}&\int_0^1\left|f''(tw+(1-t)a)\right|^q dt \\ &\le \int_0^1\left[t\left|f''(w)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt \\ &= \frac{\left|f''(w)\right|^q+\left|f''(a)\right|^q}{2}\end{aligned} \tag{13}$$

and

$$\int_0^1\left|f''(tw+(1-t)b)\right|^q dt = \frac{\left|f''(w)\right|^q+\left|f''(b)\right|^q}{2}. \tag{14}$$

Thus, if we use (13) and (14) in (12), we obtain

$$\left|I_f\left(f;\alpha,a,b\right)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\frac{\left|f''(w)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}} \\ +h^{\alpha+2}\left(\frac{\left|f''(w)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\end{array}\right],$$

this completes the proof of first part of the Theorem.

Let $a_1=a_2=\left|f''(w)\right|^q$, $b_1=\left|f''(a)\right|^q$ and $b_2=\left|f''(b)\right|^q$ for $q>1$. Using the fact that

$\sum_{i=1}^{n}(a_i+b_i)^j \le \sum_{i=1}^{n}a_i^j+\sum_{i=1}^{n}b_i^j$ for $j\in(0,1]$ and $a_k,b_k>0,\ k=1,2,\dots,n,$ we find

$$\left|I_f(f;\alpha,a,b)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}\times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\left|f''(w)\right|+\left|f''(a)\right|\right)^{\frac{1}{q}}\\ +h^{\alpha+2}\left(\left|f''(w)\right|+\left|f''(b)\right|\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le \frac{(b-a)^2}{2^{2-\alpha}}\left(\frac{2}{(\alpha+1)p+1}\right)^{\frac{1}{p}}\times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\left|f''(w)\right|+\left|f''(a)\right|\right)\\ +h^{\alpha+2}\left(\left|f''(w)\right|+\left|f''(b)\right|\right)\end{array}\right],$$

which completes the proof of last part of the Theorem. This completes the proof.

Remark 2.3. *If we choose* $h=\frac{1}{2}$ *in Theorem 2.2, then inequality* (11) *reduces to* (4).

Theorem 2.3. *Let the assumptions of Lemma 1.1 be satisfied. If* $\left|f''\right|^q$ *is convex on* $[a,b]$ *for* $q>1$, *then the following inequality holds*:

$$\left|I_f(f;\alpha,a,b)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\frac{\left|f''(w)\right|^q}{\alpha+3}+\frac{\left|f''(a)\right|^q}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\\ +h^{\alpha+2}\left(\frac{\left|f''(w)\right|^q}{(\alpha+2)(\alpha+3)}+\frac{\left|f''(b)\right|^q}{\alpha+3}\right)^{\frac{1}{q}}\end{array}\right]. \quad (15)$$

Proof. From Lemma 2.1 and using the well known power mean inequality, we have

$$\left|I_f(f;\alpha,a,b)\right| \le \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1\left|f''(tw+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ +h^{\alpha+2}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le \frac{(b-a)^2}{2^{1-\alpha}}\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ +h^{\alpha+2}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\frac{(b-a)^2}{2^{1-\alpha}}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\times\left[\begin{array}{l}(1-h)^{\alpha+2}\left(\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ +h^{\alpha+2}\left(\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]. \quad (16)$$

Since $\left|f''\right|^q$ is convex, then we have

$$\begin{aligned}&\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)a)\right|^q dt\\ &\le \int_0^1\left[t^{\alpha+2}\left|f''(w)\right|^q+(1-t)t^{\alpha+1}\left|f''(a)\right|^q\right]dt \quad (17)\\ &=\frac{\left|f''(w)\right|^q}{\alpha+3}+\frac{\left|f''(a)\right|^q}{(\alpha+2)(\alpha+3)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t^{\alpha+1}\left|f''(tw+(1-t)b)\right|^q dt\\ &=\frac{\left|f''(w)\right|^q}{(\alpha+2)(\alpha+3)}+\frac{\left|f''(b)\right|^q}{\alpha+3}.\end{aligned} \quad (18)$$

Combining equations (16), (17) and (18), we obtain inequality (15) as required.

Remark 2.4. *If we choose* $h=\frac{1}{2}$ *in Theorem 2.3, then inequality* (15) *reduces to* (5).

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