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Some Integral Inequalities of the Hermite-Hadamard Type for Strongly Quasi-convex Functions

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Abstract In the paper, the authors introduce a new notion "strongly quasi-convex function", establish an integral identity for strongly quasi-convex functions, and establish some new integral inequalities of the Hermite-Hadamard type for strongly quasi-convex functions.

Keywords: *integral identity, integral inequality, Hermite-Hadamard type, strongly quasi-convex function, Hölder inequality*

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1. Introduction

We first list some definitions concerning various convex functions.

Definition 1.1. A function $f: I \subseteq R = (-\infty, \infty) \to R$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([1]). A function $f: I \subseteq R \to R_0 = [0, \infty)$ is said to be quasi-convex convex if

$$f(\lambda x + (1-\lambda)y) \le \sup\{f(x), f(y)\} \tag{1.2}$$

for all $x, y \in I, \lambda \in [0,1]$.

Definition 1.3 ([2]). A function $f: [a,b] \to R_0$ is said to be strongly convex with modulus $c \ge 0$ if

$$\begin{aligned} & f(\lambda x + (1-\lambda)y) \\ & \le \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)(x-y)^2 \end{aligned} \tag{1.3}$$

is valid for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

The following inequalities of the Hermite-Hadamard type were established for the above convex functions.

Theorem 1.1 ([3]). Let $f: I^\circ \subseteq R \to R$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$.

(i) If $|f'|$ is convex function on $[a,b]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \le \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}; \end{aligned} \tag{1.4}$$

(ii) If $|f'|^{p/(p-1)}$ is convex function on $[a,b]$ and $p > 1$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \le \frac{b-a}{2(p+1)^{1/p}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{(p-1)/p}. \end{aligned} \tag{1.5}$$

In [4], two inequalities of the Hermite-Hadamard type for quasi-convex functions were introduced as follow.

Theorem 1.2 ([4]). Let $f: I^\circ \subseteq R \to R_0$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is quasi-convex on $[a,b]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \right| \\ & \le \frac{b-a}{4} \sup\{|f'(a)|, |f'(b)|\}. \end{aligned} \tag{1.6}$$

Theorem 1.3 ([4]). Let $f: I^\circ \subseteq R \to R_0$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|^p$ is quasi-convex on $[a,b]$ and $p > 1$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \right| \\ & \le \frac{b-a}{2(p+1)^{1/p}} \left(\sup\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{(p-1)/p}. \end{aligned} \tag{1.7}$$

Theorem 1.4 ([5, Theorem 2.3]). Let $f: I^\circ \subseteq R \to R_0$ be a differentiable mapping on I° and $a, b \in I^\circ$ with

$a<b$. If $|f'(x)|^p$ is quasi-convex on $[a,b]$ and $p>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{(p-1)/p} + \left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(\tfrac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{(p-1)/p}\right]. \tag{1.8}$$

Theorem 1.5 ([5, Theorem2.4]). Let $f: I^\circ \subseteq R \to R_0$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|^q$ is quasi-convex on $[a,b]$ and $q\ge 1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{1/q} + \left(\sup\left\{\left|f'(a)\right|^{q},\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right\}\right)^{1/q}\right]. \tag{1.9}$$

In this paper, we will introduce a new notion "strongly quasi-convex function" and establish some integral inequalities of the Hermite-Hadamard type for functions whose derivatives are of strongly quasi-convexity.

2. A Definition and a Lemma

We now introduce the notion "strongly quasi-convex functions".

Definition 2.1. A function $f:[a,b]\to R_0$ is said to be strongly quasi-convex with modulus $c\ge 0$ if

$$f(tx+(1-t)y)\le \sup\{f(x),f(y)\}-ct(1-t)(x-y)^2$$

is valid for all $x,y\in[a,b]$ and $t\in[0,1]$.

For establishing new integral inequalities of the Hermite--Hadamard type for strongly quasi-convex functions, we need the following identity.

Lemma 2.1. Let $f: I\subseteq R\to R$ be differentiable on I° and $a,b\in I$ with $a<b$. If $f'\in L([a,b])$, then

$$\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{b-a}{4}\left[\int_0^1\left(t-\frac{2}{5}\right)f'\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t + \int_0^1\left(t-\frac{3}{5}\right)f'\left((1-t)\frac{a+b}{2}+tb\right)\mathrm{d}t.\right. \tag{2.1}$$

Proof. This follows from a straightforward computation of definite integrals.

3. Some Integral Inequalities of the Hermite-Hadamard Type

Now we are in a position to establish some integral inequalities of the Hermite–-Hadamard type for functions whose derivatives are of strongly quasi-convexity.

Theorem 3.1. Let $f: I\subseteq R\to R_0$ be differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in(L[a,b])$ and $|f'|^q$ is strongly quasi-convex on $[a,b]$ for $q\ge 1$ with modulus $c\ge 0$, then

$$\left|\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{13}{50}\right)^{1-1/q}\left\{\left[\frac{13}{50}\sup\left\{|f'(a)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\} - \frac{253c}{30000}(b-a)^2\right]^{1/q} + \left[\frac{13}{50}\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\}-\frac{253c}{30000}(b-a)^2\right]^{1/q}\right\}.$$

Proof. Since $|f'|^q$ is strongly quasi-convex on $[a,b]$, using Lemma 2.1 and by Hölder's inequality, we have

$$\left|\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le \frac{b-a}{4}\left[\int_0^1\left|t-\frac{2}{5}\right|\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|\mathrm{d}t + \int_0^1\left|t-\frac{3}{5}\right|\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|\mathrm{d}t\right]$$

$$\le \frac{b-a}{4}\left[\left(\int_0^1\left|t-\frac{2}{5}\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|t-\frac{2}{5}\right|\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right)^{1/q} + \left(\int_0^1\left|t-\frac{3}{5}\right|\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|t-\frac{3}{5}\right|\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q\mathrm{d}t\right)^{1/q}\right]$$

$$\le \frac{b-a}{4}\left\{\left(\int_0^1\left|t-\frac{2}{5}\right|\mathrm{d}t\right)^{1-1/q} \times\left[\int_0^1\left|t-\frac{2}{5}\right|\left(\sup\left\{|f'(a)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\} -\frac{c}{4}t(1-t)(b-a)^2\right)\mathrm{d}t\right]^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|t-\frac{3}{5}\right|\mathrm{d}t\right)^{1-1/q}\left[\int_0^1\left|t-\frac{3}{5}\right|\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q,\right\} -\frac{c}{4}t(1-t)(b-a)^2\right)\mathrm{d}t\right]^{1/q}\right\}$$

$$=\frac{b-a}{4}\left(\frac{13}{50}\right)^{1-1/q}$$

$$\times\left\{\left[\frac{13}{50}\sup\left\{|f'(a)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}-\frac{253c}{30000}(b-a)^2\right]^{1/q}\right.$$

$$\left.+\left[\frac{13}{50}\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\}-\frac{253c}{30000}(b-a)^2\right]^{1/q}\right\}.$$

Theorem 3.1 is thus proved.

Crollary 3.1. Under conditions of Theorem 3.1, if $q=1$, then

$$\left|\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le\frac{b-a}{200}\left[13\sup\left\{|f'(a)|,\left|f'\left(\frac{a+b}{2}\right)\right|,|f'(b)|^q\right\}\right.$$

$$\left.-\frac{253c}{300}(b-a)^2\right].$$

Theorem 3.2. Let $f: I\subseteq R\to R_0$ be differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in(L[a,b])$ and $|f'|^q$ is strongly quasi-convex on $[a,b]$ for $q>1$ with modulus $c\ge 0$, then

$$\left|\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le\frac{b-a}{4}\left(\frac{2^{(2q-1)/(q-1)}+3^{(2q-1)/(q-1)}}{5^{(2q-1)/(q-1)}}\right)^{1-1/q}$$

$$\times\left\{\left[\sup\left\{|f'(a)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}-\frac{c}{24}(b-a)^2\right]^{1/q}\right.$$

$$\left.+\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\}-\frac{c}{24}(b-a)^2\right]^{1/q}\right\}.$$

Proof. By Lemma 2.1 and using Hölder's inequality and the strongly quasi-convexity of $|f'|^q$, we obtain

$$\left|\frac{1}{5}\left[f(a)+3f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$

$$\le\frac{b-a}{4}\left[\int_0^1\left|t-\frac{2}{5}\right|\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|\mathrm{d}t\right.$$

$$\left.+\int_0^1\left|t-\frac{3}{5}\right|\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|\mathrm{d}t\right]$$

$$\le\frac{b-a}{4}\left[\left(\int_0^1\left|t-\frac{2}{5}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1\left|t-\frac{3}{5}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q\mathrm{d}t\right)^{1/q}\right]$$

$$\le\frac{b-a}{4}\left(\frac{2^{(2q-1)/(q-1)}+3^{(2q-1)/(q-1)}}{5^{(2q-1)/(q-1)}}\right)^{1-1/q}$$

$$\times\left\{\left[\sup\left\{|f'(a)|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}-\frac{c}{24}(b-a)^2\right]^{1/q}\right.$$

$$\left.+\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,|f'(b)|^q\right\}-\frac{c}{24}(b-a)^2\right]^{1/q}\right\}.$$

Theorem 3.3 is proved.

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