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Integral Inequalities for Mappings Whose Derivatives Are s-Convex in the Second Sense and Applications to Special Means for Positive Real Numbers

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Abstract In this paper, the authors establish a new type integral inequalities for differentiable s-convex functions in the second sense. By the well-known Hölder inequality and power mean inequality, they obtain some integral inequalities related to the s-convex functions and apply these inequalities to special means for positive real numbers.

Keywords: *s-convexity, Hermite-Hadamard Inequality, Bullen's inequality, Special Means.*

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1. Introduction

1.1. Definitions

Definition 1. [10] *A function $\varphi : I \to \mathbb{R}$ is said to be convex on I if inequality*

$$\varphi(tx+(1-t)y) \le t\varphi(x)+(1-t)\varphi(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in (0,1]$. We say that φ is concave if $(-\varphi)$ is convex.

Definition 2. [8] *Let $s \in (0,1]$. A function $\varphi : (0,\infty] \to [0,\infty]$ is said to be s-convex in the second sense if*

$$\varphi(tx+(1-t)y) \le t^s\varphi(x)+(1-t)^s\varphi(y), \tag{1.2}$$

for all $x, y \in (0,b]$ and $t \in (0,1]$. This class of s-convex functions is usually denoted by K_s^2.

Certainly, s-convexity means just ordinary convexity when $s = 1$.

1.2. Theorems

Theorem 1. ***The Hermite-Hadamard inequality***: *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$. The following double inequality*:

$$\varphi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \varphi(x)dx \le \frac{\varphi(u)+\varphi(v)}{2} \tag{1.3}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If φ is a positive concave function, then the inequality is reversed.

Theorem 2. [6] *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$, and let $a, b \in [0,\infty)$, $a < b$. If $\varphi \in L_1([0,1])$, then the following inequalities hold:*

$$2^{s-1}\varphi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \varphi(x)dx \le \frac{\varphi(u)+\varphi(v)}{s+1}. \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4). The above inequalities are sharp. If φ is an s-concave function in the second sense, then the inequality is reversed.

Theorem 3. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$\frac{1}{v-u}\int_u^v \varphi(x)dx \le \frac{1}{2}\left[\varphi\left(\frac{u+v}{2}\right)+\frac{\varphi(u)+\varphi(v)}{2}\right]$$

is known as Bullen's inequality for convex functions [[5], p.39].

In [4], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected to Hadamard's inequality, as follow:

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$, with $a < b$. If $|f'|^q$ is convex on [a; b], then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right]. \tag{1.5}$$

In [11], Pearce and Pečarić obtained inequalities for differentiable convex mappings which are connected with Hadamard's inequality, as follow:

Theorem 5. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable mapping on* I°, *where* $a,b \in I$, *with* $a<b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *for some* $q \geq 1$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{1.6}$$

If $|f'|^q$ *is concave on* $[a,b]$ *for some* $q \geq 1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.7}$$

In [1], Alomari, Darus and Kırmacı obtained inequalities for differentiable *s*-convex and concave mappings which are connected with Hadamard's inequality, as follow:

Theorem 6. *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be differentiable mapping on* I° *such that* $f' \in [a,b]$, *where* $a,b \in I$ *with* $a<b$. *If* $|f'|^q$, $q \geq 1$ *is concave on* $[a,b]$ *for some fixed* $s \in (0,1]$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \tag{1.8}$$

In [12], Tunç and Balgeçti obtained inequalities for differentiable convex functions which are connected with a new type integral inequality, as follow:

Lemma 1. *Le* $f: J \to \mathbb{R}$ *be a differentiable function on* J°. *If* $f' \in L[a,b]$, *then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right) &= \frac{1}{4}\int_0^1 (tb+(1-t)a)\, f'\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dx \\ &+\frac{1}{4}\int_0^1 (ta+(1-t)b)\, f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dx\end{aligned} \tag{1.9}$$

for each $t \in [0,1]$ *and* $a,b \in J$.

Theorem 7. [12] *Let* $f: J \to \mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|$ *is convex on* J, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right| \leq \left(\frac{5}{48}a+\frac{7}{48}b\right)|f'(a)|+\left(\frac{7}{48}a+\frac{5}{48}b\right)|f'(b)| \tag{1.10}$$

for each $a,b \in J$.

Theorem 8. [12] *Let* $f: J \to \mathbb{R}$ *R be a differentiable function on* J°. *If* $|f'|^q$ *is convex on* $[a,b]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right| \leq \frac{1}{4^{1+1/q}}L_p(a,b)\left[\begin{array}{l}\left[|f'(b)|^q+3|f'(a)|^q\right]^{\frac{1}{q}} \\ +\left[|f'(a)|^q+3|f'(b)|^q\right]^{\frac{1}{q}}\end{array}\right]. \tag{1.11}$$

Theorem 9. [12] *Let* $f: J \to \mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|^q$ *is convex on* $[a,b]$ *and* $q \geq 1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right| \leq \frac{A^{1-\frac{1}{q}}(a,b)}{4\times 12^{\frac{1}{q}}}\left\{\begin{array}{l}\left[\begin{array}{l}|f'(b)|^q(2a+b) \\ +|f'(a)|^q(4a+5b)\end{array}\right]^q \\ +\left[\begin{array}{l}|f'(a)|^q(a+2b) \\ +|f'(b)|^q(5a+4b)\end{array}\right]^q\end{array}\right\}. \tag{1.12}$$

For recent results and generalizations concerning Hadamard's inequality and concepts of convexity and *s*-convexity see [1-12] and the references therein.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty) \subset \mathbb{R}=(-\infty,\infty)$, and $u,v \in J$ with $0<u<v$ and $f' \in L[u,v]$ and

$$A(u,v)=\frac{u+v}{2},$$

$$G(u,v)=\sqrt{uv},$$

$$L_p(u,v)=\left(\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right)^{1/p},$$

$$u \neq v,\ p \in \mathbb{R},\ p \neq -1,0$$

be the arithmetic mean, geometric mean, generalized logarithmic mean for $u,v>0$ respectively.

2. Inequalities for s-convex Functions and Applications

Theorem 10. Let $f: J \to \mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|$ *is s-convex on* $[a,b]$ *for some fixed* $s \in (0,1]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq\frac{b\left(s2^{s+1}+s+2\right)+a\left(2^{s+2}-s-2\right)}{2^{s+2}(s+1)(s+2)}\left|f'(a)\right| \tag{2.1}$$
$$+\frac{a\left(s2^{s+1}+s+2\right)+b\left(2^{s+2}-s-2\right)}{2^{s+2}(s+1)(s+2)}\left|f'(ab)\right|$$

for each $x\in[a,b]$.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is s-convex on J, then we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq\frac{1}{4}\int_0^1(tb+(1-t)a)\left|f'\left(\left(\frac{1-t}{2}\right)^s b+\left(\frac{1+t}{2}\right)^s a\right)\right|dt$$
$$+\frac{1}{4}\int_0^1(ta+(1-t)b)\left|f'\left(\left(\frac{1-t}{2}\right)^s a+\left(\frac{1+t}{2}\right)^s b\right)\right|dt$$
$$\leq\frac{1}{4}\int_0^1(tb+(1-t)a)\left[\left(\frac{1-t}{2}\right)^s\left|f'(b)\right|+\left(\frac{1+t}{2}\right)^s\left|f'(a)\right|\right]dt$$
$$+\frac{1}{4}\int_0^1(ta+(1-t)b)\left[\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|+\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|\right]dt$$
$$\leq\frac{1}{4}\int_0^1(tb+(1-t)a)\left(\frac{1-t}{2}\right)^s\left|f'(b)\right|dt$$
$$+\frac{1}{4}\int_0^1(tb+(1-t)a)\left(\frac{1+t}{2}\right)^s\left|f'(a)\right|dt$$
$$+\frac{1}{4}\int_0^1(ta+(1-t)b)\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|dt$$
$$+\frac{1}{4}\int_0^1(ta+(1-t)b)\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|dt$$
$$=\frac{1}{2^{s+2}}\frac{as+a+b}{(s+1)(s+2)}\left|f'(b)\right|$$
$$+\frac{1}{2^{s+2}}\frac{b\left(s2^{s+1}+1\right)+a\left(2^{s+2}-s-3\right)}{(s+1)(s+2)}\left|f'(a)\right|$$
$$+\frac{1}{2^{s+2}}\frac{bs+b+a}{(s+1)(s+2)}\left|f'(a)\right|$$
$$+\frac{1}{2^{s+2}}\frac{a\left(s2^{s+1}+1\right)+b\left(2^{s+2}-s-3\right)}{(s+1)(s+2)}\left|f'(b)\right|$$
$$=\frac{1}{2^{s+2}}\left(\begin{array}{l}\frac{as+a+b}{(s+1)(s+2)}\\+\frac{1}{2^{s+2}}\frac{a\left(s2^{s+1}+1\right)+b\left(2^{s+2}-s-3\right)}{(s+1)(s+2)}\end{array}\right)\left|f'(b)\right|$$
$$+\frac{1}{2^{s+2}}\left(\begin{array}{l}\frac{bs+b+a}{(s+1)(s+2)}\\+\frac{1}{2^{s+2}}\frac{b\left(s2^{s+1}+1\right)+a\left(2^{s+2}-s-3\right)}{(s+1)(s+2)}\end{array}\right)\left|f'(a)\right|$$
$$=\frac{1}{2^{s+2}}\left(\frac{a\left(s2^{s+1}+s+2\right)+b\left(2^{s+2}-s-2\right)}{(s+1)(s+2)}\right)\left|f'(b)\right|$$
$$+\frac{1}{2^{s+2}}\left(\frac{b\left(s2^{s+1}+s+2\right)+a\left(2^{s+2}-s-2\right)}{(s+1)(s+2)}\right)\left|f'(a)\right|$$

Proposition 1. *Let* $a,b\in J^\circ$, $0<a<b$ *and* $s\in(0,1]$ *then*

$$\left|L_s^s(a,b)+\frac{(s-1)G^2(a,b)L_{s-2}^{s-2}(a,b)-A^s(a,b)}{2}\right|$$
$$\leq\frac{s\left[\begin{array}{l}\left(ab^{s-1}+a^{s-1}b\right)\left(s2^{s+1}+s+2\right)\\+\left(a^s+b^s\right)\left(2^{s+2}-s-2\right)\end{array}\right]}{2^{s+2}(s+1)(s+2)}. \tag{2.2}$$

Proof. The proof follows from (2.1) applied to the s-convex function $f(x)=x^s$ and $\left|f'(x)\right|=sx^{s-1}$.

Proposition 2. *Let* $a,b\in J^\circ$, $0<a<b$, $s\in(0,1)$ *then*

$$\left|\frac{L_s^s(a,b)}{1-s}-\frac{sG^2(a,b)L_{-1-s}^{-1-s}(a,b)+A^{1-s}(a,b)}{2(1-s)}\right|$$
$$\leq\frac{1}{2^{s+2}b^s}\left(\frac{a\left(s2^{s+1}+s+2\right)+b\left(2^{s+2}-s-2\right)}{(s+1)(s+2)}\right)$$
$$+\frac{1}{2^{s+2}a^s}\left(\frac{a\left(s-2^{s+2}+2\right)-b\left(s2^{s+1}+s+2\right)}{(s+1)(s+2)}\right). \tag{2.3}$$

Proof. The proof follows from (2.1) applied to the s-convex function $f(x)=\frac{x^{1-s}}{1-s}$ and $\left|f'(x)\right|=1/x^s$ with $s\in(0,1)$.

Remark 1. *In (2.1), (2.2), if we take* $s\to1$, *then (2.1), (2.2) reduces to (1.10),* [[12], Proposition 2], *respectively.*

Theorem 11. *Let* $f:J\to\mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|$ *is s-convex on* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left(\frac{bf(a)-af(b)}{b-a}+f\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq\frac{L_p(a,b)}{4\left(2^s(s+1)\right)^{\frac{1}{q}}}\left\{\begin{array}{l}\left(\left|f'(b)\right|^q+\left(2^{s+1}-1\right)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\+\left(\left|f'(a)\right|^q+\left(2^{s+1}-1\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right\} \tag{2.4}$$

for each $x\in[a,b]$.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is s-convex on J, then we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{af(b)-bf(b)}{2(b-a)}+\frac{1}{2}f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{1}{4}\int_0^1(tb+(1-t)a)\left|f'\left(\left(\frac{1-t}{2}\right)^s b+\left(\frac{1+t}{2}\right)^s a\right)\right|dt \quad (2.5)\\&+\frac{1}{4}\int_0^1(ta+(1-t)b)\left|f'\left(\left(\frac{1-t}{2}\right)^s a+\left(\frac{1+t}{2}\right)^s b\right)\right|dt.\end{aligned}$$

Since $|f'|^q$ is *s*-convex, by the Hölder inequality, we have

$$\begin{aligned}&\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)^s b+\left(\frac{1+t}{2}\right)^s a\right)\right|^q dt\\&\leq\int_0^1\left(\left(\frac{1-t}{2}\right)^s|f'(b)|^q+\left(\frac{1+t}{2}\right)^s|f'(a)|^q\right)dt\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}&\frac{1}{4}\int_0^1(tb+(1-t)a)\left|f'\left(\left(\frac{1-t}{2}\right)^s b+\left(\frac{1+t}{2}\right)^s a\right)\right|dt\\&\leq\frac{1}{4}\left(\int_0^1(tb+(1-t)a)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)^s b+\left(\frac{1+t}{2}\right)^s a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\frac{1}{4}\left(\int_0^1(ta+(1-t)b)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)^s a+\left(\frac{1+t}{2}\right)^s b\right)\right|^q dt\right)^{\frac{1}{q}}\\&\leq\frac{1}{4}\left(\int_0^1(tb+(1-t)a)^p dt\right)^{\frac{1}{p}}\left[\int_0^1\left(\left(\frac{1-t}{2}\right)^s|f'(b)|^q+\left(\frac{1+t}{2}\right)^s|f'(a)|^q\right)dt\right]^{\frac{1}{q}}\\&+\frac{1}{4}\left(\int_0^1(ta+(1-t)b)^p dt\right)^{\frac{1}{p}}\left[\int_0^1\left(\left(\frac{1-t}{2}\right)^s|f'(a)|^q+\left(\frac{1+t}{2}\right)^s|f'(b)|^q\right)dt\right]^{\frac{1}{q}}.\end{aligned}\tag{2.7}$$

It can be easily seen that

$$\begin{aligned}&\int_0^1(tb+(1-t)a)^p dt=\int_0^1(ta+(1-t)b)^p dt\\&=\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}=L_p^p(a,b).\end{aligned}\tag{2.8}$$

If expressions (2.6)-(2.8), we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx+\frac{af(b)-bf(a)}{2(b-a)}-\frac{1}{2}f\left(\frac{a+b}{2}\right)\right|\\&=\frac{1}{4}L_p(a,b)\left[\frac{1}{2^s(s+1)}\left(|f'(b)|^q+\left(2^{s+1}-1\right)|f'(a)|^q\right)\right]^{\frac{1}{q}}\\&+\frac{1}{4}L_p(a,b)\left[\frac{1}{2^s(s+1)}\left(|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q\right)\right]^{\frac{1}{q}}.\end{aligned}$$

The proof is completed.

Proposition 3. *Let* $a,b\in J^\circ$, $0<a<b$ *and* $s\in(0,1]$, *then*

$$\begin{aligned}&\left|L_s^s(a,b)+\frac{(s-1)G^2(a,b)L_{s-2}^{s-2}(a,b)-A^s(a,b)}{2}\right|\\&\leq\frac{L_p(a,b)}{\left(2^{2q+s}(s+1)\right)^{\frac{1}{q}}}\left\{\begin{array}{l}\left(\left(sb^{s-1}\right)^q+\left(2^{s+1}-1\right)\left(sa^{s-1}\right)^q\right)^{\frac{1}{q}}\\+\left(\left(sa^{s-1}\right)^q+\left(2^{s+1}-1\right)\left(sa^{s-1}\right)^q\right)^{\frac{1}{q}}\end{array}\right\}.\end{aligned}\tag{2.9}$$

Proof. The proof follows from (2.4) applied to the *s*-convex function $f(x)=x^s$ and $|f'(x)|=sx^{s-1}$.

Proposition 4. *Let* $a,b\in J^\circ$, $0<a<b$ *and* $s\in(0,1)$, *then*

$$\begin{aligned}&\left|\frac{L_s^s(a,b)}{1-s}+\frac{sG^2(a,b)L_{-1-s}^{-1-s}(a,b)+A^{1-s}(a,b)}{2(1-s)}\right|\\&\leq\frac{L_p(a,b)}{\left(2^{2q+s}(s+1)\right)^{\frac{1}{q}}}\left\{\begin{array}{l}\left(b^{-sq}+\left(2^{s+1}-1\right)a^{-sq}\right)^{\frac{1}{q}}\\+\left(a^{-sq}+\left(2^{s+1}-1\right)b^{-sq}\right)^{\frac{1}{q}}\end{array}\right\}.\end{aligned}\tag{2.10}$$

Proof. The proof follows from (2.4) applied to the s-convex function $f(x)=\frac{x^{1-x}}{1-s}$ and $|f'(x)|=1/x^2$.

Remark 2. *In (2.4), (2.9), if we take* $s\to1$, *then (2.4), (2.9) reduces to (1.11),* [[12], Proposition 5], *respectively.*

Theorem 12. *Let* $f:J\to\mathbb{R}$ *be a differentiable function on* J°. *If* $|f'|$ *is s-convex on* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $q>1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx+\frac{1}{2}\left(\frac{af(b)-bf(a)}{(b-a)}-f\left(\frac{a+b}{2}\right)\right)\right|$$

$$\leq\frac{A^{1-\frac{1}{q}}(a,b)}{\left(2^{2q+s}(s+1)(s+2)\right)}$$

$$\times\left\{\begin{array}{l}\left[\begin{array}{l}(as+a+b)\left|f'(b)\right|^q\\+\left(\begin{array}{l}b\left(s2^{s+1}+1\right)\\+a\left(2^{s+2}-s-3\right)\end{array}\right)\left|f'(a)\right|^q\end{array}\right]^{\frac{1}{q}}\\+\left[\begin{array}{l}(bs+b+a)\left|f'(a)\right|^q\\+\left(\begin{array}{l}a\left(s2^{s+1}+1\right)\\+b\left(2^{s+2}-s-3\right)\end{array}\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right\} \quad (2.11)$$

Proof. From Lemma 1 and using the well-known power mean inequality and since $|f'|^q$ is *s*-convex on $[a,b]$, we can write

$$\left|\frac{1}{b-a}\int_a^b f(x)dx+\frac{af(b)-bf(a)}{2(b-a)}-\frac{1}{2}f\left(\frac{a+b}{2}\right)\right|$$

$$=\frac{1}{4}\left(\int_0^1(tb+(1-t)a)dx\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(tb+(1-t)a)\left|f'\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)\right|^q dx\right]^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\int_0^1(ta+(1-t)b)dx\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(ta+(1-t)b)\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dx\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{4}\left(\int_0^1(tb+(1-t)a)dx\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(tb+(1-t)a)\left(\begin{array}{l}\left(\frac{1-t}{2}\right)^s\left|f'(b)\right|^q\\+\left(\frac{1+t}{2}\right)^s\left|f'(a)\right|^q\end{array}\right)dx\right]^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\int_0^1(ta+(1-t)b)dx\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(ta+(1-t)b)\left(\begin{array}{l}\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|^q\\+\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|^q\end{array}\right)dx\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{4}\left(\frac{a+b}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\left|f'(b)\right|^q\int_0^1\left(\frac{1-t}{2}\right)^s(tb+(1-t)a)dt\\+\left|f'(a)\right|^q\int_0^1\left(\frac{1+t}{2}\right)^s(tb+(1-t)a)dt\end{array}\right]^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\frac{a+b}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\left|f'(a)\right|^q\int_0^1\left(\frac{1-t}{2}\right)^s(ta+(1-t)b)dt\\+\left|f'(b)\right|^q\int_0^1\left(\frac{1+t}{2}\right)^s(ta+(1-t)b)dt\end{array}\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{4}A^{1-\frac{1}{q}}(a,b)\left[\begin{array}{l}\frac{\left|f'(b)\right|^q}{2^s}\frac{as+a+b}{(s+1)(s+2)}\\+\frac{\left|f'(a)\right|^q}{2^s}\frac{\left[\begin{array}{l}b\left(s2^{s+1}+1\right)\\+a\left(2^{s+2}-s-3\right)\end{array}\right]}{(s+1)(s+2)}\end{array}\right]^{\frac{1}{q}}$$

$$+\frac{1}{4}A^{1-\frac{1}{q}}(a,b)\left[\begin{array}{l}\frac{\left|f'(a)\right|^q}{2^s}\frac{bs+b+a}{(s+1)(s+2)}\\+\frac{\left|f'(b)\right|^q}{2^s}\frac{\left[\begin{array}{l}a\left(s2^{s+1}+1\right)\\+b\left(2^{s+2}-s-3\right)\end{array}\right]}{(s+1)(s+2)}\end{array}\right]^{\frac{1}{q}}.$$

The proof is completed.

Proposition 5. *Let* $a,b\in J^\circ$, $0<a<b$ *and* $s\in(0,1]$, *then*

$$\left|L_s^s(a,b)+\frac{(s-1)G^2(a,b)L_{s-2}^{s-2}(a,b)-A^s(a,b)}{2}\right|$$

$$\leq\frac{A^{1-\frac{1}{q}}(a,b)}{\left(2^{2q+s}(s+1)(s+2)\right)^{\frac{1}{q}}}$$

$$\left\{\begin{array}{l}\left[\begin{array}{l}(as+a+b)\left(sb^{s-1}\right)^q\\+\left(b\left(s2^{s+1}+1\right)+a\left(2^{s+2}-s-3\right)\right)\left(sa^{s-1}\right)^q\end{array}\right]^{\frac{1}{q}}\\+\left[\begin{array}{l}(bs+b+a)\left(sa^{s-1}\right)^q\\+\left(a\left(s2^{s+1}+1\right)+b\left(2^{s+2}-s-3\right)\right)\left(sa^{s-1}\right)^q\end{array}\right]^{\frac{1}{q}}\end{array}\right\}. \quad (2.12)$$

Proof. The proof follows from (2.11) applied to the s-convex function $f(x)=x^s$ and $\left|f'(x)\right|=sx^{s-1}$.

Proposition 6. Let $a,b\in J^\circ$, $0<a<b$ and $s\in(0,1]$, then

$$\left|\frac{L_s^s(a,b)}{1-s}+\frac{sG^2(a,b)L_{-1-s}^{-1-s}(a,b)-A^{1-s}(a,b)}{2(1-s)}\right|$$

$$\le \frac{A^{1-\frac{1}{q}}(a,b)}{\left(2^{2q+s}(s+1)(s+2)\right)^{\frac{1}{q}}}$$

$$\left\{\begin{array}{l}\left[\begin{array}{l}(as+a+b)b^{-sq}\\+\left(b\left(s2^{s+1}+1\right)+a\left(2^{s+2}-s-3\right)\right)a^{-sq}\end{array}\right]^{\frac{1}{q}}\\+\left[\begin{array}{l}(bs+b+a)a^{-sq}\\+\left(a\left(s2^{s+1}+1\right)+b\left(2^{s+2}-s-3\right)\right)b^{-sq}\end{array}\right]^{\frac{1}{q}}\end{array}\right\}. \quad (2.13)$$

Proof. The proof follows from (2.11) applied to the s-convex function $f(x)=\frac{x^{1-s}}{1-s}$ and $|f'(x)|=1/x^2$.

Remark 3. *In (2.11), (2.12), if we take* $s\to 1$, *then (2.11), (2.12) reduces to (1.12),* [[12], Proposition 8] *respectively*.

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