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On Irresolute Topological Vector Spaces-II

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Abstract In this paper, we continue the study of Irresolute topological vector spaces. Notions of convex, bounded and balanced set are introduced and studied for Irresolute topological vector spaces. Along with other results, it is proved that: 1. Irresolute topological vector spaces are semi-Hausdorff spaces. 2. Every Irresolute topological vector space is semi-regular space. 3. In Irresolute topological vector spaces, $sCl(C)$ as well as $sInt(C)$ is convex if C is convex. 4. In Irresolute topological vector spaces, $sCl(E)$ is bouned if E is bounded. 5. In Irresolute topological vector spaces, $sInt(E)$ is balanced if E is balanced and $0 \in sInt(E)$. 6. In Irresolute topological vector spaces, every semi compact set is bounded.

Keywords: *topological vector space, irresolute topological vector space, irresolute mapping, semi open set*

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1. Introduction

This paper is devoted to the study of Irresolute topological vector spaces. The notion was defined by Moiz and Asad [1] in 2016. The notion is defined although analogously but is independent of linear topological space. The part of this paper deals with the boundedness, convexity and balancedness of sets in Irresolute topological vector spaces.

The motivation behind the study of this paper is to investigate such structures in which the topology is endowed upon a vector space which fails to satisfy the continuity condition for vector addition and scalar multiplication or either. We are interested to study such structures for irresolute mappings in the sense of Levine. It is proved in [1] that:

- Irresolute topological vector space is not topological vector space in general.
- Open hereditary property holds in irresolute topological vector spaces.
- Every semi-open set is translationally invariant in irresolute topological vector spaces
- A homomorphism between irresolute topological vector spaces is irresolute if it is irresolute at identity element.

In this paper, notions of convex, bounded and balanced set are introduced and studied for Irresolute topological vector spaces.

2. Preliminaries

Throughout in this paper, X and Y are always representing topological spaces on which separation axioms are not considered untill and unless stated. We will represent field by F and the set of all real numbers by $\mathbb{R}$. δ and ∈ are assumed here negligible small but positive real numbers.

Semi open sets in topological spaces were firstly appeared in 1963 in the paper of N. Levine [3]. With the invent of semi open sets and semi continuity, many interesting concepts in topology were further generalized and investigated by number of mathematicians. A subset A of a topological space X is said to be semi-open if, and only if, there exists an open set O in X such that $O \subset A \subset Cl(O)$, or equivalently if $A \subset Cl(Int(A))$. $SO(X)$ denotes the collection of all semi-open sets in the topological space (X, τ). The complement of a semi-open set is said to be semi closed; the semi closure of $A \subset X$, denoted by $sCl(A)$, is the intersection of all semi-closed subsets of X containing A [4]. It is known that $x \in sCl(A)$ if, and only if, for any semi-open set U containing x, $U \cap A$ is non-empty. Every open set is semi-open and every closed set is semi-closed. It is known that union of any collection of semi-open sets is semi-open set, while the intersection of two semi-open sets need not be semi-open. The intersection of an open set and a semi-open set is semi-open set. A subset A of a topological space X is said to be semi compact if for every cover of A by semi open sets of X, there exists a finite subcover.

Remember that, a set $U \subset X$ is a semi-open neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that $x \in A \subset U$. A set $A \subset X$ is semi open in X if, and only if, A is semi open neighbourhood of each of its points. If a semi open neighbourhood U of a point x is a semi open set, we say that U is a semi open neighbourhood of x. If $A_1 \in (X_1)$ and $A_2 \in SO(X_2)$, then $A_1 \times A_2 \in SO(X_1 \times X_2)$, where X_1 and X_2 are topological spaces and $X_1 \times X_2$ is a product space. It is worth mentioning that a set semi-open in the product space cannot be expressed as product of semi-open sets in the components spaces. Basic properties of semi-open sets are given in [3] and of semi closed sets in [4] and [5], and references there in.

If $X(F)$ is a vector space then e denotes its identity element, and *for* a fixed $x\in X$, ${}_xT:X\to X$; $y\mapsto x+y$ and $T_x:X\to X$, $y\mapsto y+x$, denote the left and the right translation by x, respectively. The addition mapping $m:X\times X\to X$ is defined by $m((x,y))=x+y$, and the scalar multiplication mapping $M:F\times X\to X$ is defined by $M((\lambda,x))=\lambda x$.

Definition 2.1: Let $f:X\to Y$ be single valued function between topological spaces (continuity not assumed). Then:

1. $f:X\to Y$ is termed as semi-continuous [3], if and only if, for each V open in Y, there exists $f^{\leftarrow}(V)\in SO(X)$.
2. $f:X\to Y$ is termed as irresolute [4], if, and only if, for each $V\in SO(Y)$, there exists $f^{\leftarrow}(V)\in SO(X)$. Note that the function $f:X\to Y$ is irresolute at $x\in X$, if for each semi open set V in Y containing $f(x)$, there exists a semi open set U in X containing x such that $f(U)\subseteq V$.

Recall that a topological vector space $(X_{(F)},\tau)$ is a vector space over a topological field F (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that:

1. Addition mapping $m:X\times X\to X$ defined by $m(x,y)=x+y$; $x,y\in X$ is continuous function.
2. Multiplication mapping $M:F\times X\to X$ defined by $M(\lambda,x)=\lambda x$; $\lambda\in F$, $x\in X$. is continuous function (where the domains of these functions are endowed with product topologies).

Equivalently, we have a topological vector space X over a topological field F (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that:

1. for each x, $y\in X$, and for each open neighbourhood W of $x+y$ in X, there exist neighbourhoods U and V of x and y respectively in X, such that $U+V\subset W$.
2. for each $\lambda\in F$, $x\in X$ and for each open neighbourhood W in X containing λx, there exist neighbourhoods U of λ in F and V of x in X such that $U\cdot V\subseteq W$.

Equivalently, we have: topological Vector Space X over the field F ($\mathbb{R}$ *or* $\mathbb{C}$) with a topology on X such that $(X,+)$ is a topological group and $M:F\times X\to X$ is a continuous mapping.

3. Irresolute Topological Vector Spaces

Definition 3.1: [2] A mapping $f:X\to Y$ form a space X onto a space Y is said to be semi-quotient provided a subset V of Y is open in Y if and only if $f^{\leftarrow}(V)$ is semi-open in X. Equivalently, every semi-quotient mapping is semi-continuous and every quotient mapping is semi-quotient.

Remark 3.2: [2] Every surjective continuous mapping $f:X\to Y$ which is either s-open or s-closed is a semi-quotient mapping. Let X be a topological space and Y a set $f:X\to Y$ be a mapping. Define, $S\tau_Q:=\{V\subset Y: f^{\leftarrow}(V)\in SO(X)\}$. It is easy to see that the family $S\tau_Q$ is a generalized topology on Y. (*i.e.*$\phi\in S$ᅟᅟ$_Q$: and union of any collection of sets in, $S\tau_Q$ is again in $S\tau_Q$ generated by f): we call it the semi-quotient generalized topology. But, $S\tau_Q$ need not be a topology in Y. It happens if X is an extremely disconnected space, because in this case the intersection of two semi-open sets in X is semi-open. It is trivial fact that in the later case, $S\tau_Q$ is the finest topology σ on Y such that $f:X\to(Y,\sigma)$ is semi-continuous.

For more detail we refer the reader to [2].

Lemma 3.3: Let $p:G\to G/H$ be a canonical projection mapping then for any subset U of G, $p^{\leftarrow}(p(U))=U\star H$. We know that, if G is a group and H is a subgroup of G then the collection of all left cosets of H in G that is $G/H=\{x\star H: x\in G\}$ is not a group until H is normal in G.

Definition 3.4: A mapping f form a topological space to itself is called irresolute-homeomorphism if it is bijective, irresolute and pre-semi-open.

Definition 3.5: A mapping is said to be semi-open if the image of a semi-open set is open.

Lemma 3.6: [1] Let A and B be subsets of an irresolute topological vector space. Then $sCl(A)+sCl(B)\subseteq sCl(A+B)$

Lemma 3.7: [1] Let $(X_{(F)},\tau)$ be an irresolute topological vector space. For given $y\in X$ and nonzero $\lambda\in F$, each translation mapping $T_y:x\to x+y$ and multiplication mapping $M_\lambda:x\to\lambda x$ where $x\in X$ is irresolute homeomorphisim onto itself.

Lemma 3.8: [1] If f: $(X,\tau_X)\to(Y,\tau_y)$ is an irresolute homeomorphism, then $sCl(f(B))=f(sCl(B))$ for all $B\subseteq X$.

Lemma 3.9: [1] Let $(X_{(F)},\tau)$ be an irresolute topological vector space:

- i. If $A\in SO(X)$, then $A+x\in SO(X)$ for every $x\in X$.
- ii. If $A\in SO(X)$, then $\lambda A\in SO(X)$ for every $\lambda\in$ F.
- iii. For every semi-open neighbourhood U of 0, there exists a semi-open neighbourhood V of 0 such that $V+V\subseteq U$.

Theorem 3.10: For a closed subspace H of an irresolute topological vector space $X(F)$, the semi-quotient mapping $q:X\to X/H$ is s-open mapping, that caries semi-open sets to open.

Proof: Let U be semi-open set in $X(F)$. Then, $q^{-1}(q(U))=q^{-1}(U+H)=U+H\in SO(X)$ because $X_{(F)}$ is an irresolute topological vector space. Therefore, $q(U)$ is open, this proves that q is s-open mapping.

Definition 3.11: Let Y be a linear subspace of $(X(F),\tau)$ which means $Y+Y\subseteq Y$ and for all $\alpha\in F$, $\alpha Y\subseteq Y$.

Theorem 3.12: Let $(X(F),\tau)$ be an irresolute topological vector space and if Y is linear subspace of X, so is $sCl(Y)$.

Proof: Let Y be a linear subspace of X, which means that, $Y+Y\subseteq Y$ and $\forall\alpha\in F$, $\alpha Y\subseteq Y$. By Lemma 3.6, $sCl(Y)+sCl(Y)\subseteq sCl(Y+Y)\subseteq sCl(Y)$. Since, scalar multiplication is an irresolute homeomorphism, therefore by Lemma 3.8, it maps the semi-closure of a set into the semi-closure of its image, namely for every $\alpha\in F$, $\alpha(sCl(Y))=sCl(\alpha Y)\subseteq sCl(Y)$. Therefore, $sCl(Y)$ is linear subspace.

Definition 3.13: A subset E of an irresolute topological vector space $(X(F),\tau)$ is said to be balanced if $\forall\alpha\in F$, $|\alpha|\leq 1$, $\alpha E\subseteq E$.

Theorem 3.14: Let $(X(F),\tau)$ be an irresolute topological vector space, For every $B\subset X$: if B is balanced, so is $sCl(B)$.

Proof: By lemma 3.8, semi-closures of a set maps onto the semi-closures of its image, thus for every $\alpha\in F$, $\alpha(sCl(B))=sCl(\alpha B)$. If B is balanced, then for $|\alpha|\leq 1$. $\alpha(sCl(B))=sCl(\alpha B)\subseteq sCl(B)$. Hence, $sCl(B)$ is balanced.

Lemma 3.15: [2] Let G be an irresolute topological group, and μ_e the collection of all semi-open neighbourhoods of e. Then:

1. For every $U\in\mu_e$, there is a $V\in\mu_e$ such that $V^{-1}\subset U$.

2. For every $U\in\mu_e$ and every $x\in U$, there is $V\in\mu_e$ such that $x\star V\subseteq U$.

Theorem 3.16: Let $(X_{(F)},\tau)$ be an irresolute topological vector space then for any $A\subset X$, $sCl(A)=\cap\{A+U:U\in SO(X,0)\}$.

Proof: Firstly we have to show that, $\cap\{A+U:U\in SO(X,0)\}\subseteq sCl(A)$. For this, assume that $x\notin sCl(A)$, this implies that there exists semi-open set U containing 0 and hence gives: $(-U+x)\cap A=\phi$. Or $x\notin A+U$. Or $x\notin\cap\{A+U:U\in SO(X,0)\}$. This shows that, $\cap\{A+U:U\in SO(X,0)\}\subseteq sCl(A)$. Conversely, Let $x\in sCl(A)$, this implies that there exists, $V\in SO(X,0)$ such that: $x+V\in SO(X,x)$ and $(x+V)\cap A\neq\phi$. Let $a\in(x+V)$ and $a\in A$. Thus $a=x+v_1$ for some $v_1\in V$. This gives $x\in a-v_1\in a-V\subseteq A+U$. This proves that, $x\in\cap\{A+U:U\in(X,0)\}$. Hence, $sCl(A)\subseteq\{A+U:U\in SO(X,0)\}$. Therefore, we have $sCl(A)=\cap\{A+U:U\in SO(X,0)\}$.

Theorem 3.17: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. Let μ_0 be the collection of all semi-open neighbourhoods of 0, then for every $x\in X$, the collection $\mu_x=\{V+x:V\in\mu\}$ is a semi-open neighbourhood system for x.

Proof: Since for the irresolute topological vector spaces the translation mappings are irresolute homeomorphism. Therefore, this result is obvious. i.e. $\mu_{(0)}=\{V+0:V\in\mu\}$. Thus by algebra we can deduce that $\mu_x=\{V+x:V\in\mu\}$. This gives that μ_x is semi-open neighbourhood system for x. The $V+x$ is semi-open by Lemma 3.9(i).

Theorem 3.18: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. μ_0 is collection of all semi-open neighbourhoods of 0, then the topology τ is semi-τ_1 space if and only if $\cap_{V\in\mu_0}V=\{0\}$.

Proof: Let $\cap_{V\in\mu_0}V=A$ so $0\in A$ because $\{0\in V:\forall\ V\in\mu_0\}$. Assume τ is semi-Hausdorff, $\forall\ x\in X-\{0\}$, the set $X-\{x\}$ is semi-open neighbourhood of 0. Now, there exists $V_x\in\mu_0$ with $V_x\subseteq X-\{x\}$. Therefore, $A\subset\cap_{x\neq0}V_x\subset\cap_{x\neq0}(X-\{x\})=\{0\}$. This implies, $A=\{0\}$. Conversely, let $A=\{0\}$ and we have to show that τ is semi-τ_1. For that let $x\in X$ and $x\neq y$ with $x-y\neq0$ and let us indicate how to construct two disjoint semi-open neighbourhoods one for x and one for y. Using translations we can assume $y=0$. Since, $0\neq x\notin\cap_{V\in\mu_0}V=\{0\}$, there exist some $V\in\mu_0$, such that $x\notin V$. Using Lemma 3.9(iii), there is some $W\in\mu_0$, such that: $W+W\subseteq V$. We still have $x\notin W+W$ and $W+W$ is semi-open. Since V is semi-open neighbourhood of 0, therefore, $-V$ is also semi-open neighbourhood of 0 by Lemma 3.9(ii). $-V+x$ would be the semi-open neighbourhood for x. Therefore, τ is semi-Hausdorff. If $y\neq0$ then we can construct $-V+y$ a semi-open neighbourhood for y. we can clearly see that $x\in-V+x$ and $y\in-V+y$ also $x\notin-V+y$ and $y\notin-V+x$. This implies, τ is semi-τ_1 space.

Theorem 3.19: Every irresolute topological vector space is semi-Hausdroff space.

Proof: We only need to separate 0 and $a\in X$ with $a\neq0$. Let $W=X-\{a\}$ is semi-open neighbourhood of 0, then by Lemma 3.9(iii), there exists semi-open neighbourhood V of 0 such that: $V+V\subseteq W$. Since V is semi-open neighbourhood of 0, then $a-V$ is semi-open neighbourhood of a by algebra. We claim that, $V\cap(a-V)=\varphi$. If $V\cap(a-V)\neq\varphi$ then $y=a-x$ and thus $x+y=a$, where x and y are some members of V. Therefore, we have $a=x+y\in V+V\subseteq W$ which is a contradiction. This proves that every irresolute topological vector space is semi-Hausdroff space.

Theorem 3.20: Every irresolute topological vector space is semi-regular space if the space is extremally disconnected.

Proof: Let A be semi-closed subset of an irresolute extremally disconnected space X and let $x\notin A$. Now $U=X-A$ is semi-open set and $U^*=U-\{x\}$ is a semi-open neighbourhood of 0. By Lemma 3.9(iii), there exists semi-open neighbourhood V of 0 such that: $V+V\subseteq U^*$. Now, $-V$ is also semi-open neighbourhood of 0, and let $W=V\cap(-V)$. Then, $W=-W\in(X,0)$ moreover $W+W\subseteq U^*$. Let $U_1=W+\{x\}$ and $U_2=W+A=\cup_{y\in A}(W+y)$. Then, $x\in U_1$, and $A\subseteq U_2$ and clearly, U_1 and U_2 are semi-open sets. Further, if $z\in U_1\cap U_2$, then we must have, $z=w_1+x$ and $z=w_2+a$, where w_1 and w_2 are some members of W and $a\in A$. But then we would have, $a=x+w_1-w_2\in x+W-W=x+W+W\subseteq x+U^*=X-A$. A contradiction to the fact that $a\in A$. Hence, $U_1\cap U_2=\varphi$. Therefore, every extremally disconnected irresolute topological vector space is semi-regular space.

Definition 3.21: A set C is said to be convex if for $t\in[0,1]$, $tC+(1-t)C\subseteq C$.

Theorem 3.22: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. If C is convex then so is $sCl(C)$.

Proof: Convexity is a purely algebraic property, but (closure) semi-closures and (interior) semi-interior are topological concepts. The convexity of C implies, $tC+(1-t)C\subseteq C$. Let $t\in[0,1]$, then $t(sCl(C))=sCl(tC)$ and $(1-t)(sCl(C))=sCl((1\text{-}t)C)$. By Lemma 3.6: $t(sCl(C))+(1-t)sCl(C)=sCl(tC)+sCl((1-t)C)\subseteq sCl(tC+(1-t)C)\subseteq sCl(C)$. Thus, $sCl(C)$ is convex.

Theorem 3.23: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. If C is convex then $sInt(C)$ is convex.

Proof: Suppose that C is convex. Let $x,y\in sCl(C)$. This means there exist semi-open neighbourhoods U and V of 0 such that: $x+U\subseteq C$ and $y+V\subseteq C$. Since, C is convex. Therefore, $t(x+U)+(1-t)(y+V)=(tx+(1-t)y)+Ut+(1-t)V\subseteq C$. Which proves that $tx+(1-t)y\in sInt(C)$. Namely $sInt(C)$ is convex.

Definition 3.24: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. A subset $E\subset X$ is said to be bounded if for all V semi-open neighbourhoods of 0, there exists $s\in R$ such that for all $t>s$, $E\subset tV$. That is, every semi-open neighborhood of zero contains E after being blown up sufficiently.

Theorem 3.25: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. If E is bounded, then $sCl(E)$ is bounded.

Proof: Let V is neighbourhood of 0 then by theorem, there exist W such that $sCl(W)\subseteq V$. Since E is bounded, $E\subseteq tW\subseteq(sCl(W))\subseteq tV$, for sufficiently large t. It follows that for large enough t, $sCl(E)\subseteq t(sCl(W))\subseteq tV$. Thus, $sCl(E)$ is bounded.

Theorem 3.26: Let $(X_{(F)},\tau)$ be an irresolute topological vector space, and let μ_0 be a collection of all semi-open neighbourhoods containing the identity. Then, for each $U\in\mu_0$, there exists $V\in\mu_0$ such that $sCl(V)\subseteq U$.

Proof: Let $U\in\mu_0$. Then by Lemma 3.9(iii), there exists $V\in\mu_0$ such that $V+V\subseteq U$. Let $x\in sCl(V)$. Since $x-V$ is semi-open neighbourhood of x, so $(x-V)\cap V\neq\varphi$. Choose, $y\in(x-V)\cap V$ then $y=x-v_1=v_2$, where v_1, $v_2\in V$. Thus, $x=v_2+v_1\in V+V\subseteq U$. Therefore, $sCl(V)\subseteq U$.

Theorem 3.27: (Balanced) Let $(X_{(F)},\tau)$ be an irresolute topological vector space, and μ_0 be a collection of all semi-open neighbourhoods, then for each $U\in\mu_0$, there exists a balanced $W\in\mu_0$ such that $W\subseteq U$.

Proof: Let $U\in\mu_0$, then by definition, there exist semi-open neighbourhood V of 0 in X and B $[0, r]$ of 0 in F such that $W=B[0, r]V\subseteq U$, Clearly, W is balanced. Furthur, W is semi-open neighbourhood of 0 (Since $rV\subseteq B[0, r]=W$) and rV is semi-open neighbourhood of 0 by Lemma 3.9(ii). Thus we have a balanced $W\in\mu_0$ and $W\subseteq U$.

Lemma 3.28: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. Then, $\alpha(sInt(B))=sInt(\alpha B)$, where $\alpha\in F$ and $B\subset X$.

Proof: Let $\alpha x\in(sInt(B))$ such that $x\in sInt(B)$ then there exists a semi-open neighbourhood U such that $x\in U\subseteq B$. Now, $\alpha x\in\alpha U\subseteq\alpha B$. As αU is semi-open by Theorem 3.9 (2). So, $\alpha x\in sInt(\alpha B)$. Therefore, $\alpha(sInt(B))\subseteq sInt(\alpha B)$. Conversely, Let $y\in sInt(\alpha B)$, where define $y=\alpha x$ for some $x\in B$, then there exists V semi-open neighbourhood and $V=\alpha W$ such that W be semi-open in B by the fact that $\alpha x\in\alpha W\subseteq\alpha B$. So, $x\in W\subseteq B$ or $x\in sInt(B)$ or $\alpha x\in\alpha(sInt(B))$. Therefore, $sInt(\alpha B)\subseteq\alpha(sInt(B))$. Hence, $\alpha(sInt(B))=sInt(\alpha B)$.

Theorem 3.29: (Semi-interior) Let $(X_{(F)},\tau)$ be an irresolute topological vector space. For every $B\subseteq X$: if B is balanced and $0\in sInt(B)$, then $sInt(B)$ is balanced.

Proof: Let B be balanced subset of X. By Lemma 3.8, for every $0<|\alpha|\leq 1$, $\alpha(sInt(B))=sInt(\alpha B)$. Since, B is balanced therefore $\alpha B\subseteq B$, $|\alpha|\leq 1$. Also, $\alpha(sInt(B))=sInt(\alpha B)\subseteq sInt(B)$. Since for $\alpha=0$, $\alpha(sInt(B))=\{0\}$, we must require $0\in sInt(B)$ for the latter to be balanced.

Theorem 3.30: (Convergent sequence) Let V be a semi-open neighbourhood of 0 in Irresolute topological vector space. Then, for every sequence $r_n\to\infty$, $\cup_{n-1}^{\infty} r_nV=X$.

Proof: Let $x\in X$ and consider the sequence $(x/(r_n))$. This sequence converges to 0 by the irresoluteness of the scalar multiplication $F\times X\to X$.
Thus, for sufficiently large n, $(x/(r_n))\in V$ i.e., $x\in r_nV$.

Theorem 3.31: Let $(X_{(F)},\tau)$ be an irresolute topological vector space. Every semi-compact set is bounded.

Proof: Let $K\subset X$ be a semi-compact. We need to prove that it is bounded, namely, that for every V semi-open neighbourhood of 0, $K\subset tV$ for sufficiently large t. Let V be semi-open neighbourhood of 0, then by theorem 3.27, there exists a balanced semi-open neighbourhood W of 0 such that $W\subset V$. By theorem 3.30, $K\subset\cup_{j=1}^{\infty} nW$. Since, K is semi-compact therefore:

$K \subset\cup_{j=1}^{K} n_jW = n_k \cup_{j=1}^{K}(n_j/n_K) W \subset n_KW$. Thus, for all $t>n$, $K\subset n_KW=(t(n_K/t))W)\subset tW\subset tV$. Which proves that K is bounded.

Theorem 3.32: A Cauchy sequence in an irresolute topological vector space is bounded.

Proof: Let (x_n) be a Cauchy sequence. Let W be the semi-open neighbourhood of 0, then by Lemma 4(iii), there exists a semi-open neighbourhood V of 0 such that: $V+V\subseteq W$. By definition of a cauchy sequence, there exists N such that for all $m,n \geq N$, $x_n-x_m\in V$ and in particular for all $n>N$, $x_n\in x_N+V$. Set $s>1$ such that $x_N\in sV$, then for all $n>N$, $x_n\in sV+V\subseteq sV+sV\subseteq sW$. Since for balanced sets $sW\subset tW$ for $s<t$, and since every semi-open neighbourhood of zero contains a balanced neighbourhood, this proves that the sequence in indeed bounded.

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