Turkish Journal of Analysis and Number Theory, 2016, Vol. 4, No. 1, 23-30
Available online at http://pubs.sciepub.com/tjant/4/1/5

DOI:10.12691/tjant-4-1-5

An Extended Coupled Coincidence Point Theorem

Esra Yolacan[1], Mehmet Kir[2,*], Hukmi Kiziltunc[3]

[1]Republic of Turkey Ministry of National Education, Mathematics Teacher, 60000 Tokat, Turkey
[2]Department of Civil Engineering, Faculty of Engineering, Şırnak University, 73000, Turkey
[3]Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey
*Corresponding author: mehmetkir@sirnak.edu.tr

Received December 29, 2015; Revised February 06, 2016; Accepted February 14, 2016

Abstract In this paper, we prove some coupled coincidence point theorem for a pair $\{F,G\}$ of mappings $F,G:C^2\to C$ without mixed G-monotone property of F. Our results improve and generalize results given by Karapinar et al. (Arab J Math (2012) 1: 329-339) and Jachymski (Nonlinear Anal. 74, 768-774 (2011)). The theoretic results are also accompanied with suitable example.

Keywords: *coupled coincidence point, generalized compatibility, ordered set*

Cite This Article: Esra Yolacan, Mehmet Kir, and Hukmi Kiziltunc, "An Extended Coupled Coincidence Point Theorem." *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 1 (2016): 23-30. doi: 10.12691/tjant-4-1-5.

1. Introduction and Preliminaries

In the sequel, let C be a non-empty set. Throughout this paper, we use indifferently the notation C^2 to denote the product space $C\times C$. $\preceq$ will denote a partial order on C, and ρ will be a metric on C. Also, with respect to abbreviated as w.r.t.

Definition 1. [1] *An element $(a,b)\in C^2$ is said to be a coupled fixed point of the mapping $F:C^2\to C$ if $F(a,b)=a$ and $F(b,a)=b$.*

Definition 2. [1] *Let $(C,\preceq)$ be a partially ordered set and $F:C^2\to C$ be a mapping. We say that F has the mixed monotone property if $F(a,b)$ is monotone nondecreasing in a and is monotone non-increasing in b; that is, for any $a,b\in C$,*

$$a_1,a_2\in C, a_1\preceq a_2 \text{ implies } F(a_1,b)\preceq F(a_2,b)$$

and

$$b_1,b_2\in C, b_1\preceq b_2 \text{ implies } F(a,b_1)\succeq F(a,b_2).$$

Lakshmikantham and Ćirić [2] introduced the concept of mixed g-monotone mapping.

Definition 3. [2] *An element $(a,b)\in C^2$ is said to be a coupled coincidence point of a mapping $F:C^2\to C$ and $g:C\to C$ if $F(a,b)=ga$ and $F(b,a)=gb$.*

Definition 4. [2] Let $(C,\preceq)$ be a partially ordered set and $F:C^2\to C$ and $g:C\to C$. We say F has the mixed g-monotone property if for any $a,b\in C$,

$$a_1,a_2\in C, ga_1\preceq g a_2 \text{ implies } F(a_1,b)\preceq F(a_2,b)$$

and

$$b_1,b_2\in C, gb_1\preceq g b_2 \text{ implies } F(a,b_1)\succeq F(a,b_2).$$

Definition 5. [2] *Let C be a nonempty set and $F:C^2\to C$ and $g:C\to C$. We say F and g are commutative if $gF(a,b)=F(gx,gy)$ for all $a,b\in C$.*

Definition 6. [4] *Let (C,ρ) be a metric space, $F:C^2\to C$ be a mapping and g be a self mapping on C. A hybrid pair F, g is compatible if*

$$\lim_{n\to\infty}\rho\left(g\left(F(a_n,b_n)\right),F(ga_n,gb_n)\right)=0$$

and

$$\lim_{n\to\infty}\rho\left(g\left(F(b_n,a_n)\right),F(gb_n,ga_n)\right)=0$$

whenever (a_n) and (b_n) are sequences in C such that

$$\begin{cases}\lim_{n\to\infty}F(a_n,b_n)=\lim_{n\to\infty}ga_n=a,\\ \lim_{n\to\infty}F(b_n,a_n)=\lim_{n\to\infty}gb_n=b\end{cases}$$

with $a,b\in C$.

Denote by Φ the set of functions $\varphi:[0,\infty)\to[0,\infty)$ satisfying:

(φ1) φ is continuous,

(φ2) $\varphi(t)<t$ for all $t>0$ and $\varphi(t)=0$ if nd only if $t=0$.

Using the concept of compatibility, Karapinar et al. [5] proved the following theorem.

Theorem 1. [5] *Let $(C,\preceq)$ be a partially ordered set, and suppose there is a metric ρ on C such that (C,ρ) is a complete metric space. Let $F:C^2\to C$ and $g:C\to C$ be two mappings having the g-mixed monotone property on C such that there exists two elements $a_0,b_0\in C$ with*

$$ga_0\preceq F(a_0,b_0) \text{ and } gb_0\succeq F(b_0,a_0).$$

Suppose there exists $\varphi \in \Phi$ *and* $L \geq 0$ *such that*

$$\rho(F(a,b),F(c,d)) \leq \varphi(\max\{\rho(ga,gb),\rho(gb,ga)\}) + L\min\left\{\begin{matrix}\rho(F(a,b),gc),\rho(F(c,d),ga),\\ \rho(F(a,b),ga),\rho(F(c,d),gc)\end{matrix}\right\}$$

for all $a,b,c,d \in C$ *with* $ga \succeq gc$ *and* $gb \preceq gd$. *Suppose* $F(C^2) \subseteq g(C)$, *g is continuous and compatible with* F.

Also suppose either

(a) F is continuous or;

(b) C has the following properties:

(1) *if a non-decreasing sequence* $\{a_n\} \to a$, *then* $ga_n \preceq ga$ *for all n;*

(2) if a non-increasing sequence $\{b_n\} \to b$, *then* $gb \preceq gb_n$ *for all n:*

Then there exists $a,b \in C$ *such that* $F(a,b) = ga$ *and* $F(b,a) = gb$, *that is, F and g have a coupled coincidence point in C.*

Hussain et al. [3] introduced the concept of G-increasing mappings and concept of generalized compatibility for the pair $\{F,G\}$. Also, they introduced some coupled coincidence point results.

Definition 7. [3] *Suppose that* $F: C^2 \to C$ *are two mappings. F is said to be G-increasing w.r.t* $\preceq$ *if for all* $a,b,c,d \in C$, *with* $G(a,b) \preceq G(c,d)$ *we have* $F(a,b) \preceq F(c,d)$.

Definition 8. [3] *An element* $(a,b) \in C^2$ *is said to be a coupled coincidence point of a mappings* $F,G: C^2 \to C$ *if* $F(a,b) = G(a,b)$ *and* $F(b,a) = G(b,a)$.

Definition 9. [3] *Let* $F,G: C^2 \to C$. *We say that pair* $\{F,G\}$ *is generalized compatible if*

$$\begin{cases}\rho\left(\begin{matrix}F(G(a_n,b_n),G(b_n,a_n)),\\ G(F(a_n,b_n),F(b_n,a_n))\end{matrix}\right) \to 0 \text{ as } n \to \infty,\\ \rho\left(\begin{matrix}F(G(b_n,a_n),G(a_n,b_n)),\\ G(F(b_n,a_n),F(a_n,b_n))\end{matrix}\right) \to 0 \text{ as } n \to \infty,\end{cases}$$

whenever (a_n) *and* (b_n) *are sequences in C such that*

$$\begin{cases}\lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} G(a_n,b_n) = t_1,\\ \lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} G(b_n,a_n) = t_2.\end{cases}$$

Definition 10. [3] *Let* $F,G: C^2 \to C$ *be two maps. We say that the pair* $\{F,G\}$ *is commuting if* $F(G(a,b),G(b,a)) = G(F(a,b),F(b,a))$ *for all* $a,b \in C$.

Remark 1. [3] *A commuting pair is a generalized compatible but not conversely in general.*

In this paper, we prove the existence of a coupled coincidence point theorem for a pair $\{F,G\}$ of mapping $F,G: C^2 \to C$ with contraction (2.1) in complete metric space without mixed G-monotone property of F. Therefore provided coupled fixed point results need not the mixed monotone property of F. Our results represent new version of results given by Karapinar et al. [5] and Jachymski [6]. The theoretic results are also accompanied with suitable example.

2. Main Results

Theorem 2. *Let* $(C,\preceq)$ *be a partially ordered set, and suppose there is a metric* ρ *on C such that* (C,ρ) *is a complete metric space. Assume that* $F,G: C^2 \to C$ *are two generalized compatible mappings such that F is G-increasing w.r.t* $\preceq$, *G is continuous and has the mixed monotone property, and there exists two elements* $a_0,b_0 \in C$ *with*

$$G(a_0,b_0) \preceq F(a_0,b_0) \text{ and } G(b_0,a_0) \succeq F(b_0,a_0).$$

Suppose there exists non-negative real numbers $\varphi \in \Phi$ *and* $L \geq 0$ *such that*

$$\rho(F(a,b),F(c,d)) \leq \varphi\left(\max\left\{\begin{matrix}\rho(G(a,b),G(c,d)),\\ \rho(G(b,a),G(d,c))\end{matrix}\right\}\right) + L\min\left\{\begin{matrix}\rho(F(a,b),G(c,d)),\rho(F(c,d),G(a,b)),\\ \rho(F(a,b),G(a,b)),\rho(F(c,d),G(c,d))\end{matrix}\right\} \tag{2.1}$$

for all $a,b,c,d \in C$ *with* $G(a,b) \preceq G(c,d)$ *and* $G(b,a) \succeq G(d,c)$. *Suppose that for any* $a,b \in C$, *there exists* $c,d \in C$ *such that*

$$\begin{cases}F(a,b) = G(c,d),\\ F(b,a) = G(d,c).\end{cases} \tag{2.2}$$

Also suppose that either

(a) F is continuous or;

(b) C has the following properties:

(1) if a non-decreasing sequence $\{a_n\} \to a$, *then* $a_n \preceq a$ *for all n,*

(2) if a non-increasing sequence $\{b_n\} \to b$, *then* $b \preceq b_n$ *for all n.*

Then F and G have a coupled coincidence point in C.

Proof. Let $a_0,b_0 \in C$ be such that $G(a_0,b_0) \preceq F(a_0,b_0)$ and $G(b_0,a_0) \succeq F(b_0,a_0)$. By (2.2), there exists $a_1,b_1 \in C^2$ such that $F(a_0,b_0) = G(a_1,b_1)$ and $F(b_0,a_0) = G(b_1,a_1)$. Continuing this process, we construct sequences $\{a_n\}$ and $\{b_n\}$ in C such that

$$F(a_n,b_n)=G(a_{n+1},b_{n+1}) \text{ and } F(b_n,a_n)=G(b_{n+1},a_{n+1}) \text{ for all } n\in\mathbb{N}. \tag{2.3}$$

Since F is G-increasing w.r.t $\preceq$ and using the mathematical induction, we have

$$G(a_n,b_n)\preceq G(a_{n+1},b_{n+1}) \text{ and } G(b_{n+1},a_{n+1})\preceq G(b_n,a_n) \text{ for all } n\in\mathbb{N}. \tag{2.4}$$

Since $G(a_n,b_n)\succeq G(a_{n-1},b_{n-1})$ and $G(b_n,a_n)\preceq G(b_{n-1},a_{n-1})$, from (2.1) and (2.3), we have

$$\begin{aligned}
&\rho(G(a_{n+1},b_{n+1}),G(a_n,b_n))\\
&=\rho(F(a_n,b_n),F(a_{n-1},b_{n-1}))\\
&\le\varphi\left(\max\left\{\begin{matrix}\rho(G(a_n,b_n),G(a_{n-1},b_{n-1})),\\ \rho(G(b_n,a_n),G(b_{n-1},a_{n-1}))\end{matrix}\right\}\right)\\
&\quad+L\min\left\{\begin{matrix}\rho(F(a_n,b_n),G(a_{n-1},b_{n-1})),\\ \rho(F(a_{n-1},b_{n-1}),G(a_n,b_n)),\\ \rho(F(a_n,b_n),G(a_n,b_n)),\\ \rho(F(a_{n-1},b_{n-1}),G(a_{n-1},b_{n-1}))\end{matrix}\right\}\\
&=\varphi\left(\max\left\{\begin{matrix}\rho(G(a_n,b_n),G(a_{n-1},b_{n-1})),\\ \rho(G(b_n,a_n),G(b_{n-1},a_{n-1}))\end{matrix}\right\}\right).
\end{aligned}\tag{2.5}$$

Similarly, we have

$$\begin{aligned}
&\rho(G(b_n,a_n),G(b_{n+1},a_{n+1}))\\
&=\rho(F(b_{n-1},a_{n-1}),F(b_n,a_n))\\
&\le\varphi\left(\max\left\{\begin{matrix}\rho(G(b_{n-1},a_{n-1}),G(b_n,a_n)),\\ \rho(G(a_{n-1},b_{n-1}),G(a_n,b_n))\end{matrix}\right\}\right)\\
&\quad+L\min\left\{\begin{matrix}\rho(F(b_{n-1},a_{n-1}),G(b_n,a_n)),\\ \rho(F(b_n,a_n),G(b_{n-1},a_{n-1})),\\ \rho(F(b_{n-1},a_{n-1}),G(b_{n-1},a_{n-1})),\\ \rho(F(b_n,a_n),G(b_n,a_n))\end{matrix}\right\}\\
&=\varphi\left(\max\left\{\begin{matrix}\rho(G(b_{n-1},a_{n-1}),G(b_n,a_n)),\\ \rho(G(a_{n-1},b_{n-1}),G(a_n,b_n)),\end{matrix}\right\}\right).
\end{aligned}\tag{2.6}$$

By (2.5) and (2.6), we obtain

$$\begin{aligned}
&\max\left\{\begin{matrix}\rho(G(a_{n+1},b_{n+1}),G(a_n,b_n)),\\ \rho(G(b_n,a_n),G(b_{n+1},a_{n+1}))\end{matrix}\right\}\\
&\le\varphi\left(\max\left\{\begin{matrix}\rho(G(a_n,b_n),G(a_{n-1},b_{n-1})),\\ \rho(G(b_n,a_n),G(b_{n-1},a_{n-1}))\end{matrix}\right\}\right).
\end{aligned}\tag{2.7}$$

Owing to (φ2), by (2.7), we have

$$\begin{aligned}
&\max\left\{\begin{matrix}\rho(G(a_{n+1},b_{n+1}),G(a_n,b_n)),\\ \rho(G(b_n,a_n),G(b_{n+1},a_{n+1}))\end{matrix}\right\}\\
&\le\max\left\{\begin{matrix}\rho(G(a_n,b_n),G(a_{n-1},b_{n-1})),\\ \rho(G(b_n,a_n),G(b_{n-1},a_{n-1}))\end{matrix}\right\}.
\end{aligned}$$

Set

$$z_n=\max\left\{\begin{matrix}\rho(G(a_{n+1},b_{n+1}),G(a_n,b_n)),\\ \rho(G(b_n,a_n),G(b_{n+1},a_{n+1}))\end{matrix}\right\},$$

then sequence $\{z_n\}$ is non-increasing. Hence, there is some $z_n\ge 0$ such that $\lim_{n\to\infty}z_n=z$. We claim that $z=0$. Suppose, to the contrary, that $z>0$, then by (2.7) and using the property of φ, we have

$$z\le\varphi(z)\le z$$

which is a contradiction. Therefore $z=0$, i.e.,

$$\lim_{n\to\infty}z_n=\lim_{n\to\infty}\max\left\{\begin{matrix}\rho(G(a_{n+1},b_{n+1}),G(a_n,b_n)),\\ \rho(G(b_n,a_n),G(b_{n+1},a_{n+1}))\end{matrix}\right\}=0. \tag{2.8}$$

Now, we show that $\rho(G(a_n,b_n),G(b_n,a_n))$ is Cauchy sequence in C^2 endowed with the metric Γ defined by

$$\Gamma((a,b),(c,d))=\max\{\rho(a,c),\rho(b,d)\} \tag{2.9}$$

for all $(a,b),(c,d)\in C^2$. If $(G(a_n,b_n),G(b_n,a_n))$ is not a Cauchy sequence in (C^2,Γ). Then there exists $\varepsilon>0$ for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integer k with $n(k)>m(k)>k$, we have

$$\left\{\begin{aligned}
&\Gamma\left(\begin{matrix}(G(a_{m(k)},b_{m(k)}),G(b_{m(k)},a_{m(k)})),\\ G(a_{n(k)},b_{n(k)}),G(b_{n(k)},a_{n(k)})\end{matrix}\right)>\varepsilon,\\
&\Gamma\left(\begin{matrix}(G(a_{m(k)},b_{m(k)}),G(b_{m(k)},a_{m(k)})),\\ G(a_{n(k)-1},b_{n(k)-1}),G(b_{n(k)-1},a_{n(k)-1})\end{matrix}\right)\le\varepsilon.
\end{aligned}\right. \tag{2.10}$$

From (2.9), we get

$$\rho_k=\max\left\{\begin{matrix}\rho(G(a_{m(k)},b_{m(k)}),G(a_{n(k)},b_{n(k)})),\\ \rho(G(b_{m(k)},a_{m(k)}),G(b_{n(k)},a_{n(k)}))\end{matrix}\right\} \tag{2.11}$$
$$>\varepsilon$$

and

$$\max\left\{\begin{matrix}\rho(G(a_{m(k)},b_{m(k)}),G(a_{n(k)-1},b_{n(k)-1})),\\ \rho(G(b_{m(k)},a_{m(k)}),G(b_{n(k)-1},a_{n(k)-1}))\end{matrix}\right\} \tag{2.12}$$
$$\le\varepsilon.$$

From (2.12) and using triangle inequality, we have

$$\begin{aligned}
&\rho(G(a_{n(k)},b_{n(k)}),G(a_{m(k)},b_{m(k)}))\\
&\le\rho(G(a_{n(k)},b_{n(k)}),G(a_{n(k)-1},b_{n(k)-1}))\\
&\quad+\rho(G(a_{n(k)-1},b_{n(k)-1}),G(a_{m(k)},b_{m(k)}))\\
&\le\rho(G(a_{n(k)},b_{n(k)}),G(a_{n(k)-1},b_{n(k)-1}))+\varepsilon
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}&\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\\&\le\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\\&+\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\\&\le\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)+\varepsilon.\end{aligned}\tag{2.14}$$

From (2.11), (2.13), (2.14), we have

$$\varepsilon<\rho_k\le\max\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\end{array}\right\}+\varepsilon.\tag{2.15}$$

Letting $k\to\infty$ in (2.15) and by (2.8), we obtain

$$\lim_{n\to\infty}\rho_k=\varepsilon.\tag{2.16}$$

From triangle inequality

$$\begin{aligned}&\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{m(k)},b_{m(k)}\right)\right)\\&\le\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right)\\&+\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\\&+\rho\left(G\left(a_{m(k)-1},b_{m(k)-1}\right),G\left(a_{m(k)},b_{m(k)}\right)\right)\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\\&\le\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\\&+\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\\&+\rho\left(G\left(b_{m(k)-1},a_{m(k)-1}\right),G\left(b_{m(k)},a_{m(k)}\right)\right).\end{aligned}\tag{2.18}$$

From (2.11), (2.17) and (2.18), we have

$$\begin{aligned}\varepsilon<\rho_k&<\max\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\\rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\end{array}\right\}\\&+\max\left\{\begin{array}{l}\rho\left(G\left(a_{m(k)-1},b_{m(k)-1}\right),G\left(a_{m(k)},b_{m(k)}\right)\right),\\\rho\left(G\left(b_{m(k)-1},a_{m(k)-1}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\end{array}\right\}\\&+\max\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right),\\\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\end{array}\right\}.\end{aligned}\tag{2.19}$$

Again, from the triangle inequality,

$$\begin{aligned}&\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\\&\le\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)},b_{m(k)}\right)\right)\\&+\rho\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\\&\le\varepsilon+\rho\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\end{aligned}\tag{2.20}$$

and

$$\begin{aligned}&\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\\&\le\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\\&+\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\\&<\varepsilon+\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right).\end{aligned}\tag{2.21}$$

Thus,

$$\begin{aligned}&\max\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right),\\\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\end{array}\right\}\\&<\varepsilon+\max\left\{\begin{array}{l}\rho\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right),\\\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\end{array}\right\}.\end{aligned}\tag{2.22}$$

Letting $k\to\infty$ in (2.19) and by (2.8), (2.22), (2.16), we get

$$\max\left(\begin{array}{l}\rho\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right),\\\rho\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right)\end{array}\right)=\varepsilon\ \ as\ \ k\to\infty.\tag{2.23}$$

In view of $n(k)>m(k)$,

$$G\left(a_{n(k)-1},b_{n(k)-1}\right)\succeq G\left(a_{m(k)-1},b_{m(k)-1}\right)$$

and $G\left(b_{n(k)-1},a_{n(k)-1}\right)\preceq G\left(b_{m(k)-1},a_{m(k)-1}\right)$, from (2.1) and (2.3), we have

$$\begin{aligned}&\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{m(k)},b_{m(k)}\right)\right)\\&=\rho\left(F\left(a_{n(k)-1},b_{n(k)-1}\right),F\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\\&\le\varphi\left(\max\left\{\begin{array}{l}\rho\left(\begin{array}{l}G\left(a_{n(k)-1},b_{n(k)-1}\right),\\G\left(a_{m(k)-1},b_{m(k)-1}\right)\end{array}\right),\\\rho\left(\begin{array}{l}G\left(b_{n(k)-1},a_{n(k)-1}\right),\\G\left(b_{m(k)-1},a_{m(k)-1}\right)\end{array}\right)\end{array}\right\}\right)\end{aligned}$$

$$+L\min\left\{\begin{array}{l}\rho\left(F\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right),\\ \rho\left(F\left(a_{m(k)-1},b_{m(k)-1}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\ \rho\left(F\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\ \rho\left(F\left(a_{m(k)-1},b_{m(k)-1}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\end{array}\right\}$$

$$\leq\varphi\left(\max\left\{\begin{array}{l}\rho\left(\begin{array}{l}G\left(a_{n(k)-1},b_{n(k)-1}\right),\\ G\left(a_{m(k)-1},b_{m(k)-1}\right)\end{array}\right),\\ \rho\left(\begin{array}{l}G\left(b_{n(k)-1},a_{n(k)-1}\right),\\ G\left(b_{m(k)-1},a_{m(k)-1}\right)\end{array}\right)\end{array}\right\}\right) \quad (2.24)$$

$$+L\min\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\ \rho\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\end{array}\right\}.$$

Similarly, we have

$$\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)$$

$$\leq\varphi\left(\max\left\{\begin{array}{l}\rho\left(\begin{array}{l}G\left(a_{n(k)-1},b_{n(k)-1}\right),\\ G\left(a_{n(k)-1},b_{n(k)-1}\right)\end{array}\right),\\ \rho\left(\begin{array}{l}G\left(b_{n(k)-1},a_{n(k)-1}\right),\\ G\left(b_{m(k)-1},a_{m(k)-1}\right)\end{array}\right)\end{array}\right\}\right) \quad (2.25)$$

$$+L\min\left\{\begin{array}{l}\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right),\\ \rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\end{array}\right\}.$$

Using (2.24) and (2.25), we get

$$\max\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{m(k)},b_{m(k)}\right)\right),\\ \rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{m(k)},a_{m(k)}\right)\right)\end{array}\right\}$$

$$\leq\varphi\left(\max\left\{\begin{array}{l}\rho\left(\begin{array}{l}G\left(a_{n(k)-1},b_{n(k)-1}\right),\\ G\left(a_{m(k)-1},b_{m(k)-1}\right)\end{array}\right),\\ \rho\left(\begin{array}{l}G\left(b_{n(k)-1},a_{n(k)-1}\right),\\ G\left(b_{m(k)-1},a_{m(k)-1}\right)\end{array}\right)\end{array}\right\}\right) \quad (2.26)$$

$$+L\min\left\{\begin{array}{l}\rho\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right),\\ \rho\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)-1},b_{m(k)-1}\right)\right)\end{array}\right\}$$

$$+L\min\left\{\begin{array}{l}\rho\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)-1},a_{m(k)-1}\right)\right),\\ \rho\left(G\left(b_{n(k)},a_{n(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\end{array}\right\}.$$

Taking the limit as $k\to\infty$ in (2.26), and from (2.8), (2.16), (2.23) and (φ 2), we obtain

$$\varepsilon\leq\varphi(\varepsilon)+L\min\{0,0\}+L\min\{0,0\}<\varepsilon$$

which is a contradiction. Therefore, $\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)$ is Cauchy sequence in $\left(C^2,\Gamma\right)$ which implies that $G\left(a_n,b_n\right)$ and $G\left(b_n,a_n\right)$ are Cauchy sequence in (C,ρ). Since C is a complete metric space, there exists $a,b\in C$ such that

$$\begin{array}{l}\lim_{n\to\infty}G\left(a_n,b_n\right)=\lim_{n\to\infty}F\left(a_n,b_n\right)=a,\\ \lim_{n\to\infty}G\left(b_n,a_n\right)=\lim_{n\to\infty}F\left(b_n,a_n\right)=b.\end{array} \quad (2.27)$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility, by (2.27), we have

$$\begin{array}{l}\rho\left(\begin{array}{l}F\left(a_n,b_n\right),G\left(b_n,a_n\right),\\ G\left(F\left(a_n,b_n\right),F\left(b_n,a_n\right)\right)\end{array}\right)\to 0\ as\ n\to\infty\\ \rho\left(\begin{array}{l}F\left(b_n,a_n\right),G\left(a_n,b_n\right),\\ G\left(F\left(b_n,a_n\right),F\left(a_n,b_n\right)\right)\end{array}\right)\to 0\ as\ n\to\infty.\end{array} \quad (2.28)$$

Suppose the assumption (a) holds. For all $n\in\mathbb{N}$, we get

$$\begin{array}{l}\rho\left(G(a,b),F\left(G\left(a_n,b_n\right),\left(b_n,a_n\right)\right)\right)\\ \leq\rho\left(G(a,b),G\left(F\left(a_n,b_n\right),\left(b_n,a_n\right)\right)\right)\\ +\rho\left(\begin{array}{l}G\left(F\left(a_n,b_n\right),\left(b_n,a_n\right)\right),\\ F\left(G\left(a_n,b_n\right),\left(b_n,a_n\right)\right)\end{array}\right).\end{array}$$

Taking the limit as $n\to\infty$ in (2.27), by (2.28), and since F and G are continuous, we have

$$G(a,b)=F(a,b). \quad (2.29)$$

Similarly, we show that

$$G(b,a)=F(b,a). \quad (2.30)$$

Hence (a,b) is a coupled coincidence point of F and G.

Next, suppose the assumption (b) holds. From (2.4) and (2.27), we obtain $\left(G\left(a_n,b_n\right)\right)$ is non-decreasing sequence, $G\left(a_n,b_n\right)\to a$ as $n\to\infty$ and $\left(G\left(b_n,a_n\right)\right)$ is non-increasing sequence, $G\left(b_n,a_n\right)\to b$ as $n\to\infty$. Therefore, we get

$$G\left(a_n,b_n\right)\preceq a\ and\ G\left(b_n,a_n\right)\succeq b. \quad (2.31)$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility and G is continuous, from (2.28), we obtain

$$\begin{array}{l}\lim_{n\to\infty}G\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)\\ =G(a,b)\\ =\lim_{n\to\infty}G\left(F\left(a_n,b_n\right),F\left(b_n,a_n\right)\right)\\ =\lim_{n\to\infty}F\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)\end{array} \quad (2.32)$$

and

$$\begin{aligned}&\lim_{n\to\infty} G\left(G\left(b_n,a_n\right),G\left(a_n,b_n\right)\right)=G\left(b,a\right)\\&=\lim_{n\to\infty} G\left(F\left(b_n,a_n\right),F\left(a_n,b_n\right)\right)\\&=\lim_{n\to\infty} F\left(G\left(b_n,a_n\right),G\left(a_n,b_n\right)\right).\end{aligned} \tag{2.33}$$

Next, we have

$$\begin{aligned}&\rho\left(G\left(a,b\right),F\left(a,b\right)\right)\\&\leq \lim_{n\to\infty} \rho\left(G\left(F\left(a_n,b_n\right),F\left(b_n,a_n\right)\right),F\left(a,b\right)\right)\\&= \lim_{n\to\infty} \rho\left(F\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right),F\left(a,b\right)\right).\end{aligned}$$

Since G has the mixed monotone property, it follows from (2.31) that

$$G\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)\preceq G\left(a,b\right)$$

and $G\left(G\left(b_n,a_n\right),G\left(a_n,b_n\right)\right)\succeq G\left(b,a\right)$. From (2.1), (2.32) and (2.33), we obtain

$$\begin{aligned}&\rho\left(G\left(a,b\right),F\left(a,b\right)\right)\\&\leq\varphi\left(\max\left\{\begin{array}{l}\rho\left(G\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right),G\left(a,b\right)\right)\right),\\ \rho\left(G\left(G\left(b_n,a_n\right),G\left(a_n,b_n\right),G\left(b,a\right)\right)\right)\end{array}\right\}\right)\\&+L\min\left\{\begin{array}{c}\rho\left(G\left(a,b\right),G\left(a,b\right)\right),\\ \rho\left(F\left(a,b\right),G\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)\right),\\ \rho\left(G\left(a,b\right),G\left(G\left(a_n,b_n\right),G\left(b_n,a_n\right)\right)\right),\\ \rho\left(F\left(a,b\right),G\left(a,b\right)\right)\end{array}\right\}.\end{aligned}$$

Then we get $G\left(a,b\right)=F\left(a,b\right)$. Similarly, $G\left(b,a\right)=F\left(b,a\right)$.

By Remark 1, we have the following Corrollary.

Corollary 1. Under the assumption of Theorem 2, suppose that $F,G:C^2\to C$ are two commuting mappings such that F is G-increasing w.r.t $\preceq$, G is continuous and has the mixed monotone property, then F and G have a coupled coincidence point in C.

Definition 11. [3] *Let* $(C,\preceq)$ *be a partially ordered set and* $F:C^2\to C$ *and* $g:C\to C$. *We say* F *is g-increasing w.r.t* $\preceq$ *if for any* $a,b\in C$,

$$a_1,a_2\in C,\ ga_1\preceq ga_2 \text{ implies } F\left(a_1,b\right)\preceq F\left(a_2,b\right)$$

and

$$b_1,b_2\in C,\ gb_1\preceq gb_2 \text{ implies } F\left(a,b_1\right)\preceq F\left(a,b_2\right).$$

The consequence of the main results of Karapinar et al. [5] (Theorem 1) without g-mixed monotone property of F is given in the following corollary.

Corollary 2. *Let* $F:C^2\to C$ *and* $g:C\to C$ *be two mappings such that* F *is g-increasing w.r.t* $\preceq$. *Under the assumption of Theorem* 1, *suppose that the pair* $\{F,g\}$ *is compatible, then* F *and* g *have a coupled coincidence point in C.*

Corollary 3. *Let* $F:C^2\to C$ *and* $g:C\to C$ *be two mappings such that* F *is g-increasing w.r.t* $\preceq$. *Under the assumption of Theorem* 1, *suppose that the pair* $\{F,g\}$ *is compatible, then* F *and* g *have a coupled coincidence point in C.*

Corollary 4. *Taking* L = 0 *in (2.1), then Corrollary 2 and 3 provides the conclusion of the main results of Jachymski* [6].

Now, we shall prove the uniqueness of coupled fixed point. Note that if $(C,\preceq)$ is a partially ordered set, then we endow the product $C\times C$ with the following partial order relation:

$$\begin{aligned}&\left(a,b\right)\preceq\left(c,d\right)\Leftrightarrow G\left(a,b\right)\preceq G\left(c,d\right)\\&\text{and } G\left(b,a\right)\succeq G\left(d,c\right),\end{aligned}$$

where $G:C\times C\to C\times C$ is one-one.

Theorem 3. *In addition to the hypotheses of Theorem 2, suppose that for every* $\left(a,b\right),\left(e,f\right)\in C^2$, *there exists another* $\left(c,d\right)\in C^2$ *which is comparable to* $\left(a,b\right)$ *and* $\left(e,f\right)$. *Then F and G have a unique coupled coincidence point.*

Proof. Owing to Theorem 2, the set of coupled coincidence points of F and G is nonempty. Suppose $\left(a,b\right)$ and $\left(e,f\right)$ are coupled coincidence points of F and G, that is,

$$\begin{cases}F\left(a,b\right)=G\left(a,b\right),\\ F\left(b,a\right)=G\left(b,a\right)\end{cases}$$

and

$$\begin{cases}F\left(e,f\right)=G\left(e,f\right),\\ F\left(f,e\right)=G\left(f,e\right).\end{cases}$$

By assumption, there exists $\left(c,d\right)\in C^2$ such that $\left(c,d\right)$ is comparable to $\left(a,b\right)$ and $\left(e,f\right)$. We define sequences $\{c_n\},\{d_n\}$ as follows

$$\begin{aligned}&c_0=c,d_0=d,F\left(c_n,d_n\right)=G\left(c_{n+1},d_{n+1}\right)\\&\text{and } F\left(d_n,c_n\right)=G\left(d_{n+1},c_{n+1}\right) \text{ for all } n\in\mathbb{N}.\end{aligned}$$

Since $\left(c,d\right)$ is comparable with $\left(a,b\right)$, we assume that $\left(a,b\right)\preceq\left(c,d\right)=\left(c_0,d_0\right)$ which implies $G\left(a,b\right)\preceq G\left(c_0,d_0\right)$ and $G\left(b,a\right)\succeq G\left(d_0,c_0\right)$. We assume that $\left(a,b\right)\preceq\left(c_n,d_n\right)$ for some $n\in\mathbb{N}$. We derive that

$$\left(a,b\right)\preceq\left(c_{n+1},d_{n+1}\right) \text{ for all } n\in\mathbb{N}.$$

Since F is G increasing, we have $G\left(a,b\right)\preceq G\left(c_n,d_n\right)$ implies $F\left(a,b\right)\preceq F\left(c_n,d_n\right)$ and $G\left(b,a\right)\succeq G\left(d_n,c_n\right)$ implies $F\left(b,a\right)\succeq F\left(d_n,c_n\right)$. Then, we have

$$G\left(a,b\right)=F\left(a,b\right)\preceq F\left(c_n,d_n\right)=G\left(c_{n+1},d_{n+1}\right)$$

and

$$G\left(b,a\right)=F\left(b,a\right)\succeq F\left(d_n,c_n\right)=G\left(d_{n+1},c_{n+1}\right).$$

Hence we obtain

$$(a,b)\preceq(c_n,d_n) \text{ for all } n\in\mathbb{N}. \tag{2.34}$$

By (2.1) and (2.34), we have

$$\begin{aligned}
&\rho\left(G(a,b),G(c_{n+1},d_{n+1})\right)\\
&=\rho\left(F(a,b),F(c_n,d_n)\right)\\
&\le\varphi\left(\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix}\right)\\
&+L\min\begin{Bmatrix}\rho\left(F(a,b),G(c_n,d_n)\right),\\ \rho\left(F(c_n,d_n),G(a,b)\right),\\ \rho\left(F(a,b),G(a,b)\right),\\ \rho\left(F(c_n,d_n),G(c_n,d_n)\right)\end{Bmatrix}\\
&=\varphi\left(\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix}\right).
\end{aligned}\tag{2.35}$$

Similarly, we have

$$\begin{aligned}
&\rho\left(G(d_{n+1},c_{n+1}),G(b,a)\right)\\
&=\rho\left(F(d_n,c_n),F(b,a)\right)\\
&\le\varphi\left(\max\begin{Bmatrix}\rho\left(G(d_n,c_n),G(b,a)\right),\\ \rho\left(G(c_n,d_n),G(a,b)\right)\end{Bmatrix}\right)\\
&+L\min\begin{Bmatrix}\rho\left(F(d_n,c_n),G(b,a)\right),\\ \rho\left(F(b,a),G(d_n,c_n)\right),\\ \rho\left(F(d_n,c_n),G(d_n,c_n)\right),\\ \rho\left(F(b,a),G(b,a)\right)\end{Bmatrix}\\
&=\varphi\left(\max\begin{Bmatrix}\rho\left(G(d_n,c_n),G(b,a)\right),\\ \rho\left(G(c_n,d_n),G(a,b)\right)\end{Bmatrix}\right).
\end{aligned}\tag{2.36}$$

From (2.35) and (2.36), we get

$$\begin{aligned}
&\max\begin{Bmatrix}\rho\left(G(a,b),G(c_{n+1},d_{n+1})\right),\\ \rho\left(G(d_{n+1},c_{n+1}),G(b,a)\right)\end{Bmatrix}\\
&\le\varphi\left(\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix}\right).
\end{aligned}\tag{2.37}$$

Owing to (φ2), by (2.37), we have

$$\begin{aligned}
&\max\begin{Bmatrix}\rho\left(G(a,b),G(c_{n+1},d_{n+1})\right),\\ \rho\left(G(d_{n+1},c_{n+1}),G(b,a)\right)\end{Bmatrix}\\
&\le\left(\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix}\right).
\end{aligned}$$

Set

$$\alpha_n=\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix},$$

then sequence $\{\alpha_n\}$ is decreasing. Hence, there is some $\alpha_n\ge 0$ such that $\lim_{n\to\infty}\alpha_n=\alpha$.

We claim that $\alpha=0$. Suppose, to the contrary, that $\alpha>0$. Taking the limit as $n\to\infty$ in (2.37) and using the property of φ, we have

$$\alpha\le\varphi(\alpha)<\alpha$$

which is a contradiction. Therefore $\alpha=0$, i.e.,

$$\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\max\begin{Bmatrix}\rho\left(G(a,b),G(c_n,d_n)\right),\\ \rho\left(G(b,a),G(d_n,c_n)\right)\end{Bmatrix}=0.$$

This implies that

$$\begin{aligned}
&\lim_{n\to\infty}\rho\left(G(a,b),G(c_n,d_n)\right)\\
&=\lim_{n\to\infty}\rho\left(G(b,a),G(d_n,c_n)\right)=0.
\end{aligned}\tag{2.38}$$

Similarly, we get

$$\begin{aligned}
&\lim_{n\to\infty}\rho\left(G(e,f),G(c_n,d_n)\right)\\
&=\lim_{n\to\infty}\rho\left(G(f,e),G(d_n,c_n)\right)=0.
\end{aligned}\tag{2.39}$$

From (2.38) and (2.39), we have $G(a,b)=G(e,f)$ and $G(b,a)=G(f,e)$.

Next, we discuss an example to support Theorem 2.

Example 1. *Let $C=[1,3)$ with the usual metric $\rho(a,b)=|a-b|$, for all $a,b\in C$. We consider the following order relation on C*

$$\begin{aligned}
&a,b\in C, a\preceq b\Leftrightarrow a=b\\
&\text{or } (a,b)\in\{(1,1),(1,2),(2,2)\}.
\end{aligned}$$

Let $F,G:C^2\to C$ be defined by

$$F(a,b)=\begin{cases}\dfrac{a^2-b^2}{4} & \text{if } a\ge b,\\ 0 & \text{if } a<b\end{cases}$$

and

$$G(a,b)=\begin{cases}a^2-b^2 & \text{if } a\ge b,\\ 0 & \text{if } a<b.\end{cases}$$

Clearly, G is continuous and has the mixed monotone property. Moreover, F is G-increasing.

Now, we prove that for any $a,b\in C$, there exists $c,d\in C$ such that $F(a,b)=G(c,d)$ and $F(b,a)=G(d,c)$. It is easy to see the following cases.

Case 1: If $a=b$, then we have $F(a,b)=0=G(a,b)$ and $F(b,a)=0=G(b,a)$.

Case 2: If $a>b$, then we have $F(a,b)=\dfrac{a^2-b^2}{4}=G\left(\dfrac{a}{2},\dfrac{b}{2}\right)$ and $F(b,a)=0=G\left(\dfrac{b}{2},\dfrac{a}{2}\right)$.

Case 3: If $a<b$, then we have $F(a,b)=0=G\left(\dfrac{a}{2},\dfrac{b}{2}\right)$ and $F(b,a)=\dfrac{b^2-a^2}{4}=G\left(\dfrac{b}{2},\dfrac{a}{2}\right)$.

Now, we show that the pair $\{F,G\}$ *satisfies the generalized compatibility hypothesis. Let* (a_n) *and* (b_n) *be two sequences in C such that*

$$\begin{cases} \lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} G(a_n,b_n) = t_1, \\ \lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} G(b_n,a_n) = t_2. \end{cases}$$

Then we must have $t_1 = 0 = t_2$ *and one can easly prove that*

$$\begin{cases} \rho\begin{pmatrix} F(G(a_n,b_n),G(b_n,a_n)), \\ G(F(a_n,b_n),F(b_n,a_n)) \end{pmatrix} \to 0 \text{ as } n\to\infty, \\ \rho\begin{pmatrix} F(G(b_n,a_n),G(a_n,b_n)), \\ G(F(b_n,a_n),F(a_n,b_n)) \end{pmatrix} \to 0 \text{ as } n\to\infty. \end{cases}$$

Let $\varphi:[0,\infty)\to[0,\infty)$ *be defined by* $\varphi(t)=\frac{t}{2}$ *for all* $t\geq 0$. *Now, we verify the contraction (2.1) for all* $a,b,c,d\in C$, *with* $G(a,b)\preceq G(c,d)$ *and* $G(d,c)\preceq G(b,a)$. *We have the following cases.*

Case 1: $(a,b)=(c,d)$ *or* $(a,b)=(1,1)$, $(c,d)=(2,2)$, *we have* $\rho(F(a,b),F(c,d))=0$ *Thus, (2.1) holds.*

Case 2: $(a,b)=(1,1)$, $(c,d)=(1,2)$, *we have*

$$\rho(F(1,1),F(1,2))=\frac{3}{4}$$
$$<\frac{3}{2}=\varphi(\max\{\rho(0,3),\rho(0,3)\})+L.0.$$

Thus, (2.1) holds.

Case 3: $(a,b)=(1,2)$, $(c,d)=(2,2)$, *we have*

$$\rho(F(1,2),F(2,2))=\frac{3}{4}$$
$$<\frac{3}{2}=\varphi(\max\{\rho(0,3),\rho(0,3)\})+L.0.$$

Thus, (2.1) holds.

Therefore, all the conditions of Theorem 2 are satisfied and $(0,0)$ *is a coupled coincidence point of F and G.*

Acknowledgement

The authors would like to thank the editor and referees for their valuable comments and suggestions.

References

[1] Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 65, 1379-1393 (2006).

[2] Lakshmikantham, V, Ćirić, LB: Coupled .xed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. 70, 4341-4349 (2009).

[3] Hussain et al.: Coupled coincidence point results for a generalized compatible pair with applications. Fixed Point Theory and Applications 2014, 2014: 62.

[4] Choudhury, B, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. 73, 2524-2531 (2010).

[5] Karapinar, E, Luong, NV, Thuan, NX: Coupled coincidence point for mixed monotone operators in partially ordered metric spaces. Arab J Math (2012) 1: 329-339.

[6] Jachymski, J: Equivalent conditions for generalized contractions on (ordered) metric spaces. Nonlinear Anal. 74, 768-774 (2011).