Turkish Journal of Analysis and Number Theory, 2016, Vol. 4, No. 1, 20-22
Available online at http://pubs.sciepub.com/tjant/4/1/4
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DOI:10.12691/tjant-4-1-4

Some Identities of Tribonacci Polynomials

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Received July 31, 2015; Revised February 03, 2016; Accepted February 12, 2016

Abstract The Tribonacci polynomial is famous for possessing wonderful and amazing properties. Tribonacci polynomials $t_n(x)$ defined *by the recurrence relation* $t_{n+3}(x)=x^2t_{n+2}(x)+xt_{n+1}(x)+t_n(x)$ *for* $n\ge o$ with $t_o(x)=o$, $t_1(x)=1$, $t_2(x)=x^2$. In this paper, we introduce some identities Tribonacci polynomials by standard techniques.

Keywords: *fibonacci polynomials, tribonacci polynomials, generating function of tribonacci polynomials*

Cite This Article: Yogesh Kumar Gupta, V. H. Badshah, Mamta Singh, and Kiran Sisodiya, "Some Identities of Tribonacci Polynomials." *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 1 (2016): 20-22. doi: 10.12691/tjant-4-1-4.

1. Introduction

Mathematics can be considered as the underlying order of the universe, and the Fibonacci numbers is one of the most fascinating discovery made in the mathematical world. Among numerical sequences, the Fibonacci sequence has achieved a kind of celebrity status and has been studied extensively in number theory, applied mathematics, physics, computer science, and biology [2]. The Fibonacci numbers are famous for possessing wonderful and amazing properties. A similar interpretation also exists for Lucas sequence. The Fibonacci numbers have been studied both for their applications and the mathematical beauty of rich and interesting identities that they satisfy.

The Fibonacci sequence $\{F_n\}$ of number F_n is defined by the recurrence relation

$$F_n=F_{n-1}+F_{n-2} \text{ for } n\ge o,$$
$$F_o(x)=o,\ F_1(x)=1 \tag{1.1}$$

Binet Formula for Fibonacci number is defined by

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{1.2}$$

In 1883, Catalan, E [2] was considered related set of polynomials which satisfies the recurrence relations.

$$y_{k+2}(x)=xy_{k+1}(x)+y_k(x)$$

The name Fibonacci polynomials is also given to the solution of the relation

$$Z_{k+2}(x)=Z_{k+1}(x)+xZ_k(x). \tag{1.3}$$

With $Z_o(x)=o,\ Z_1(x)=1$, investigated by Jacasthal, E [9]. Byrd, P.F [3] defined the Fibonacci Polynomials $\varphi_k(x)$ by the recurrence relation.

$$\varphi_{k+2}(x)=2x\varphi_{k+1}(x)+\varphi_k(x) \tag{1.4}$$

with initial conditions $\varphi_0(x)=o,\ \varphi_1(x)=1$

Swamy, M.N.S [15] and Hoggatt, V.E. [9] almost simultaneously defined the Fibonacci polynomials $\{f_n(x)\}$ by

$$f_{n+1}(x)=xf_n(x)+f_{n-1}(x) \tag{1.5}$$

with $f_o(x)=o,\quad f_1(x)=1$.

Here if we put x=1 in $f_n(x)$ we get $F_n(1)$ which if Fibonacci sequence.

Generating function of Fibonacci Polynomials is defined by

$$\sum_{n=o}^{\infty} f_n(x)\,t^n=\frac{t}{\left(1-xt-t^2\right)}.$$

Hyper geometric form of generating function of Fibonacci polynomials is

$$\sum_{n=o}^{\infty} f_n(x)\frac{t^n}{n!}=te^{xt}\,2F_1\left(n+1,1;1;t^2\right).$$

Tribonacci number T_n [10] defined by

$$\begin{aligned}&t_{n+3}=t_{n+2}+t_{n+1}+t_n\\&\text{with?}T_o=o,\ T_1=1 \text{ and } T_2=1.\end{aligned} \tag{1.6}$$

The Tribonacci numbers are (0, 1, 1, 2, 4, 7, 13, 24…).

The Tribonacci polynomial $t_n(x)$ [17] satisfies the following recurrence relation:

$$t_{n+3}(x)=x^2t_{n+2}(x)+xt_{n+1}(x)+t_n(x) \text{ for } n\ge o$$

and $t_o(x)=o$, $t_1(x)=1$ $t_2(x)=x^2$.

In this chapter we present some identities of Tribonacci Polynomials by standard methods.

2. Tribonacci Polynomials

The Tribonacci polynomial $t_n(x)$ [17] satisfies the following recurrence relation:

$$t_{n+3}(x) = x^2 t_{n+2}(x) + x t_{n+1}(x) + t_n(x) \ \ for\ n \ge o$$

and $t_o(x)=o$, $t_1(x)=1$, $t_2(x)=x^2$

The first few Tribonacci polynomials are as follows.

$t_1(x) = 1$
$t_2(x) = x^2$
$t_3(x) = x^4 + x$
$t_4(x) = x^6 + 2x^3 + 1$
$t_5(x) = x^8 + 3x^5 + 3x^2$
$t_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$
$t_7(x) = x^{12} + 5x^9 + 10x^4 + 7x^3 + 1$
$t_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2$…and so on.

Obviously, T_n (1) is Just the classical Tribonacci number T_n several basic properties its simple form the Tribonacci polynomials have many interesting properties.

The Generating function of Tribonacci polynomials is defined by

$$\sum_{n=0}^{\infty} t_n(x).t^n = \frac{t}{1-x^2t-xt^2-t^3}. \qquad (2.1)$$

Now we will define the Hypergeometric form of generating function of Tribonacci polynomials. In this chapter we shall use following results.

(1) $(1-x)^{-1} = \sum_{n=o}^{\infty} x^n$

(2) $(x+y)^n = \sum_{m=o}^{n} \binom{n}{m} x^{n-m} y^m$

(3) $(\alpha)_n = \frac{\Gamma \alpha + n}{\Gamma \alpha}$

(4) $e^x = \sum_{n=o}^{\infty} \frac{x^n}{n!}.$

3. Some Identities of Tribonacci Polynomials

Now we state and prove some Identities of Tribonacci polynomials theorem.

Theorem (3.1). $\sum_{n=o}^{\infty} t_n(x)\frac{t^n}{n!} = e^{\,xt(x+t)} t 2F_1\left(n+1,1;1;t^3\right)$

Proof. We know that the generating function of Tribonacci polynomials is

$$\sum_{n=o}^{\infty} t_n(x).t^n = \frac{t}{1-x^2t-xt^2-t^3} = t\left(1-x^2t-xt^2-t^3\right)^{-1}$$

$$= t\left[1-t\left(x^2+xt+t^2\right)\right]^{-1} = t\sum_{n=o}^{\infty} t^n\left(x^2+xt+t^2\right)^n$$

$$= t\sum_{n=o}^{\infty}\sum_{m-o}^{n} \frac{n!}{m!\,n-m!}\left(x^2+xt\right)^{n-m} t^{2m+n}$$

$$= t\sum_{n=o}^{\infty} t^n \sum_{m=o}^{n} \frac{n!}{m!\ n-m!}\left(x^2+xt\right)^{n-m} t^{2m}$$

$$n \to n+m$$

$$= t\sum_{n=o}^{\infty}\sum_{m=o}^{\infty} \frac{n+m!}{m!\ n!}\left(x^2+xt\right)^n t^{3m+n}$$

$$= t\sum_{n=o}^{\infty} \frac{(xt)^n}{n!}(x+t)^n \sum_{m=o}^{\infty} \frac{n+m!}{m!\,n!} t^{3m} \quad n!$$

$$= t\sum_{n=o}^{\infty} \frac{\left(x^2t+xt^2\right)^n}{n!} \sum_{m=o}^{\infty} \frac{(n+1)_m (1)_m}{(1)_m\ m!} t^{3m} \quad n!$$

$$= te^{x^2t+xt^2} \quad n!\sum_{m=o}^{\infty} \frac{(n+1)_m (1)_m}{(1)_m} \frac{t^{3m}}{m!}$$

$$= te^{xt(x+t)} \quad n!\sum_{m=o}^{\infty} \frac{(n+1)_m (1)_m}{(1)_m} \frac{t^{3m}}{m!}$$

$$\sum_{n=o}^{\infty} t_n(x)\frac{t^n}{n!} = e^{xt(x+t)}\, t\sum_{m=o}^{\infty} \frac{(n+1)_m (1)_m}{(1)_m} \frac{t^{3m}}{m!}$$

$$\sum_{n=o}^{\infty} t_n(x)\frac{t^n}{n!} = e^{xt(x+t)} t\, 2F_1\left(n+1,1;1;t^3\right)$$

Theorem (3.2). Recurrence relations by the generating function of Tribonacci polynomials we can easily get following recurrence relations.

$$t'_{n+1}(x) - x^2 t'_n(x) - x\,t'_{n-1}(x) - t'_{n-2}(x)$$
$$= 2xt_n(x) + t_{n-1}(x)$$

and

$$t'_n(x) - x^2 t'_{n-1}(x) - x\,t'_{n-2}(x) - t'_{n-3}(x)$$
$$= 2xt_{n-1}(x) + t_{n-2}(x).$$

Proof. The generating function of Tribonacci polynomials is

$$\sum_{n=o}^{\infty} t_n(x)t^n = \frac{t}{1-x^2t-xt^2-t^3}.$$

Differentiating (2.1) w.r. to $'x'$ we get

$$\sum_{n=o}^{\infty} t'_n(x)t^n = -t\left(1-x^2t-xt^2-t^3\right)^{-2}\left(-2xt-t^2\right)$$

$$= \frac{t\left(2xt+t^2\right)}{\left(1-x^2t-xt^2-t^3\right)^2} = \frac{t^2(2x+t)}{\left(1-x^2t-xt^2-t^3\right)^2}$$

$$= (2x+t)\frac{t}{\left(1-x^2t-xt^2-t^3\right)}.\frac{t}{\left(1-x^2t-xt^2-t^3\right)}$$

$$\left(1-x^2t-xt^2-t^3\right)\sum_{n=o}^{\infty} t'_n(x)t^n = t(2x+t)\sum_{n=o}^{\infty} t_n(x)t^n$$

$$= (2x+t)\sum_{n=o}^{\infty} T_n(x)t^{n+1}$$

$$\sum_{n=o}^{\infty} t'_n(x).t^n - x^2\sum_{n=o}^{\infty} t'_n(x)t^{n+1} - x\sum_{n=o}^{\infty} t'_n(x).t^{n+2}$$

$$-\sum_{n=o}^{\infty} t'_n(x)t^{n+3} = 2x\sum_{n=o}^{\infty} t_n(x)t^{n+1} + \sum_{n=o}^{\infty} t_n(x)t^{n+2}.$$

Equating the coefficient of t^{n+1} and t^n, we get

$$t'_{n+1}(x)-x^2t'_n(x)-xt'_{n-1}(x)-t'_{n-2}(x)$$
$$=2xt_n(x)+t_{n-1}(x)$$

and

$$t'_n(x)-x^2t'_{n-1}(x)-xt'_{n-2}(x)-t'_{n-3}(x)$$
$$=2xt_{n-1}(x)+t_{n-2}(x).$$

Theorem (3.3). Prove that $t_{3n}(o)=o$ and $t_{3n+1}(o)=1$.

Proof.

The Generating function of Tribonacci Polynomials is defined by

$$\sum_{n=o}^{\infty} t_n(x).t^n=\frac{t}{1-x^2t-xt^2-t^3}.$$

Put x=0 in eq^n (2.1) we get

$$\sum_{n=o}^{\infty} t_n(o)t^n=\frac{t}{1-t^3}=t\left(1-t^3\right)^{-1}=t\sum_{n=o}^{\infty} t^{3n}$$
$$\sum_{n=o}^{\infty} t_n(o)t^n=\sum_{n=o}^{\infty} t^{3n+1}.$$

Equating the coefficients of 't^{3n}' and 't^{3n+1}' on both sides respectively, we get

$$t_{3n}(o)=o$$

and $t_{3n+1}(o)=1$.

Theorem (3.4).Prove that

$$t'_{3n}(o)=0 \text{ and } t'_{3n+3}(o)=\frac{(2)n}{n}.$$

Proof . Differentiating equation (2.1) w.r.t ‘x', we get

$$\sum_{n=o}^{\infty} t'_n(x)t^n=(-1)t\left(1-x^2t-xt^2-t^3\right)^{-2}\left(-2xt-t^2\right)$$

$$\sum_{n=o}^{\infty} t'_n(x)t^n=\left(2xt^2+t^3\right)\left(1-x^2t-xt^2-t^3\right)^{-2}.$$

Put x=o, we get

$$\sum_{n=o}^{\infty} t'_n(o)t^n=t^3\left(1-t^3\right)^{-2}=t^3\sum_{n=o}^{\infty}(2)n\frac{t^{3n}}{n!}$$

$$\sum_{n=o}^{\infty} t'_n(o)t^n=\sum_{n=o}^{\infty}\frac{(2)_n}{n!}t^{3n+3}$$

on both side respectively, we get

$$t'_{3n}(o)=o \ \ and\ t'_{3n+3}(o)=\frac{(2)_n}{n!}.$$

4. Conclusion

In this paper, we introduce some identities Tribonacci polynomials by standard techniques. Some basic identities are obtained by method of generating function. Also some identities are obtained in hyper geometric form.

Acknowledgement

The authors would like to thanks the anonymous referee for carefully reading the paper and for their comments.

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