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Schur-Convexity for a Class of Symmetric Functions

Shu-Hong wang*, Shu-Ping Bai

College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China
*Corresponding author: shuhong7682@163.com

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Abstract In this paper, we discuss Schur convexity for a class of symmetric functions.

Keywords: *Symmetric function, Schur- convex function, Schur –concave function, convex function, continuous function, majorized*

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1. Introduction

Throughout this paper, let I be a non-empty open interval in $\mathbb{R}$. A class of functions $S: I^2 \to \mathbb{R}$ are defined as

$$S(x,y) = \begin{cases} \lambda[f(x)+f(y)]+(1-2\lambda)f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases} \quad (1.1)$$

where $(x,y) \in I^2, \lambda \geq 0$.

If $\lambda = 0$ or $\lambda = 1/2$, in [1] chu ed al. proved

Theorem A([1, Theorem 1.1]). Suppose that I is an open interval and $f: I \to \mathbb{R}$ is a continuous function. The function

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right), & x,y \in I, x \neq y, \\ 0, & x = y \in I \end{cases} \quad (1.2)$$

is Schur-convex (concave) on I^2 if and only if f is convex (concave) on I.

Theorem B ([1, Theorem 1.2.]). Suppose that I is an open interval and $f: I \to \mathbb{R}$ is a continuous function. The function

$$G(x,y) = \begin{cases} \frac{1}{2}[f(x)+f(y)] - \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y \\ 0, & x = y \in I \end{cases} \quad (1.3)$$

is Schur-convex (concave) on I^2 if and only if f is convex (concave) on I.

If $\lambda = 1/6$, in [2] I. Franjić and J. Pečarić proved

Theorem C ([2, Theorem 4.]). If $f \in C^4(I)$, then the following statements are equivalent:

(a) The function

$$N(x,y) = \begin{cases} \frac{1}{6}[f(x)+f(y)]+\frac{2}{3}f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y \\ 0, & x = y \in I \end{cases} \quad (1.4)$$

is Schur-convex on I^2.

(b) For all $x, y \in I, x < y$, we have

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \frac{1}{6}[f(x)+f(y)]+\frac{2}{3}f\left(\frac{x+y}{2}\right).$$

(c) The function f is 4-convex on I.

Remark ([2, Remark3.]). In [2], I. Franjić and J. Pečarić have proven that f being a convex function is not a sufficient condition for $N(x,y)$ to be Schur-convex.

Schur-convexity has aroused the interest of many researchers, and numerous papers have been devoted to it. For example, in [3,4,6,7], some related results were given.

The purpose of this paper is to prove the following results:

Theorem 1.1. Let $I \subset \mathbb{R}$ be an open interval, and $f: I \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $\lambda \geq 1/4$ and f is convex (concave) on I, then function $S(x,y)$ is Schur-convex (concave) on I^2.

Theorem 1.2. Let $I \subset \mathbb{R}$ be an open interval, and $f: I \to \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $\lambda \geq 1/2$, and the function $S(x,y)$ is Schur-convex (concave) on I^2, then f is convex (concave) on I.

2. Definitions and Lemmas

In order to prove our result, we shall need several Definitions and Lemmas, which we present in this section.

Definition 2.1([5]). Let $x=(x_1,x_2,\cdots,x_n)$, $x=(y_1,y_2,\cdots,y_n)$,. x is said to be majorized by y (in symbols $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]},\ k=1,2,\cdots,n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are rearrangements of x and y in a descending order.

Definition 2.2([5]). Let $\Omega \subset \mathbb{R}^n$. The function $\varphi:\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \le \varphi(y)$. $\varphi(x)$ is said to be a Schur-concave function on Ω if and only if $-\varphi(x)$ is Schur-convex.

Lemma 2.1 (**Schur-Ostrowski Theorem**) ([5]). Let $\Omega \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior, and $\varphi\colon \Omega \to \mathbb{R}$ be a continuous symmetric function on Ω. If φ is differentiable in Ω°. Then φ is Schur-convex (concave) on Ω if and only if

$$H(x,y)=(y-x)\left(\frac{\partial\varphi}{\partial y}-\frac{\partial\varphi}{\partial x}\right)\ge(\le)0$$

for all $x\in\Omega^\circ$.

Lemma 2.2. Let $I\subset\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$. If $\lambda\ge 0$ and $f:I\to\mathbb{R}$ is a twice differentiable mapping such that f'' is integrable, then the following identity holds

$$\begin{aligned}&\lambda[f(a)+f(b)]+(1-2\lambda)f\left(\frac{a+b}{2}\right)\\&-\frac{1}{b-a}\int_a^b f(x)dx \qquad (2.1)\\&=\frac{(b-a)^2}{2}\int_0^1 K(t)f''(ta+(1-t)b)dt,\end{aligned}$$

where

$$K(t)=\begin{cases}t(2\lambda-t), & 0\le t\le 1/2,\\ (t-1)(1-t-2\lambda), & 1/2\le t\le 1.\end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}I&=\int_0^1 K(t)f''(ta+(1-t)b)dt\\&=\int_0^{1/2} t(2\lambda-t)f''(ta+(1-t)b)dt \qquad (2.2)\\&+\int_{1/2}^1 (t-1)(1-t-2\lambda)f''(ta+(1-t)b)dt.\end{aligned}$$

From integrating by part the right-hand sides of (2.2), we can state:

$$\begin{aligned}&\int_0^{1/2} t(2\lambda-t)f''(ta+(1-t)b)dt\\&=\frac{1}{a-b}\Big[t(2\lambda-t)f'(ta+(1-t)b)\Big|_0^{1/2}\\&\quad-\int_0^{1/2}(2\lambda-2t)f'(ta+(1-t)b)dt\Big]\\&=\frac{1}{2(a-b)}\left(2\lambda-\frac{1}{2}\right)f'\left(\frac{a+b}{2}\right)\\&\quad-\frac{1}{(a-b)^2}\Big[(2\lambda-2t)f(ta+(1-t)b)\Big|_0^{1/2}\\&\quad+2\int_0^{1/2}f(ta+(1-t)b)dt\Big]\\&=\frac{1}{2(a-b)}\left(2\lambda-\frac{1}{2}\right)f'\left(\frac{a+b}{2}\right)\\&\quad-\frac{1}{(a-b)^2}\left[(2\lambda-1)f\left(\frac{a+b}{2}\right)-2\lambda f(b)\right]\\&\quad-\frac{2}{(a-b)^2}\int_0^{1/2}f(ta+(1-t)b)dt, \qquad (2.3)\end{aligned}$$

and

$$\begin{aligned}&\int_{1/2}^1 (t-1)(1-t-2\lambda)f''(ta+(1-t)b)dt\\&=\frac{1}{a-b}\Big[(t-1)(1-t-2\lambda)f'(ta+(1-t)b)\Big|_{1/2}^1\\&\quad-\int_{1/2}^1(2-2\lambda-2t)f'(ta+(1-t)b)dt\Big]\\&=\frac{1}{2(a-b)}\left(\frac{1}{2}-2\lambda\right)f'\left(\frac{a+b}{2}\right)\\&\quad-\frac{1}{(a-b)^2}\Big[(2-2\lambda-2t)f(ta+(1-t)b)\Big|_{1/2}^1\\&\quad+2\int_{1/2}^1 f(ta+(1-t)b)dt\Big]\\&=\frac{1}{2(a-b)}\left(\frac{1}{2}-2\lambda\right)f'\left(\frac{a+b}{2}\right)\\&\quad-\frac{1}{(a-b)^2}\left[-2\lambda f(a)-(1-2\lambda)f\left(\frac{a+b}{2}\right)\right]\\&\quad-\frac{2}{(a-b)^2}\int_{1/2}^1 f(ta+(1-t)b)dt. \qquad (2.4)\end{aligned}$$

Using (2.3) and (2.4) in (2.2), it follows that

$$\begin{aligned}I=&\frac{2}{(a-b)^2}\Big\{\lambda\left[f(a)+f(b)\right]\\&+(1-2\lambda)f\left(\frac{a+b}{2}\right)-\int_0^1 f(at+(1-t)b)dt\Big\}.\end{aligned}$$

Thus, using the change of the variable $x=ta+(1-t)b$ for $t\in[0,1]$ and by multiplying both sides by $(a-b)^2/2$, we have the conclusion (2.1).

This completes the proof.

Lemma 2.3. *Let $I \subset \mathbb{R}$ be an open interval, and $f: I \to \mathbb{R}$ be a twice differentiable mapping such that f' is integrable. If $\lambda \geq \frac{1}{2}$, and $x, y \in I$ with $x < y$. Then it follows that*

$$\lambda[f(x)+f(y)]+(1-2\lambda)f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y f(t)dt = \frac{y-x}{2}\left(\frac{1}{4}-\lambda\right)(f'(\xi_1)-f'(\xi_2))$$

where $\xi_1 \in \left(x, \frac{x+y}{2}\right), \xi_2 \in \left(\frac{x+y}{2}, y\right), \xi \in (\xi_1, \xi_2)$.

Proof. From $x < y$ and $\lambda \geq 1/2$, then

$$1-\lambda-\frac{y-t}{y-x} \leq 0, t \in \left(x, \frac{x+y}{2}\right), \tag{2.5}$$

and

$$\lambda-\frac{y-t}{y-x} \geq 0, t \in \left(\frac{x+y}{2}, y\right). \tag{2.6}$$

So, by Integral mean value theorem, we have

$$\begin{aligned}
&\lambda[f(x)+f(y)]+(1-2\lambda)f\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y f(t)dt \\
&= \int_x^{\frac{x+y}{2}}\left(1-\lambda-\frac{y-t}{y-x}\right)f'(t)dt + \int_{\frac{x+y}{2}}^y\left(\lambda-\frac{y-t}{y-x}\right)f'(t)dt \\
&= f'(\xi_1)\int_x^{\frac{x+y}{2}}\left(1-\lambda-\frac{y-t}{y-x}\right)dt + f'(\xi_2)\int_{\frac{x+y}{2}}^y\left(\lambda-\frac{y-t}{y-x}\right)dt \\
&= f'(\xi_1)\frac{y-x}{2}\left(1-\lambda-\frac{3}{4}\right) + f'(\xi_2)\frac{y-x}{2}\left(\lambda-\frac{1}{4}\right) \\
&= \frac{y-x}{2}\left(\frac{1}{4}-\lambda\right)(f'(\xi_1)-f'(\xi_2)) \\
&= \frac{y-x}{2}\left(\lambda-\frac{1}{4}\right)f''(\xi)(\xi_2-\xi_1),
\end{aligned}$$

where $\xi_1 \in \left(x, \frac{x+y}{2}\right), \xi_2 \in \left(\frac{x+y}{2}, y\right), \xi \in (\xi_1, \xi_2)$.

This completes the proof.

Lemma 2.4. *Suppose that $S(x,y)$ be defined as in (1.1). If f has continuous second order derivatives on I, then*

$$\left.\frac{\partial S}{\partial x}\right|_{(t_0,t_0)} = \left.\frac{\partial S}{\partial y}\right|_{(t_0,t_0)} = 0$$

for all $t_0 \in I$.

Proof. For any $t_0 \in I$, from (1) together with the L'Hospital's rule we clearly see that

$$\begin{aligned}
\left.\frac{\partial S}{\partial x}\right|_{(t_0,t_0)} &= \lim_{t\to 0}\frac{S(t_0+t,t_0)-S(t_0,t_0)}{t} \\
&= \lim_{t\to 0}\frac{1}{t}\left\{\lambda[f(t_0+t)+f(t_0)] + (1-2\lambda)f\left(t_0+\frac{t}{2}\right)-\frac{1}{t}\int_{t_0}^{t_0+t}f(t)dt\right\} \\
&= \lim_{t\to 0}\frac{1}{t^2}\left\{\lambda t[f(t_0+t)+f(t_0)] + (1-2\lambda)tf\left(t_0+\frac{t}{2}\right)-\int_{t_0}^{t_0+t}f(t)dt\right\} \\
&= \lim_{t\to 0}\frac{1}{2}\left\{2\lambda f'(t_0+t)+(1-2\lambda)f'\left(t_0+\frac{t}{2}\right) + \lambda tf''(t_0+t)+\frac{(1-2\lambda)t}{2}f''\left(t_0+\frac{t}{2}\right) - f'(t_0+t)\right\} \\
&= 0.
\end{aligned}$$

Making use of similar arguments for $\frac{\partial S}{\partial y}$, and we get

$$\left.\frac{\partial S}{\partial y}\right|_{(t_0,t_0)} = 0.$$

This completes the proof.

3. Proof of Theorems

Proof of Theorem 1.1. The proof is divided into three cases.

Case 1. If $x = y \in I$, then Lemma 2.4 leads to

$$H(x,y) = 0.$$

Case 2. If $x < y \in I$, then (1.1) leads to

$$\frac{\partial S}{\partial y} = \lambda f'(y)+\frac{1-2\lambda}{2}f'\left(\frac{x+y}{2}\right) + \frac{1}{(y-x)^2}\int_x^y f(t)dt-\frac{1}{y-x}f(y), \tag{3.1}$$

and

$$\frac{\partial S}{\partial x} = \lambda f'(x)+\frac{1-2\lambda}{2}f'\left(\frac{x+y}{2}\right) - \frac{1}{(y-x)^2}\int_x^y f(t)dt+\frac{1}{y-x}f(x). \tag{3.2}$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&H(x,y) \\
&= \lambda(y-x)[f'(y)-f'(x)] + \frac{2}{y-x}\int_x^y f(t)dt-[f(y)+f(x)]
\end{aligned}$$

$$=\lambda\left(y-x\right)^2\int_0^1 f''(tx+(1-t)y)dt$$
$$-\left(y-x\right)^2\int_0^1 t(1-t)f''(tx+(1-t)y)dt$$
$$=\left(y-x\right)^2\int_0^1\left[\lambda-t(1-t)\right]f''(tx+(1-t)y)dt.$$

If $\lambda\geq\frac{1}{4}$ and f is convex, then $H(x,y)\geq 0$.

Therefore, $S(x,y)$ is Schur convex on I^2 from Lemma 2.1.

Case 3. If $x>y\in I$, since $S(x,y)$ is a symmetric function, we also can conclude that $S(x,y)$ is Schur convex on I^2 from $\lambda\geq\frac{1}{4}$ and the convexity of f.

It follows from the similar arguments as above that $S(x,y)$ is Schur-concave on I^2 if $\lambda\geq\frac{1}{4}$ and f is concave on I, which completes the proof.

Proof of Theorem 1.2. If $S(x,y)$ is Schur-convex on I^2, then from (1.1) and Definition 1 together with the fact that $\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y)$ we have

$$\lambda\left[f(x)+f(y)\right]+(1-2\lambda)f\left(\frac{x+y}{2}\right)-\frac{1}{y-x}\int_x^y f(t)dt\geq 0.$$

If $x,y\in I, x<y,$ by Lemma 2.3, we have

$$\lambda\left[f(x)+f(y)\right]+(1-2\lambda)f\left(\frac{x+y}{2}\right)$$
$$-\frac{1}{y-x}\int_x^y f(t)dt$$
$$=\frac{y-x}{2}\left(\lambda-\frac{1}{4}\right)\left(\xi_2-\xi_1\right)f''(\xi)\geq 0.$$

where $\xi_1\in\left(x,\frac{x+y}{2}\right),\xi_2\in\left(\frac{x+y}{2},y\right),\xi\in(\xi_1,\xi_2)$.

Namely $f''(\xi)\geq 0$, and so, since x and y are arbitrary, we conclude that f is convex.

Since $S(x,y)$ is a symmetric function, if $x>y$ we also can conclude that f is convex.

It follows from the similar arguments as above that f is concave on I if $S(x,y)$ is Schur- concave on I^2, which completes the proof.

4. Conclusion

In this paper, we discuss Schur convexity for a class of symmetric functions and obtain the following results:

Theorem 1.1. Let $I\subset\mathbb{R}$ be an open interval, and $f:I\to\mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $\lambda\geq 1/4$ and f is convex (concave) on I, then function $S(x,y)$ is Schur-convex (concave) on I^2.

Theorem 1.2. Let $I\subset\mathbb{R}$ be an open interval, and $f:I\to\mathbb{R}$ be a twice differentiable mapping such that f'' is integrable. If $\lambda\geq 1/2$, and the function $S(x,y)$ is Schur-convex (concave) on I^2, then f is convex (concave) on I.

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Competing Interests

The authors declare that they have no competing interests.

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