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$\mathscr{D}$–sets and Structure-Preserving Maps

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Abstract This paper investigates $\mathscr{D}$–sets of groups in relation to structure-preserving maps. It shows connections between non-involutions of groups and the concept of $\mathscr{D}$–sets. In particular, we prove that the existence of a semigroup isomorphism between the families of $\mathscr{D}$–sets of two groups is equivalent to an existence of a special type of bijection between the subsets containing all elements of orders greater than two of the groups.

Keywords: *$\mathscr{D}$–sets, non-involutions, morphism*

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1. Introduction

The elements of a group of order two play a very important role not only in group theory but in other branches of mathematics, they are known as involutions. We call elements of order greater than two as non-involutions in this paper. The structure called $\mathscr{D}$–set is constructed with the concept of inverses and reveal some properties related to involutions [7]. In fact, a group has only one $\mathscr{D}$–set if and only if it is an elementary abelian 2-group. A subset D of a group G is a $\mathscr{D}$–set whenever every element of G not in D has its inverse in D. This paper shows results that would lead to the comparison of the numbers of non-involutions of two arbitrary groups. We study connections of structural-preserving mappings between groups and their corresponding $\mathscr{D}$–set families.

We borrow concepts and notations from set theory [5]. Let X and Y be sets, then $X \setminus Y = \{x \in X \mid x \notin Y\}$ is the complement of Y in X. If $f : X \to Y$ is a function with $A \subseteq X$ then $f[A] = \{f(a) \mid a \in A\}$, called the image of A in f. The cardinality of a set X is denoted by $|X|$. We denote the set of all involutions of a group G together with the identity element by S_G; that is,

$$S_G = \{x \in G \mid x^2 = e\}.$$

A $\mathscr{D}$–set D of group G is a **minimum** $\mathscr{D}$– set if and only if the inverse of each $x \in D \setminus S_G$ is not in D [1]. Note that for a finite group G, this idea coincides with the minimum $\mathscr{D}$–sets mentioned in [6]. We write T_G as the family of all $\mathscr{D}$–sets of a group G and $T_{\min(G)}$ the subset containing all minimum $\mathscr{D}$–sets [1]. It was shown in [7] that T_G is a semigroup with respect to union of sets.

We deviate a little to discuss the motivation of $\mathscr{D}$–set and some related literature. The definition of $\mathscr{D}$–set is based on dominating sets of graphs. Let $\mathscr{G}(V,E)$ be a graph and $D \subseteq V$. D is said to dominate $\mathscr{G}$ if for any $u \in V \setminus D$, there exists $v \in D$ such that $\{u,v\} \in E$ (see [2]). As mentioned in [1], a special type of graph constructed from a group was introduced by Kandasamy and Smarandache [4] in 2009. An identity graph of a nontrivial group G is an undirected graph formed by adjoining every non-identity element to the identity e of G and $x, y \in G$ are connected whenever $xy = e$. In view of identity graphs of finite groups, the points contained in a minimum $\mathscr{D}$–set form a special type of induced subgraph called stars [1]. Hence, we can view $T_{\min(G)}$ as a family of stars related to the group.

2. Results

We start by showing how T_G can be generated from the corresponding $T_{\min(G)}$.

Proposition 1 *Let G be a group. Then $T_{\min(G)}$ generates T_G as a semigroup. Moreover, if $|T_G| \neq 1$,*

$$T_G = T^2_{\min(G)}$$

where

$$T^2_{\min(G)} = \{X \cup Y \mid X, Y \in T_{\min(G)}\}.$$

Proof: Let D_k be in T_G. If $D_k \setminus S_G = \varnothing$ then $D_k = S_G = G$ and we only have one $\mathscr{D}$–set in this case. That is, $T_G = T_{\min(G)}$. Assume $D_k \setminus S_G \neq \varnothing$ and denote

$\Delta = \{x \in D_k \setminus S_G \mid x^{-1} \in D_k\}$. Consider an nonempty subset A of Δ such that, for each $x \in A$, $x^{-1} \in \Delta \setminus A$. We observe that D_k can be expressed as

$$D_k = ((D_k \setminus \Delta) \cup A) \cup ((D_k \setminus \Delta) \cup (\Delta \setminus A))$$

where $(D_k \setminus \Delta) \cup A$ and $(D_k \setminus \Delta) \cup (\Delta \setminus A)$ are in $T_{\min(G)}$. Thus, $T_{\min(G)}$ generates T_G.

We remark that a minimum $\mathscr{D}$–set cannot be written as a union of two distinct $\mathscr{D}$–sets. Let x be in G. Then we write

$$T(x) = \{D \in T_G \mid x \in D\}$$

and

$$T_{\min(x)} = \{D \in T_{\min(G)} \mid x \in D\}.$$

The following lemma in [7] gives a certain characterization of the involutions in G.

Lemma 1 [7] *Let x be a non-identity element of a group G. Then x is an involution if and only if* $T(x) = T_G$.

The following proposition is a refinement of Lemma 1.

Proposition 2 *Let G be a nontrivial group. A non-identity element x in G is an involution if and only if* $T_{\min(x)} = T_{\min(G)}$.

Proof: Let x be an involution in G. Since $T(x) = T_G$, then $T_{\min(x)} = T_{\min(G)}$. Suppose $T_{\min(x)} = T_{\min(G)}$. Let $D \in T_G$ and by Proposition 1, $D = D_{m_1} \cup D_{m_2}$ where D_{m_1} and D_{m_2} are elements of $T_{\min(G)}$. By assumption, D_{m_1} and D_{m_2} are both in $T_{\min(x)}$. Hence, $X \in D$. This means that $T(x) = T_G$, and by Lemma 1, x is an involution.

The proposition below proves that an isomorphism of families of $\mathscr{D}$–sets preserves the *minimality* property.

Proposition 3 *Let G and H be groups and* $\varphi : T_G \to T_H$ *be a semigroup isomorphism. Then D is a minimum* $\mathscr{D}$*–set of G if and only if* $\varphi(D)$ *is a minimum* $\mathscr{D}$*–set of H.*

Proof: The case $G = S_G$ is trivial. Suppose $G \neq S_G$. Assume D_i is minimum while $\varphi(D_i)$ is not. Then there exists at least one pair $y, \neq y^{-1}$ both in $\varphi(D_i)$. As in the proof of Proposition 1, there exist D'_j and D'_k in $T_{\min(H)}$ such that $\varphi(D_i) = D'_j \cup D'_k$ with $y \in D'_j$ and $y^{-1} \in D'_k$. It follows that there exist distinct D_j and D_k in T_G such that $\varphi(D_j) = D'_j$ and $\varphi(D_k) = D'_k$. But this implies that $\varphi(D_i) = \varphi(D_j) \cup \varphi(D_k) = \varphi(D_j \cup D_k)$ and so $D_i = D_j \cup D_k$. This is a contradiction to a remark following Proposition 1.

For the converse, suppose $\varphi(D_i)$ is a minimum while D_i is not. There exist distinct D_j and D_k in $T_{\min(G)}$ where $D_i = D_j \cup D_k$. Hence, $\varphi(D_i) = \varphi(D_j \cup D_k) = \varphi(D_j) \cup \varphi(D_k)$ where $\varphi(D_j) \neq \varphi(D_k)$, this is absurd.

Proposition 4 Let G and H be groups and $\varphi : T_G \to T_H$ be a semigroup isomorphism. If $D \in T_{\min(G)}$ and $x \in G \setminus D$ then $\varphi(D \cup \{x\}) = \varphi(D) \cup \{y\}$ where $y \in H \setminus \varphi(D)$.

Proof: Suppose D is in $T_{\min(G)}$ not containing an element x of G. Then $(D \cup \{x\})$ is an element of T_G where $\{x, x^{-1}\}$ is the only pair of inverses in this $\mathscr{D}$–set. As in the proof of Proposition 1,

$$D \cup \{x\} = D \cup ((D \setminus \{x^{-1}\}) \cup \{x\})$$

where $(D \setminus \{x^{-1}\}) \cup \{x\}$ is also in $T_{\min(G)}$. The homomorphic property of φ implies that

$$\varphi(D \setminus \{x\}) = \varphi(D) \cup \varphi(((D \setminus \{x^{-1}\}) \cup \{x\}))$$

where $\varphi(D), \neq \varphi(((D \setminus \{x^{-1}\}) \cup \{x\}))$ in $T_{\min(H)}$ by Proposition 3. Further, there must exist $y, \neq y^{-1}$ in $H \setminus S_H$ where (WLOG)

$$y^{-1} \in \varphi(D) \text{ and } y \in \varphi((D \setminus \{x^{-1}\}) \cup \{x\}).$$

Suppose there exists another element z which shares the same characteristic with y.

We may assume that y^{-1} and z^{-1} are in $\varphi(D)$ while y and z are in $\varphi((D \setminus \{x^{-1}\}) \cup \{x\})$. As a consequence of the above argument, $\varphi(D \cup \{x\})$ can be expressed as

$$\begin{aligned}\varphi(D \cup \{x\}) = &((\varphi(D) \setminus \{y^{-1}\}) \cup \{y\}) \\ &\cup ((\varphi(D) \setminus \{z^{-1}\}) \cup \{z\}) \\ &\cup \varphi((D \setminus \{x^{-1}\}) \cup \{x\})\end{aligned}$$

where the three factors are distinct elements of $T_{\min(H)}$. By the surjective property of φ and Proposition 3, there exist D_i and D_j in $T_{\min(G)}$ such that

$$\varphi(D_i) = (\varphi(D) \setminus \{y^{-1}\}) \cup \{y\}$$

$$\text{and } \varphi(D_j) = (\varphi(D) \setminus \{z^{-1}\}) \cup \{z\}.$$

This means that

$$\varphi(D \cup \{x\}) = \varphi(D_i) \cup \varphi(D_j) \cup \varphi((D \setminus \{x^{-1}\}) \cup \{x\}).$$

By the properties of φ, we have

$$\varphi\left(D\cup\{x\}\right)=\varphi\left(D_i\cup D_j\cup\left(\left(D\setminus\left\{x^{-1}\right\}\right)\cup\{x\}\right)\right)$$

and so

$$D\cup\{x\}=D_i\cup D_j\cup\left(\left(D\setminus\left\{x^{-1}\right\}\right)\cup\{x\}\right). \qquad (\S)$$

Since the three factors on the right handside of equation (§) are distinct elements of $T_{\min(G)}$, we get at least two pairs of inverses. But we only have x and x^{-1} from the left handside of (§), this is absurd. Hence, y and y^{-1} must be the only pair of inverses in $\varphi(D)\cup\varphi\left(\left(D\setminus\left\{x^{-1}\right\}\right)\cup\{x\}\right)$ and so

$$\varphi\left(D\cup\{x\}\right)=\varphi(D)\cup\{y\}.$$

Let us now state and prove the main result of this paper.

Theorem 1 Let G and H be groups with $G\setminus S_G\neq\varnothing$. Then T_G is isomorphic to T_H if and only if there exists a bijection $\sigma:(G\setminus S_G)\to(H\setminus S_H)$ such that $\sigma\left(x^{-1}\right)=\sigma(x)^{-1}$ for any x in $G\setminus S_G$.

Proof: Let $\varphi:T_G\to T_H$ be an isomorphism. We form the bijection $\sigma:(G\setminus S_G)\to(H\setminus S_H)$ such that $\sigma\left(x^{-1}\right)=\sigma(x)^{-1}$ for any x in $G\setminus S_G$. Firstly, we choose a fix D in $T_{\min(G)}$. Let x be in $G\setminus S_G$, then either $x\notin D$ or $x\in D$. If $x\notin D$, then the pair x and x^{-1} is unique in $D\cup\{x\}$. By Proposition 4, there exists a unique pair $y\neq y^{-1}$ in $H\setminus S_H$ where $\varphi\left(D\cup\{x\}\right)=\varphi(D)\cup\{y\}$. We can now form $\sigma(x)=y$ and $\sigma\left(x^{-1}\right)=y^{-1}=\sigma(x)^{-1}$. If $x\in D$ then $x^{-1}\notin D$ and we proceed as in the first case. Therefore, if $x\in G\setminus S_G$ then there exists a unique $y\in H\setminus S_H$ such that $\sigma(x)=y$ and $\sigma\left(x^{-1}\right)=y^{-1}$.

We show that σ is an injection by way of contradiction. Suppose that $a\neq b$ in $G\setminus S_G$ such that $\sigma(a)=\sigma(b)$. Since a is mapped to $\sigma(a)$ and a^{-1} to $\sigma(a)^{-1}$ where $\sigma(a)\in H\setminus S_H$, then $b\neq a^{-1}$. Now, we form D_j and D_k in $T_{\min(G)}$:

- If $a\notin D$ then $D_j=D$;
- If $a\in D$ then $D_j=\left(D\setminus\{a\}\right)\cup\left\{a^{-1}\right\}$;
- If $b\notin D$ then $D_k=D$;
- If $b\in D$ then $D_k=\left(D\setminus\{b\}\right)\cup\left\{b^{-1}\right\}$.

Hence, we have the following cases:

Case 1: $a\notin D$ and $b\notin D$

- $\varphi\left(D_j\cup\{a\}\right)=\varphi\left(D\cup\left(\left(D\setminus\left\{a^{-1}\right\}\right)\cup\{a\}\right)\right)$
  $\Rightarrow\varphi\left(D_j\cup\{a\}\right)=\varphi(D)\cup\varphi\left(\left(D\setminus\left\{a^{-1}\right\}\right)\cup\{a\}\right)$
- $\varphi\left(D_k\cup\{b\}\right)=\varphi\left(D\cup\left(\left(D\setminus\left\{b^{-1}\right\}\right)\cup\{b\}\right)\right)$
  $\Rightarrow\varphi\left(D_k\cup\{b\}\right)=\varphi(D)\cup\varphi\left(\left(D\setminus\left\{b^{-1}\right\}\right)\cup\{b\}\right)$.

Case 2: $a\notin D$ and $b\in D$

- $\varphi\left(D_j\setminus\{a\}\right)=\varphi(D)\cup\varphi\left(\left(D\setminus\left\{a^{-1}\right\}\right)\cup\{a\}\right)$
- $\varphi\left(D_k\cup\{b\}\right)=\varphi\left(\left(\left(D\setminus\{b\}\right)\cup\left\{b^{-1}\right\}\right)\cup\{D\}\right)$
  $\Rightarrow\varphi\left(D_k\cup\{b\}\right)=\varphi\left(\left(D\setminus\{b\}\right)\cup\left\{b^{-1}\right\}\right)\cup\varphi(D)$.

Case 3: $a\in D$ and $b\notin D$

- $\varphi\left(D_j\cup\{a\}\right)=\varphi\left(\left(\left(D\setminus\{a\}\right)\cup\left\{a^{-1}\right\}\right)\cup D\right)$
  $\Rightarrow\varphi\left(D_j\cup\{a\}\right)=\varphi\left(\left(D\setminus\{a\}\right)\cup\left\{a^{-1}\right\}\right)\cup\varphi(D)$
- $\varphi\left(D_k\cup\{b\}\right)=\varphi(D)\cup\varphi\left(\left(D\setminus\left\{b^{-1}\right\}\right)\cup\{b\}\right)$.

Case 4: $a\in D$ and $b\in D$

- $\varphi\left(D_j\cup\{a\}\right)=\varphi\left(\left(D\setminus\{a\}\right)\cup\left\{a^{-1}\right\}\right)\cup\varphi(D)$
- $\varphi\left(D_k\cup\{b\}\right)=\varphi\left(\left(D\setminus\{b\}\right)\cup\left\{b^{-1}\right\}\right)\cup\varphi(D)$.

Note that in any of the cases above,

$$\varphi\left(D_j\cup\{a\}\right)=\varphi(D)\cup\varphi(D_n)$$
$$\text{and } \varphi\left(D_k\cup\{b\}\right)=\varphi(D)\cup\varphi(D_m),$$

for some $D_n,D_m\in T_{\min(G)}\setminus\{D\}$.

Now, the only pair of inverses in $\varphi\left(D_j\cup\{a\}\right)$ is $\sigma(a)\neq\sigma(a)^{-1}$ while only $\sigma(b)\neq\sigma(b)^{-1}$ in $\varphi\left(D_k\cup\{b\}\right)$. Let $y\in\left\{\sigma(a),\sigma(a)^{-1}\right\}\setminus\varphi(D)$. Since $\sigma(a)=\sigma(b)$, then $y\in\varphi(D_n)\setminus\varphi(D)$ and $y\in\varphi(D_m)\setminus\varphi(D)$. Hence,

$$\varphi\left(D_j\cup\{a\}\right)=\varphi(D)\cup\varphi(D_n)$$
$$=\varphi(D)\cup\{y\}=\varphi(D)\cup\varphi(D_m)=\varphi\left(D_k\cup\{b\}\right).$$

Since φ is injective, we have

$$D_j\cup\{a\}=D_k\cup\{b\}.$$

This further implies that a and a^{-1} are both in D_k, this is a contradiction.

To show that it is surjective, assume an element $y\in H\setminus S_H$. Using $\varphi(D)$ in T_H, either $y\notin\varphi(D)$ or $y\in\varphi(D)$. If $y\in\varphi(D)$, then $y^{-1}\in\varphi(D)$ and the pair $y\neq y^{-1}$ is unique in $\varphi(D)\cup\{y\}$. Further,

$$\varphi(D)\cup\{y\}=\varphi(D)\cup\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$$

where $\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$ is a minimum $\mathcal{S}$–set of H. By Proposition 4 and the isomorphism $\varphi^{-1}:T_H\to T_G$, we have $\varphi^{-1}(\varphi(D)\cup\{y\})$ in T_G which contains a unique pair $x \neq x^{-1}$. However,

$$\varphi^{-1}(\varphi(D)\cup\{y\}) = \varphi^{-1}\left(\varphi(D)\cup\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$$

$$\Rightarrow \varphi^{-1}(\varphi(D)\cup\{y\}) = D\cup\varphi^{-1}\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$$

where $\varphi^{-1}\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$ is in $T_{\min(G)}$. WLOG, we may have $x^{-1} \in D$ and $x \in \varphi^{-1}\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$. Thus, we take $\sigma(x) = y$ and $\sigma\left(x^{-1}\right) = y^{-1} = \sigma(x)^{-1}$ in which

$$\varphi(D\cup\{x\}) = \varphi\left(D\cup\varphi^{-1}\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$$

$$\Rightarrow \varphi(D\cup\{x\}) = \varphi(D)\cup\left(\left(\varphi(D)\setminus\left\{y^{-1}\right\}\right)\cup\{y\}\right)$$

$$\Rightarrow \varphi(D\cup\{x\}) = \varphi(D)\cup\{y\}.$$

On the other hand, given that $y \in \varphi(D)$, then $y^{-1} \notin \varphi(D)$. We proceed as above knowing that $y \neq y^{-1}$ is the only pair of inverses in $\varphi(D)\cup\left\{y^{-1}\right\}$. By following the same pattern of reasoning, we will still obtain a unique pair x and x^{-1} from $G \setminus S_G$ in which we can write $\sigma(x) = y$ and $\sigma\left(x^{-1}\right) = y^{-1} = \sigma(x)^{-1}$. Hence, σ is surjective. Summing up, we have the required bijection.

For the converse, suppose there exists a bijection $\sigma:(G\setminus S_G)\to(H\setminus S_H)$ such that $\sigma\left(x^{-1}\right) = \sigma(x)^{-1}$ for any x in $G\setminus S_G$. We form a semigroup isomorphism $\varphi: T_G \to T_H$. Let D be in T_G, then $D = S_G \cup X$ where $X \subseteq G\setminus S_G$. We define φ by

$$\varphi(D) = S_H \cup \sigma[X]$$

where $\sigma[X]$ is the image of X with respect to σ. The verification that φ is an isomorphism is a routine.

We prove more properties involving morphisms and $\mathscr{P}$–sets.

Proposition 5 *Let* $\gamma: G \to H$ *be a monomorphism of groups G and H. Then*

i. If $\gamma[D]$ *is a* $\mathscr{P}$*–et of H then D is a* $\mathscr{P}$*– set of G;*

ii. If $\gamma[D]$ *is a minimum* $\mathscr{P}$*–set of H then D is a minimum* $\mathscr{P}$*– set of G.*

Proof: (i) Let $\gamma[D]$ be a $\mathscr{P}$–set of H and $x \in G\setminus D$. Since γ is injective, then $\gamma(x)$ must not be in $\gamma[D]$. By assumption, $\gamma\left(x^{-1}\right) = \gamma(x)^{-1}$ is in $\gamma[D]$. This implies that $x^{-1} \in D$.

(ii) Suppose $\gamma[D]$ is a minimum $\mathscr{P}$–set of H. By part (i), D is a $\mathscr{P}$–set of G. If $x \in D\setminus S_G$ then $\gamma(x) \in \gamma[D]$. By assumption, $\gamma\left(x^{-1}\right) = \gamma(x)^{-1} \notin \gamma[D]$. Thus, $x^{-1} \notin D$ and this proves our claim.

We observe that if T_G s a singleton semigroup (that is, $G\setminus S_G = \varnothing$) then the following hold true vacuously.

Lemma 2 *Let* $\gamma: G \to H$ *be a mapping of groups G and H where* $G\setminus S_G \neq \varnothing$*. If an isomorphism* $\varphi: T_G \to T_H$ *has the property that* $\varphi(D\cup\{y\}) = \varphi(D)\cup\{\gamma(y)\}$*, for* $D \in T_{\min(G)}$ *and* $y \in G\setminus D$*, then* $\gamma(y) \notin \varphi(D)$*.*

Proof: Let $D \in T_{\min(G)}$, $y \in G\setminus D$, and $\varphi: T_G \to T_H$ be an isomorphism such that $\varphi(D\cup\{y\}) = \varphi(D)\cup\{\gamma(y)\}$ with γ as above. From the proof of Proposition 4,

$$\varphi(D\cup\{y\}) = \varphi(D)\cup\varphi\left(\left(\left(D\cup\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$$

where $\varphi(D) \neq \varphi\left(\left(\left(D\cup\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$ in $T_{\min(H)}$. Now we have $\varphi(D)\cup\{\gamma(y)\} = \varphi(D)\cup\varphi\left(\left(\left(D\cup\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$, implying that $\gamma(y)$ cannot be in $\varphi(D)$. Otherwise, we will get $\varphi(D) = \varphi(D)\cup\varphi\left(\left(\left(D\cup\left\{y^{-1}\right\}\right)\cup\{y\}\right)\right)$ which is absurd.

Theorem 2 *Let* $\gamma: G \to H$ *be a monomorphism of groups G and H where* $G\setminus S_G \neq \varnothing$*. Then there exists an isomorphism* $\varphi: T_G \to T_H$ *such that* $\varphi(D\cup\{y\})$ $= \varphi(D)\cup\{\gamma(y)\}$*, for every* $D \in T_{\min(G)}$ *and* $y \in G\setminus D$*, if and only if* $\gamma[G\setminus S_G] = H\setminus S_H$*.*

Proof: Suppose $\varphi: T_G \to T_H$ is an isomorphism such that $\varphi(D\cup\{y\}) = \varphi(D)\cup\{\gamma(y)\}$, for every $D \in T_{\min(G)}$ and $y \in G\setminus D$. If $z \in \gamma[G\setminus S_G]$ then $z = \gamma(x)$ for some $x \in G\setminus S_G$. Thus, $z^{-1} = \gamma(x)^{-1} = \gamma\left(x^{-1}\right)$. Assuming that $z = z^{-1}$ would imply $\gamma(x)^{-1} = \gamma(x)$ which means $x = x^{-1}$ since γ is injective. This is a contradiction. Hence, z must be in $H\setminus S_H$. Now, if $z \in H\setminus S_H$ then choose a minimum $\mathscr{P}$–set of H, say D' not containing z . By Proposition 4 and φ^{-1},

$$\varphi^{-1}(D'\cup\{z\}) = \varphi^{-1}(D')\cup\{y\}$$

for some $y \in G\setminus\varphi^{-1}(D')$. By property of φ,

$$D'\cup\{z\} = \varphi\left(\varphi^{-1}(D')\cup\{y\}\right) = D'\cup\{\gamma(y)\}.$$

As in Lemma 2, $\gamma(y) \notin D'$. It is now evident that $z \in \gamma[G\setminus S_G]$.

For the converse, assume that $\gamma[G\setminus S_G] = H\setminus S_H$. We now have a bijection $\gamma: G\setminus S_G \to H\setminus S_H$ such that

$\gamma\left(x^{-1}\right)=\gamma(x)^{-1}$ for all $x \in G \setminus S_G$. By Theorem 1, we have the isomorphism $\varphi : T_G \to T_H$ defined by

$$\varphi(D)=S_H \cup \gamma[X]$$

where $D \in T_G$ with $D=S_G \cup X$ for some $X \subseteq G \setminus S_G$. Let D be in $T_{\min(G)}$ and $y \in G \setminus D$. Suppose $D=S_G \cup X$ where $X \subseteq G \setminus S_G$. Then $\varphi(D \cup \{y\})$ $=\varphi(S_G \cup X \cup \{y\})=S_H \cup \gamma[X \cup \{y\}]$. But we have $\gamma[X \cup \{y\}]=\gamma[X] \cup \gamma\{y\}$. Consequently,

$$\begin{aligned}\varphi(D \cup \{y\}) &= S_H \cup \gamma[X \cup \{y\}] \\ &= S_H \cup \gamma[X] \cup \gamma\{y\} = \varphi(D) \cup \{\gamma(y)\}.\end{aligned}$$

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