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# A Note on Hermite poly-Bernoulli Numbers and Polynomials of the Second Kind

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**Abstract** In the paper, we introduce a new concept of poly-Bernoulli numbers and polynomials of the second kind which is called Hermite poly-Bernoulli numbers and polynomials of the second kind. We also investigate and analyse its applications in number theory, combinatorics and other fields of mathematics. The results derived here are a generalization of some known summation formulae earlier studied by Jolany et al. [17,18], Dattoli et al [14] and Pathan et al [29,30].

***Keywords:*** *Hermite polynomials, poly-Bernoulli polynomials of the second kind, Hermite poly-Bernoulli polynomials of the second kind, summation formulae, symmetric identities*

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## 1. Introduction

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [13] and [14] reads

$$H_n(x,y) = n!\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y = -1$ and $x$ is replaced by $2x$.

The classical Bernoulli numbers $B_n$ . Bernoulli polynomials $B_n(x)$ and their generalization $B_n^{(\alpha)}(x)$ (of real or complex) of order $\alpha$ are usually defined by means of the following generating functions ([2,40,41,42]).

$$\left(\frac{t}{e^t-1}\right) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1.3}$$

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.4}$$

and

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}\left(|t|<2\pi;1^{\alpha}=1\right). \tag{1.5}$$

The $B_n$ are rational numbers and in particular $B_n^{(1)}(0) = B_n(0) = B_n$.

As is well known, the Bernoulli polynomials of the second kind [35] are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind. The first few Bernoulli numbers $bn$ of the second kind are $b_0 = 1$, $b_1 = 1/2$, $b_3 = 1/24$, $b_4 = -19/720$, $b_5 = 3/160$, ....

In [22], Kaneko introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. poly-Bernoulli numbers $B_n^{(k)}$ with $k \in z$ and $n \in N$ , appear in the following power series

$$\frac{Li_k\left(1-e^{-t}\right)}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \tag{1.7}$$

where $k \in z$ and

$$Li_k = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, |z|<1$$

so for $k \le 1$,

$$Li_k = -\ln(1-z), Li_0(z) = \frac{z}{1-z}, Li_{-1} = \frac{z}{(1-z)^2}, \ldots$$

Moreover when $k \ge 1$, the left hand side of (1.1) can be written in the form

$$e^t\frac{1}{e^t-1}\int_0^t\frac{1}{e^t-1}\cdots\int_0^t\frac{1}{e^t-1}\int_0^t\frac{t}{e^t-1}dtdt\cdots dt=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}.$$

In the special case, one can see

$$B_n^{(1)}=B_n.$$

Recently, Jolany et al [17,18] generalized the concept of poly-Bernoulli polynomials is defined as follows.

Let $a$, $b$, $c>0$ and $a\neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(x,a,b)$ and the polynomials $B_n^{(k)}(x,a,b,c)$ are appeared in the following series respectively

$$\frac{Li_k\left(1-(ab)^{-t}\right)}{b^t-a^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}(a,b)\frac{t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|}\quad(1.8)$$

$$\frac{Li_k\left(1-(ab)^{-t}\right)}{b^t-a^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x,a,b)\frac{t^n}{n!},\quad(1.9)$$

$$|t|<\frac{2\pi}{|\ln a+\ln b|}$$

$$\frac{Li_k\left(1-(ab)^{-t}\right)}{b^t-a^{-t}}c^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x,a,b,c)\frac{t^n}{n!},\quad(1.10)$$

$$|t|<\frac{2\pi}{|\ln a+\ln b|}$$

One can easily see that

$$B_n^{(k)}(0,1,e)=B_n^{(k)},B_n^{(k)}(x)=1+x$$

and

$$B_n^{(k)}(x)=B_n^{(k)}\left(e^{x+1},e^x\right)\quad(1.11)$$

where $B_n^{(k)}$ re generalized poly-Bernoulli numbers. For more information about poly-Bernoulli numbers and poly-Bernoulli polynomials, we refer to [15-20].

In [29-34] Pathan et al introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ is defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}\quad(1.12)$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al [[14], p.386(1.6)] in the form

$$\left(\frac{t}{e^t-1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!}.\quad(1.13)$$

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,(n\geq 0)\quad(1.14)$$

and the Stirling number of the second kind is defined by generating function to be

$$\left(e^t-1\right)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}.\quad(1.15)$$

Recently many mathematicians have studied the symmetric identities on some special polynomials see for details [29-34,43,44]. Some of mathematicians also investigated some applications of poly-Bernoulli numbers and polynomials of the second kind cf. [26,27,28,36,37,38,39]. For more information about these polynomials, look at [1-44] and the references cited therein.

In this paper, we first give definitions of the Hermite poly-Bernoulli polynomials ${}_Hb_n^{(k)}(x,y)$ and we give some formulae of those polynomials related to the Stirling numbers of the second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite poly-Bernoulli numbers and polynomials of the second kind studied by Dattoli et al, Zhang et al, Yang, Khan, Pathan and Khan.

## 2. Hermite poly-Bernoulli Numbers and Polynomials of the Second Kind

For $k\in z$, we consider the Hermite poly-Bernoulli polynomials ${}_Hb_n^{(k)}(x,y)$ of the second kind

$$\frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y=\sum_{n=0}^{\infty}{}_Hb_n^{(k)}(x,y)\frac{t^n}{n!}\quad(2.1)$$

so that

$${}_Hb_n^{(k)}(x,y)=\sum_{m=0}^{n}\binom{n}{m}b_{n-m}^{(k)}H_m(x,y)\quad(2.2)$$

when $x=y=0$, $b_n^{(k)}=b_n^{(k)}(0,0)$ are called the poly-Bernoulli numbers of the second kind. Indeed, for $k=1$ in (2.1), we have

$$\frac{Li_1\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y=\sum_{n=0}^{\infty}{}_Hb_n(x,y)\frac{t^n}{n!}.\quad(2.3)$$

Thus by (2.1) and (2.3), we get

$${}_Hb_n^{(k)}(x,y)={}_Hb_n(x,y),(n\geq 0).$$

For $y=0$ in (2.1), the result reduces to the poly-Bernoulli polynomials of the second kind Kim et al [[26], p.Eq.(7)2] is defined as

$$\frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n^{(k)}(x)\frac{t^n}{n!},(k\in z).\quad(2.4)$$

**Theorem 2.1.** For $n\geq 0$, we have

$${}_Hb_n^{(2)}(x,y)=\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}{}_Hb_{n-m}(x,y).\quad(2.5)$$

**Proof.** Applying Definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{Li_1\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y$$

$$= \frac{(1+t)^x\left(1+t^2\right)^y}{\log(1+t)}$$

$$\int_0^t \frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t \frac{z}{e^z-1}dzdz\cdots dz$$

In particular $k=2$, we have

$$\sum_{n=0}^{\infty} {}_H b_n^{(2)}(x,y)\frac{t^n}{n!} = \frac{(1+t)^x\left(1+t^2\right)^y}{\log(1+t)}\int_0^t \frac{z}{e^z-1}dz$$

$$= \left(\frac{t}{\log(1+t)}\right)(1+t)^x\left(1+t^2\right)^y\left(\sum_{m=0}^{\infty}\frac{t^m B_m}{(m+1)m!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_H b_n(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m B_m}{(m+1)m!}\right).$$

Replacing n by n-m in above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n}{m}\frac{B_m}{m+1}\, {}_H b_{n-m}(x,y)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in the above equation, we get the result (2.5).

**Remark 1.** For $y=0$ in Theorem (2.1), the result reduces to known result of Kim et al [[26], p. 3, Theorem (2.1)].

**Corollary 1.** For $n \geq 0$, we have

$$b_n^{(2)}(x) = \sum_{n=0}^{m}\binom{n}{m}\frac{B_m}{m+1}b_{n-m}(x). \tag{2.6}$$

**Theorem 2.2.** For $n \geq 0$, we have

$$\begin{aligned} &{}_H b_n^{(k)}(x,y) \\ &= \sum_{p=0}^{n}\binom{n}{p}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}\, l!\, S_2(p+1,l)}{l^k(p+1)}\right) {}_H b_{n-p}(x,y) \end{aligned} \tag{2.7}$$

**Proof.** From equation (2.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y \\ &= \frac{t}{\log(1+t)}\frac{Li_k\left(1-e^{-t}\right)}{t}(1+t)^x\left(1+t^2\right)^y. \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} &\frac{1}{t}Li_k\left(1-e^{-t}\right) = \frac{1}{t}\sum_{l=1}^{\infty}\frac{\left(1-e^{-t}\right)^l}{l^k} = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}\left(1-e^{-t}\right)^l \\ &= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}l!\sum_{p=l}^{\infty}(-1)^p S_2(p,l)\frac{t^p}{p!} \\ &= \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!\, S_2(p,l)\frac{t^p}{p!} \\ &= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}. \end{aligned} \tag{2.9}$$

Thus by equations (2.3), (2.8) and (2.9), we get

$$\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\sum_{n=0}^{\infty} {}_H b_n(x,y)\frac{t^n}{n!}.$$

Replacing n by n-p in the r.h.s of above equation and comparing the coefficients of $t_n$, we get the result (2.7).

**Remark 2.** For $y=0$ in Theorem (2.2), the result reduces to known result of Kim et al [[26].,p. 4, Theorem (2.2)].

**Corollary 2.** For $n \geq 0$, we have

$$\begin{aligned} &b_n^{(k)}(x) \\ &= \sum_{p=0}^{n}\binom{n}{p}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}\, l!\, S_2(p+1,l)}{l^k(p+1)}\right)b_{n=p}(x) \end{aligned} \tag{2.10}$$

**Theorem 2.3.** For $n \geq 1$, we have

$$\begin{aligned} &{}_H b_n^{(k)}(x+1,y) - {}_H b_n^{(k)}(x,y) \\ &= \sum_{p=1}^{n}\sum_{l=1}^{p}\binom{n}{p}\frac{(-1)^{l+p}}{l^k}l!\, S_2(p,l)\, {}_H b_{n-p}(x,y) \end{aligned} \tag{2.11}$$

**Proof.** Using the Definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H b_n^{(k)}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H b_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$= \frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^{x+1}\left(1+t^2\right)^y$$

$$- \frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y$$

$$= \frac{tLi_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y$$

$$= \left(\frac{t}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y\right)$$

$$\sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!\, S_2(p,l)\right)\frac{t^p}{p!}$$

$$= \left(\sum_{n=0}^{\infty} {}_H b_n(x,y)\frac{t^n}{n!}\right)$$

$$\left(\sum_{p=1}^{\infty}\left(\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!\, S_2(p,l)\right)\frac{t^p}{p!}\right).$$

Replacing n by n-p in the above equation and comparing the coefficients of $t^n$, we get the result (2.11).

**Remark 3.** For $y=0$ in Theorem (2.3), the result reduces to known result of Kim et al [[26].,p. 4, Theorem (2.3)].

**Corollary 3.** For $n \geq 1$, we have

$$\begin{aligned} &b_n^{(k)}(x+1) - b_n^{(k)}(x) \\ &= \sum_{p=1}^{n}\sum_{l=1}^{p}\binom{n}{p}\frac{(-1)^{l+p}}{l^k}l!\, S_2(p,l)b_{n-p}(x). \end{aligned} \tag{2.12}$$

## 3. Implicit Summation Formulae Involving Hermite poly-Bernoulli Num-bers and Polynomials of the Second Kind

For the derivation of implicit formulae involving poly-Bernoulli polynomials of the second kind $b_n^{(k)}(x)$ and Hermite poly-Bernoulli polynomials of the second kind ${}_Hb_n^{(k)}(x,y)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [21] and Hermite-Bernoulli polynomials in Pathan and Khan [29-34] holds as well. First we prove the following results involving Hermite poly-Bernoulli polynomials of the second kind ${}_Hb_n^{(k)}(x,y)$.

**Theorem 3.1.** For $x, y \in R$ and $n \geq 0$. Then

$$ {}_Hb_n^{(k)}(x+u,y)=\sum_{j=0}^{n}\binom{n}{j}(-1)^j(-u)_j\,{}_Hb_{n-j}^{(k)}(x,y) \quad (3.1)$$

**Proof.** Since

$$\sum_{n=0}^{\infty}{}_Hb_n^{(k)}(x+u,y)\frac{t^n}{n!}$$

$$=\frac{Li_k\left(1-(e)^{-t}\right)}{\log(1+t)}(1+t)^{(x+u)}\left(1+t^2\right)^y$$

$$=\left(\sum_{n=0}^{\infty}{}_Hb_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}(-1)^j(-u)_j\frac{t^j}{j!}\right).$$

Now replacing n by n-j and comparing the coefficients of $t^n$, we get the result (3.1).

**Remark.** Set $u = 1$ in the above theorem to get

**Corollary.** For $x, y \in R$ and $n \geq 0$. Then

$${}_Hb_n^{(k)}(x+1,y)=\sum_{j=0}^{n}\binom{n}{j}(-1)^j\,{}_Hb_{n-j}^{(k)}(x,y). \quad (3.2)$$

**Theorem 3.2.** For $x, y \in R$ and $n \geq 0$. Then

$$\begin{aligned}&{}_Hb_n^{(k)}(x+u,y+w)\\&=\sum_{m=0}^{n}\binom{n}{m}{}_Hb_{n-m}^{(k)}(x,y)H_m(u,w).\end{aligned} \quad (3.3)$$

**Proof.** By the definition of poly-Bernoulli polynomials of the second kind and the definition (1.2), we have

$$\frac{Li_k\left(1-(e)^{-t}\right)}{\log(1+t)}(1+t)^{(x+u)}\left(1+t^2\right)^{(y+w)}$$

$$=\left(\sum_{n=0}^{\infty}{}_Hb_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(u,w)\frac{t^m}{m!}\right).$$

Now replacing n by n-m and comparing the coefficients of $t^n$, we get the result (3.3).

**Theorem 3.3.** For $x, y \in R$ and $n \geq 0$. Then

$$H^{b_n^{(k)}}(x,y)=\sum_{m=0}^{n-2j}\sum_{j=0}^{\left[\frac{n}{2}\right]}\left[\begin{matrix}b_m^{(k)}(-y)_j(-1)^{n-m-j}\\(-x)_{n-m-2j}\dfrac{n!}{m!\,j!(n-m-2j)!}\end{matrix}\right]. \quad (3.4)$$

**Proof.** Applying the definition (2.1) to the term $\frac{Li_k\left(1-(e)^{-t}\right)}{\log(1+t)}$ and expanding the exponential function $(1+t)^x\left(1+t^2\right)^y$ at $t = 0$ yields

$$\frac{Li_k\left(1-(e)^{-t}\right)}{\log(1+t)}(1+t)^x\left(1+t^2\right)^y$$

$$=\left(\sum_{m=0}^{\infty}b_m^{(k)}(x,y)\frac{t^m}{m!}\right)$$

$$\left(\sum_{n=0}^{\infty}(-x)_n(-1)^n\frac{t^n}{n!}\right)$$

$$\left(\sum_{j=0}^{\infty}(-y)_j(-1)^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n-m}\binom{n}{m}b_m^{(k)}(-1)^{n-m}(-x)_{n-m}\frac{t^n}{n!}\sum_{j=0}^{\infty}(-y)_j\frac{t^{2j}}{j!}.$$

Replacing n by n-2j, we have

$$\sum_{n=0}^{\infty}H^{b_n^{(k)}}(x,y)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n-2j}\sum_{j=0}^{\left[\frac{n}{2}\right]}\left[\begin{matrix}b_m^{(k)}(-y)_j(-1)^{n-m-j}\\(-x)_{n-m-2j}\dfrac{t^n}{m!\,j!(n-m-2j)!}\end{matrix}\right]. \quad (3.5)$$

Equating their coefficients of $t^n$, we get the result (3.4).

**Theorem 3.4.** For $x, y \in R$ and $n \geq 0$. Then

$${}_Hb_n^{(k)}(x,y)=\sum_{r=0}^{n}\binom{n}{r}(-1)^r(-z)_r\,{}_Hb_{n-r}^{(k)}(x-z,y) \quad (3.6)$$

**Proof.** Use the definition (2.1), we have

$$\frac{Li_k\left(1-e^{-t}\right)}{\log(1+t)}(1+t)^{x-z}\left(1+t^2\right)^y(1+t)^z$$

$$=\left(\sum_{n=0}^{\infty}{}_Hb_n^{(k)}(x-z,y)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}(-1)^r(-z)_r\frac{t^r}{r!}\right).$$

Replacing n by n-r in the above equation and comparing the coefficients of $t^n$, we get the result (3.6).

## 4. General Symmetry Identity

In this section, we give general symmetry identity for the poly-Bernoulli polynomials of the second kind $b_n^{(k)}(x)$ and the Hermite poly-Bernoulli polynomials of the second kind ${}_Hb_n^{(k)}(x,y)$ by applying the generating function(2.1) and (2.4). The results extend some known identities of Zhang and Yang [44], Yang [[43], Eqs.(9)], Khan [23,24,25] and Pathan et al [29-34].

**Theorem 4.1.** Let $a, b > 0$ and $a \neq b$. For $x, y \in R$ and $n \geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}\,{}_HG_{n-m}^{(k)}\left(bx,b^2y\right){}_HG_m^{(k)}\left(ax,a^2y\right)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}\,{}_HG_{n-m}^{(k)}\left(ax,a^2y\right){}_HG_m^{(k)}\left(bx,b^2y\right) \tag{4.1}$$

**Proof.** Start with

$$g(t)=\left(\frac{\left(Li_k\left(1-e^{-t}\right)\right)^2}{\left(\log(1+at)\right)\left(\log(1+bt)\right)}\right)\times(1+abt)^x\left(1+a^2b^2t^2\right)^y. \tag{4.2}$$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$g(t)=\frac{1}{ab}\sum_{n=0}^{\infty}{}_Hb_n^{(k)}\left(bx,b^2y\right)\frac{(at)^n}{n!}\times\sum_{m=0}^{\infty}{}_Hb_m^{(k)}\left(ax,a^2y\right)\frac{(bt)^m}{m!}$$
$$=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\binom{n}{m}a^{n-m}b^m\,{}_Hb_{n-m}^{(k)}\left(bx,b^2y\right){}_Hb_m^{(k)}\left(ax,a^2y\right)t^n\right)$$

On the similar lines we can show that

$$g(t)=\frac{1}{ab}\sum_{n=0}^{\infty}{}_Hb_n^{(k)}\left(ax,a^2y\right)\frac{(at)^n}{n!}\times\sum_{m=0}^{\infty}{}_Hb_m^{(k)}\left(bx,b^2y\right)\frac{(bt)^m}{m!}$$
$$=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\binom{n}{m}a^{m}b^{n-m}\,{}_Hb_{n-m}^{(k)}\left(ax,a^2y\right){}_Hb_m^{(k)}\left(bx,b^2y\right)t^n\right)$$

Comparing the coefficients of $t^n$ on the right hand sides of the last two equations we arrive the desired result.

**Remark 1.** By setting $b = 1$ in Theorem 4.1, we immediately following result

**Corollary.**

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m\,{}_Hb_{n-m}^{(k)}(x,y)\,{}_Hb_m^{(k)}\left(ax,a^2y\right)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^m\,{}_Hb_{n-m}^{(k)}\left(ax,a^2y\right){}_Hb_m^{(k)}(x,y). \tag{4.3}$$

# 5. Conclusion

Based on the definition of Hermite polynomials and poly logarithmic function, we introduced a new class of Hermite poly-Bernoulli numbers and polynomials of the second kind. By using Jolany's methods introduced in [17] and [18], we gave Hermite poly-Bernoulli numbers and polynomials of the second kind with two variable, and also we analyse its behaviors including general symmetric properties.

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