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Some Curvature Properties on a Special Paracontact Kenmotsu Manifold with Respect to Semi-Symmetric Connection

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Abstract The object of the present paper is to study some properties of curvature tensor $\tilde{R}$ of a semi-symmetric non-metric connection $\tilde{\nabla}$ in a type of special paracontact Kenmotsu (briefly SP-Kenmotsu) manifold. We have deduced the expressions for curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ of M_n with respect to semi-symmetric non-metric connection $\tilde{\nabla}$. It is proved that in an SP-Kenmotsu manifold if the curvature tensor of the semi-symmetric non-metric connection vanishes then the manifold is projectively flat.

Keywords: *curvature tensor, ricci tensor, projective curvature tensor, non-metric connection, sp-kenmotsu manifold*

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1. Introduction

Friedmann and Schouten [1,2] introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden [3] introduced semi-symmetric metric connection on a Riemannian manifold and it was further developed by Yano [4]. Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [1]. In 1975, Prvanovi´c [5] introduced the concept of semi-symmetric non-metric connection with the name pseudo-metric, which was further studied by Andonie [6,7]. The study of semi-symmetric non-metric connection is much older than the nomenclature ’non-metric’ was introduced. In 1992, Agashe and Chafle [8] introduced a semi-symmetric connection $\tilde{\nabla}$ satisfying $\tilde{\nabla}_X g \neq 0$ on a Riemannian manifold, and called such a connection as semi-symmetric non-metric connection. Later, the curvature properties of the connection in an SP-Sasakian manifold were studied by Bhagwat Prasad [9], and many others.

On the other hand, in 1976, Sato [10] defined the notions of an almost paracontact Riemannian manifold. After that, T. Adati and K. Matsumoto [11] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, in 1972, Kenmotsu [12] defined a class of almost contact Riemannian manifolds satisfying some special conditions. In 1995, Sinha and Sai Prasad [13] have defined a class of almost paracontact metric manifolds namely para Kenmotsu (briefly P-Kenmotsu) and special para Kenmotsu (briefly SP-Kenmotsu) manifolds.

In 1970, Pokhariyal and Mishra [14] have introduced new tensor fields, called W and E-tensor fields in a Riemannian manifold and studied their properties. In the present paper, we consider the W-curvature tensor of a semi-symmetric non-metric connection and obtained a relation connecting the curvature tensors of Mn with respect to semi-symmetric non-metric connection and the Riemannian connection. It is proved that in an SP-Kenmotsu manifold if the curvature tensor of the semi-symmetric non-metric connection vanishes then the manifold is protectively flat.

Let M_n be an n-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1, 1), ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \tag{1.1}$$

$$\Phi^2(X) = X - \eta(X)\xi; \overline{X} = \Phi X \tag{1.2}$$

Then M_n is called an almost paracontact manifold.

Let g be the Riemannian metric in an n-dimensional almost paracontact manifold M_n such that

$$g(X,\xi) = \eta(X) \tag{1.3}$$

$$\Phi\xi = 0, \eta(\Phi X) = 0, \text{rank } \Phi = n-1 \tag{1.4}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.5}$$

for all vector fields X and Y on M_n. Then the manifold M_n [10] is said to admit an almost paracontact Riemannian structure (Φ , ξ, η, g) and the manifold is called an almost paracontact Riemannian manifold.

A manifold M_n with Riemannian metric 'g' admitting a tensor field Φ of type (1, 1), a vector field ξ and 1-form η satisfying equations (1.1), (1.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \quad (1.6)$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z) \quad (1.7)$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \quad (1.8)$$

is called a para Kenmotsu manifold or briefly P-Kenmotsu manifold [13], where ∇ is the covariant differentiation with respect to g.

It is known that [13] on a P-Kenmotsu manifold the following relations hold:

$$Ric(X,\xi) = -(n-1)\eta(X) \quad (1.9)$$

$$g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)] = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \quad (1.10)$$

where R is the Riemannian curvature.

Let (M_n, g) be an n-dimensional Riemannian manifold admitting a tensor field Φ of type (1, 1), a vector field ξ and 1-form η satisfying

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) \quad (1.11)$$

$$g(X,\xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \phi(\overline{X},Y), \text{ where } \phi \text{ is an associate of } \Phi \quad (1.12)$$

is called a special para Kenmotsu manifold or briefly SP-Kenmotsu manifold [13].

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M_n is said to be semi-symmetric connection if its torsion tensor T satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y. \quad (1.13)$$

A semi-symmetric non-metric connection $\tilde{\nabla}$ in an almost paracontact metric manifold with torsion tensor (1.13) is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X \quad (1.14)$$

where ∇ is a Riemannian connection with respect to metric g [8].

Apart from conformal curvature tensor, the projective curvature tensor is an other important tensor from the differential geometric point of view. The Weyl-projective curvature tensor W of type (1, 3) of a Riemannian manifold M_n with respect to the Riemannian connection is defined by [14]

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\begin{bmatrix} Ric(Y,Z)X \\ -Ric(X,Z)Y \end{bmatrix} \quad (1.15)$$

for X, Y, Z $\in T(M)$, where R is the curvature tensor and *Ric* is the Ricci tensor. If there exists a one-to-one correspondence between each coordinate neighbourhood of a Riemannian manifold M_n and a domain in Eucledian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Eucledian space, then M_n is said to be locally projectively flat. For $n \geq 3$, M_n is locally projectively flat if and only if the projective curvature tensor W vanishes. For n = 2, the projective curvature tensor identically vanishes.

2. Curvature Tensor

The manifold M_n is considered to be an SP-Kenmotsu manifold. Let us denote the curvature tensor of the semi-symmetric non-metric connection $\tilde{\nabla}$ by $\tilde{R}$ and the curvatre tensor of ∇ by R. By straight forward calculation, we get

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + (\tilde{\nabla}_X \eta)(Z)Y - (\tilde{\nabla}_Y \eta)(Z)X. \quad (2.1)$$

As a consequence of equations (1.11) and (1.14), equation (2.1) reduces to

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + g(X,Z)Y - g(Y,Z)X \quad (2.2)$$

which is the relation between the curvature tensors of M_n with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ and the Riemannian connection ∇.

It is well known that a Riemannian manifold is of constant curvature if and only if it is projectively flat or conformally flat [15] and in general, the necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold [16].

As an example, if M_n is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M_n is projectively flat [8]. Analogus to this, we prove the following for an SP-Kenmotsu manifold which is Riemannian.

Theorem 2.1: If in an SP-Kenmotsu manifold M_n the curvature tensor of a semi-symmetric non-metric connection $\tilde{\nabla}$ vanishes, then the manifold is projectively flat.

Proof: Since $\tilde{R} = 0$, then equation (2.2) gives

$$R(X,Y,Z) = g(Y,Z)X - g(X,Z)Y. \quad (2.3)$$

On contracting the above equation, we get

$$Ric(Y,Z) = (n-1)g(Y,Z). \quad (2.4)$$

Then, by (2.3) and (2.4), we get

$$R(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y] = 0 \quad (2.5)$$

or W = 0 from (1.15), proves that the manifold is projectively flat.

Theorem 2.2: If in an SP-Kenmotsu manifold the Ric tensor of a semi-symmetric non-metric connection $\tilde{\nabla}$ vanishes, then the curvature tensor of $\tilde{\nabla}$ is equal to the projective curvature tensor of the manifold M_n.

Proof: From equation (2.2), we have

$$\tilde{R}(X,Y,Z,U) = R(X,Y,Z,U) + g(X,Z)g(Y,U) - g(Y,Z)g(X,U). \quad (2.6)$$

On contracting the above equation, we get

$$'Ric(Y,Z) = Ric(Y,Z) - (n-1)g(Y,Z). \quad (2.7)$$

Since $'Ric = 0$, we have

$$g(Y,Z) = \frac{1}{n-1}[Ric\,(Y,Z)]. \quad (2.8)$$

From equations (2.2) and (2.8), we have $\tilde{R} = W$.

Theorem 2.3: In an SP-Kenmotsu manifold the projective curvature tensor of a semi-symmetric non-metric connection $\tilde{\nabla}$ is equal to the projective curvature tensor of the manifold.

Proof: From equations (2.2) and (2.7), we get

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}\begin{bmatrix} 'Ric\,(Y,Z) \\ -Ric\,(Y,Z) \end{bmatrix} X - \frac{1}{n-1}['Ric\,(X,Z) - Ric\,(X,Z)]Y. \quad (2.9)$$

The terms of the equation (2.9) can be rearranged as

$$\tilde{R}(X,Y,Z) - \frac{1}{n-1}['Ric\,(Y,Z)X - 'Ric\,(X,Z)Y] = R(X,Y,Z) - \frac{1}{n-1}[Ric\,(Y,Z)X - Ric\,(X,Z)Y] \quad (2.10)$$

which is ‘W = W, where ‘W is the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection.

Theorem 2.4: In an SP-Kenmotsu manifold with semi-symmetric non-metric connection $\tilde{\nabla}$ we have

a) $\tilde{R}(X,Y,Z) + \tilde{R}(Y,Z,X) + \tilde{R}(Z,X,Y) = 0$

b) $'\tilde{R}(X,Y,Z,U) + '\tilde{R}(X,Y,U,Z) = 0$

Proof: Using the Bianchi’s first identity with respect to the Riemannian connection equation (2.2) gives (a). From equation (2.6) we get (b).

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Competing Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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