

Diagonal Function of k-Lucas Polynomials

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Abstract The Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, Diagonal function of k-Lucas Polynomials is introduced and defined by $G_{n+1}(x) = kxG_n(x) + G_{n-1}(x), n \geq 1$, with $G_0(x) = 2$, and $G_1(x) = 1$. Some Lucas Polynomials, rising & descending diagonal function and generating matrix established and derived by standard methods.

Keywords: Lucas Polynomials, rising diagonal function, descending diagonal function and generating matrix

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$$P_{n+1}(x) = kxP_n(x) + P_{n-1}(x), n \geq 1$$

$$P_0 = k, P_1 = kx. \quad (x \neq 0) \quad (1.3)$$

1. Introduction

The sequence 1,1,2,3,5..., got its name as name as Fibonacci sequence by the Famous Mathematics Francois Edouard Lucas in 1876 [7].

Lucas also discovered a new Fibonacci like sequence with different initial condition call it, Lucas Sequence

$$L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

with initial condition $L_0 = 2, L_1 = 1$.

In 1965 Hoggatt, V.E. [5] has defined Lucas polynomials by recurrence relation.

$L_{n+1}(x) = xL_n(x) + L_{n-1}(x)$, where

$$L_0(x) = 2, L_1(x) = x, \quad (1.1)$$

The first few Lucas polynomials are

$$L_1(x) = 1.x^1,$$

$$L_2(x) = 1.x^2 + 2.x^0,$$

$$L_3(x) = 1.x^3 + 3.x^1,$$

$$L_4(x) = 1.x^4 + 4.x^2 + 2.x^0,$$

$$L_5(x) = 1.x^5 + 5.x^2 + 5.x^1,$$

$$L_6(x) = 1.x^6 + 6.x^4 + 9.x^2 + 2.x^0,$$

$$L_7(x) = 1.x^7 + 7.x^5 + 4.x^3 + 7.x^1, \dots$$

In this paper, we are using the pair of sequence $\{G_n\}$ and $\{P_n\}$ for which,

$$G_{n+1}(x) = kxG_n(x) + G_{n-1}(x), n \geq 1. \quad (1.2)$$

$$G_0 = 2, G_1 = 1 \quad (x \neq 0)$$

where k is any positive integer. $k=0, 1, 2, 3, \dots$

Using the equation (1.1) and (1.2) and made a rising diagonal function and descending Diagonal Functions.

2. Sequence $\{G_n\}$ and $\{P_n\}$

We have the pair of sequence $\{G_n\}$ and $\{P_n\}$ for which,

$$G_{n+1}(x) = kxG_n(x) + G_{n-1}(x), n \geq 1.$$

$$G_0 = 2, G_1 = 1 \quad (x \neq 0),$$

$$P_{n+1}(x) = kxP_n(x) + P_{n-1}(x), n \geq 1.$$

$$P_0 = k, P_1 = kx. \quad (x \neq 0)$$

The first few terms of the sequence $\{G_n\}$ are

$$2$$

$$1$$

$$kx + 2$$

$$k^2x^2 + 2kx + 1$$

$$k^3x^3 + 2k^2x^2 + 2kx + 2$$

$$k^4x^4 + 2k^3x^3 + 3k^2x^2 + 4kx + 1$$

$$k^5x^5 + 2k^4x^4 + 4k^3x^3 + 6k^2x^2 + 3kx + 2 \quad (2.1)$$

$$k^6x^6 + 2k^5x^5 + 5k^4x^4 + 8k^3x^3 + 6k^2x^2 + 6kx + 1$$

$$k^7x^7 + 2k^6x^6 + 6k^5x^5 + 10k^4x^4$$

$$+ 10k^3x^3 + 12k^2x^2 + 4kx + 2$$

$$k^8x^8 + 2k^7x^7 + 7k^6x^6 + 12k^5x^5 + 15k^4x^4$$

$$+ 20k^3x^3 + 10k^2x^2 + 8kx + 1.$$

The first few terms of the sequence $\{P_n\}$ are

$$\begin{aligned}
& k \\
& kx \\
& k^2x^2 + k \\
& k^3x^3 + k^2x + kx \\
& k^4x^4 + k^3x^2 + 2k^2x^2 + k \\
& k^5x^5 + k^4x^3 + 3k^3x^3 + 2k^2x + kx, \\
& k^6x^6 + k^5x^4 + 4k^4x^4 + 3k^3x^2 + 3k^2x^2 + k \\
& k^7x^7 + k^6x^5 + 5k^5x^5 + 4k^4x^3 \\
& \quad + 6k^3x^3 + 3k^2x + kx \\
& k^8x^8 + k^7x^6 + 6k^6x^6 + 5k^5x^4 \\
& \quad + 10k^4x^4 + 6k^3x^2 + 4k^2x^2 + k.
\end{aligned} \tag{2.2}$$

3. Rising Diagonal Function

Consider the rising diagonal function of x , $U_n(x)$, $u_n(x)$ for (2.1) and (2.2) respectively,

$$\begin{aligned}
U_1(x) &= 1 \\
U_2(x) &= kx \\
U_3(x) &= k^2x^2 + 2 \\
U_4(x) &= k^3x^3 + 2kx \\
U_5(x) &= k^4x^4 + 2k^2x^2 + 1 \\
U_6(x) &= k^5x^5 + 2k^3x^3 + 2kx \\
U_7(x) &= k^6x^6 + 2k^4x^4 + 3k^2x^2 + 2 \\
U_8(x) &= k^7x^7 + 2k^5x^5 + 4k^3x^3 + 4kx \\
U_9(x) &= k^8x^8 + 2k^6x^6 + 5k^4x^4 + 6k^2x^2 + 1.
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
u_1(x) &= k \\
u_2(x) &= kx \\
u_3(x) &= k^2x^2 \\
u_4(x) &= k^3x^3 + k \\
u_5(x) &= k^4x^4 + k^2x \\
u_6(x) &= k^5x^5 + k^3x^3 + kx \\
u_7(x) &= k^6x^6 + k^4x^3 + 2k^2x^2 \\
u_8(x) &= k^7x^7 + k^5x^4 + 3k^3x^3 + k \\
u_9(x) &= k^8x^8 + k^6x^5 + 4k^4x^4 + 2k^2x^2
\end{aligned} \tag{3.2}$$

Now, we define

$$U_0(x) = u_0(x) = 0. \tag{3.3}$$

from equation (3.1), (3.2) and (3.3) we get the following theorem:

Theorem (1). If $U_n(x)$ and $u_n(x)$ are rising diagonal functions of x for sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for, $n \geq 4$

$$U_n(x) = kxU_{n-1}(x) + U_{n-4}(x). \tag{3.4}$$

Proof can be obtained by PMI's method so it is obvious.

Special Case-I

If $U_n(x)$ and $u_n(x)$ are rising diagonal functions of x for sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for $n=3$, $n=4$.

$$u_n(x) = kxu_{n-1}(x) + u_{n-3}(x). \tag{3.5}$$

4. Descending Diagonal Function

From (2.1) and (2.2), the descending diagonal function of x , $Q_i(x)$, $q_i(x)$ are

$$\begin{aligned}
Q_1(x) &= 1 \\
Q_2(x) &= kx + 1 \\
Q_3(x) &= (kx + 1)^2 \\
Q_4(x) &= (kx + 1)^3 \\
Q_5(x) &= (kx + 1)^4 \\
Q_6(x) &= (kx + 1)^5 \\
Q_7(x) &= (kx + 1)^6 \\
Q_8(x) &= (kx + 1)^7.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
q_1(x) &= k \\
q_2(x) &= kx + k \\
q_3(x) &= (kx + k)(kx + 1) \\
q_4(x) &= (kx + k)(kx + 1)^2 \\
q_5(x) &= (kx + k)(kx + 1)^3 \\
q_6(x) &= (kx + k)(kx + 1)^4 \\
q_7(x) &= (kx + k)(kx + 1)^5 \\
q_8(x) &= (kx + k)(kx + 1)^6
\end{aligned} \tag{4.2}$$

Now, we define

$$Q_0(x) = q_0(x) = 0. \tag{4.3}$$

from (4.1), (4.2) and (4.3) we get for $n \geq 2$.

$$Q_n(x) = (kx + 1)Q_{n-1} = (kx + 1)^{n-1}. \tag{4.4}$$

$$q_n(x) = (kx + 1)q_{n-1}. \tag{4.5}$$

from (4.4) and (4.5) we get the following theorem:

Theorem (2). If $Q_n(x)$ and $q_n(x)$ are descending diagonal function of x for Sequence $\{G_n\}$ and $\{P_n\}$ respectively, than for $n > 2$.

$$\begin{aligned}
a) \quad \frac{Q_n}{Q_{n-1}} &= \frac{q_n}{q_{n-1}} = (kx + 1). \\
b) \quad \frac{Q_n}{q_n} &= \frac{(kx + 1)}{(kx + k)}.
\end{aligned}$$

5. Generating Matrix

For the sequence $\{G_n\}$ defined in equation (1.1) we consider the matrix

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.1}$$

Since, the elements of this matrix are the member of the sequence of Fibonacci Polynomials. We call this matrix as Fibonacci matrix.

Theorem (3). For sequence $\{G_n\}$ we define $n \geq 1, p \geq 0$.

$$G_{n+p}(x) = G_{n+1}(x)G_p(x) + G_n(x)G_{p-1}(x)$$

$$G_{n+p}(x) = G_n(x)G_{p+1}(x) + G_{n-1}(x)G_p(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

Since, determinant of matrix A is -1, therefore,

$$\det A^n = (\det A)^n \tag{5.2}$$

$$\det A^n = (-1)^n$$

$$A^n = \begin{bmatrix} G_{n+1}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix} \tag{5.3}$$

Form equation (3.5.2) and (3.5.3), we get

$$G_{n+1}(x)G_{n-1}(x) - G_n^2(x) = (-1)^n$$

Since $A^{n+p} = A^n A^p$

$$\begin{bmatrix} G_{n+p+1}(x) & G_{n+p}(x) \\ G_{n+p}(x) & G_{n+p-1}(x) \end{bmatrix} = \begin{bmatrix} G_{n+1}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix} \begin{bmatrix} G_{p+1}(x) & G_p(x) \\ G_p(x) & G_{p-1}(x) \end{bmatrix}$$

After multiplying the matrices and equating the corresponding elements, we get

$$G_{n+p}(x) = G_{n+1}(x)G_p(x) + G_n(x)G_{p-1}(x)$$

$$G_{n+p}(x) = G_n(x)G_{p+1}(x) + G_{n-1}(x)G_p(x).$$

Theorem (4). For sequence $\{G_n\}$ we define $n \geq 1, p \geq 0$

$$G_n(x) = G_{n+p+1}(x)G_{-n}(x) + G_{n+p}(x)G_{-(p+1)}(x)$$

$$G_n(x) = G_{n+p}(x)G_{-p+1}(x) + G_{n+p-1}(x)G_{-p}(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

If A is any square Matrix, then we know that

$$AA^{-1} = I \tag{5.4}$$

Where I is identity matrix from equation (5.4) we get

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & -kx \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} G_0(x) & G_{-1}(x) \\ G_{-1}(x) & G_{-2}(x) \end{bmatrix}.$$

By Mathematical induction, we have

$$A^{-p} = \begin{bmatrix} G_{-(p-1)}(x) & G_{-p}(x) \\ G_{-p}(x) & G_{-(p+1)}(x) \end{bmatrix}$$

Since $A^n = A^{n+p}A^{-p}$

$$\begin{bmatrix} G_{n+p}(x) & G_n(x) \\ G_n(x) & G_{n-1}(x) \end{bmatrix} = \begin{bmatrix} G_{n+p+1}(x) & G_{n+p}(x) \\ G_{n+p}(x) & G_{n+p-1}(x) \end{bmatrix} \begin{bmatrix} G_{-(p-1)}(x) & G_{-p}(x) \\ G_{-p}(x) & G_{-(p+1)}(x) \end{bmatrix}$$

After multiplying the matrices and equating the corresponding elements, we get

$$G_n(x) = G_{n+p+1}(x)G_{-n}(x) + G_{n+p}(x)G_{-(p+1)}(x)$$

$$G_n(x) = G_{n+p}(x)G_{-p+1}(x) + G_{n+p-1}(x)G_{-p}(x).$$

6. Generating Matrix

For the sequence $\{P_n\}$ defined in equation (1.1) we consider the matrix

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix} \tag{6.1}$$

since, the elements of this matrix are the members of the sequence of Fibonacci polynomials. We call this matrix as Fibonacci Matrix.

Theorem (5). For sequence $\{P_n\}$ we define $n \geq 1, r \geq 0$

$$P_{n+r}(x) = P_{n+1}(x)P_r(x) + P_r(x)P_{r-1}(x)$$

$$P_{n+r}(x) = P_n(x)P_{r+1}(x) + P_{n-1}(x)P_r(x).$$

Proof. For sequence $\{P_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

Since, determinant of matrix A is -1, there for,

$$\det A^n = (\det A)^n \tag{6.2}$$

$$\det A^n = (-1)^n.$$

By mathematical induction

$$A^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix}. \tag{6.3}$$

Form equation (3.6.2) and (3.6.3), we get

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n$$

Since $A^{n+r} = A^n A^r$

$$\begin{aligned} & \begin{bmatrix} P_{n+r+1}(x) & P_{n+r}(x) \\ P_{n+r}(x) & P_{n+r-1}(x) \end{bmatrix} \\ &= \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \begin{bmatrix} P_{r+1}(x) & P_r(x) \\ P_r(x) & P_{r-1}(x) \end{bmatrix} \end{aligned}$$

After multiplying the matrices and equating the corresponding elements, we get

$$P_{n+r}(x) = P_{n+1}(x)P_r(x) + P_n(x)P_{r-1}(x).$$

$$P_{n+r}(x) = P_n(x)P_{r+1}(x) + P_{n-1}(x)P_r(x).$$

Theorem (6). For sequence $\{P_n\}$ we define $n \geq 1, r \geq 0$.

$$P_n(x) = P_{n+r+1}(x)P_{-r}(x) + P_{n+r}(x)P_{-(r+1)}(x).$$

$$P_n(x) = P_{n+r}(x)P_{-r+1}(x) + P_{n+r-1}(x)P_{-r}(x).$$

Proof. For sequence $\{G_n\}$, we have

$$A = \begin{bmatrix} kx & 1 \\ 1 & 0 \end{bmatrix}.$$

If A is any square Matrix, then we know that

$$AA^{-1} = I \quad (6.4)$$

Where I is identity matrix from equation (5.4) we get

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -kx \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} P_n(x) & P_{-1}(x) \\ P_{-1}(x) & P_{-2}(x) \end{bmatrix}$$

By mathematical indication, we have

$$A^{-r} = \begin{bmatrix} P_{-(r-1)}(x) & P_{-r}(x) \\ P_{-r}(x) & P_{-(r+1)}(x) \end{bmatrix}$$

Since $A^n = A^{n+r}A^{-r}$

$$\begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} = \begin{bmatrix} P_{n+r+1}(x) & P_{n+r}(x) \\ P_{n+r}(x) & P_{n+r-1}(x) \end{bmatrix} \begin{bmatrix} P_{-(r-1)}(x) & P_{-r}(x) \\ P_{-r}(x) & P_{-(r+1)}(x) \end{bmatrix}.$$

After multiplying the matrices and equating the corresponding elements, we get

$$P_n(x) = P_{n+r+1}(x)P_{-r}(x) + P_{n+r}(x)P_{-(r+1)}(x).$$

$$P_n(x) = P_{n+r}(x)P_{-r+1}(x) + P_{n+r-1}(x)P_{-r}(x).$$

7. Conclusions

In this paper Diagonal function k-Lucas Polynomials. Some basic rising diagonal function and descending diagonal function and generating matrix derived by standard method.

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