

# The Rogers-Ramanujan Identities

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**Abstract** In 1894, Rogers found the two identities for the first time. In 1913, Ramanujan found the two identities later and then the two identities are known as The Rogers-Ramanujan Identities. In 1982, Baxter used the two identities in solving the Hard Hexagon Model in Statistical Mechanics. In 1829 Jacobi proved his triple product identity; it is used in proving The Rogers-Ramanujan Identities. In 1921, Ramanujan used Jacobi's triple product identity in proving his famous partition congruences. This paper shows how to generate the generating function for  $C'(n)$ ,  $C_1'(n)$ ,  $C''(n)$  and  $C_1''(n)$ , and shows how to prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity. This paper shows how to prove the Remark 3 with the help of various auxiliary functions and shows how to prove The Rogers-Ramanujan Identities with help of Ramanujan's device of the introduction of a second parameter  $a$ .

**Keywords:** at most, auxiliary function, convenient, expansion, minimal difference, operator, Ramanujan's device

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## 1. Introduction

In this article, we give some related definitions of  $P(n)$ ,  $C'(n)$ ,  $P_m(n-m^2)$ ,  $C_1'(n)$ ,  $C''(n)$ ,  $P_m(n-m(m+1))$  and  $C_1''(n)$ . We describe the generating functions for  $C'(n)$ ,  $P_m(n-m^2)$ ,  $C_1'(n)$ ,  $C''(n)$ ,  $P_m(n-m(m+1))$  and  $C_1''(n)$ , and establish the Remarks 1 and 2 with numerical examples and also prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity [3]. We transfer the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter  $a$  [5],

i.e.,

$$G_k(a, x) = \prod_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1 - a^k x^{2kn}) C_n$$

to

$$G_1(x, x) = \sum_{m=0}^{\infty} (1 - x^{5m+1}) (1 - x^{5m+4}) (1 - x^{5m+5}),$$

where  $k=1$ , and  $a=x$ , it is used in proving The Rogers-Ramanujan Identity 1. We prove The Rogers-Ramanujan Identities with the help of auxiliary functions.

## 2. Some Related Definitions

$P(n)$  [7]: The number of partitions of  $n$  like: 4, 3+1, 2+2, 2+1+1, 1+1+1+1  $\therefore P(4)=5$ .

$C'(n)$  [6]: The number of partitions of  $n$  into parts each of which is of one of the forms  $5m+1$  and  $5m+4$ .

$P_m(n-m^2)$ : The number of partitions of  $n-m^2$  into  $m$  parts at most.

$C''(n)$ : The number of partitions of  $n$  into parts of the forms  $5m+2$  and  $5m+3$ .

$C_1'(n)$ : The number of partitions of  $n$  into parts without repetitions or parts whose minimal difference is 2.

$P_m(n-m(m+1))$ : The number of partitions of  $n-m(m+1)$  into  $m$  parts at most.

$C_1''(n)$ : The number of partitions of  $n$  into parts not less than 2 and with minimal difference 2.

## 3. Generating Functions for $C'(n)$ and $C''(n)$

In this section we describe the generating functions for  $C'(n)$  and  $C''(n)$  respectively. The generating function for  $C'(n)$  is of the form [5];

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})} \\
&= \frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9) \dots \infty} \quad (1) \\
&= 1+x+x^2+x^3+2x^4+2x^5+3x^6+\dots \infty \\
&= 1+\sum_{n=1}^{\infty} C'(n)x^n
\end{aligned}$$

where the coefficient  $C'(n)$  of  $x^n$  is the number of partitions of  $n$  into parts each of which is of one of these forms  $5m+1$  and  $5m+4$ .

Now we consider a special function, which is given below:

$$\begin{aligned}
& \frac{x^{m^2}}{(1-x)(1-x^2) \dots (1-x^m)} \\
&= x^{m^2} \sum_{n=m^2}^{\infty} P_m(n-m^2)x^{n-m^2} \\
&= \sum_{n=m^2}^{\infty} P_m(n-m^2)x^n
\end{aligned}$$

It is convenient to define  $P_m(0)=1$ . The coefficient  $P_m(n-m^2)$  of  $x^n$  in the above expansion is the number of partitions of  $n-m^2$  into  $m$  parts at most. Another special function, which is defined as;

$$\begin{aligned}
& 1+\sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2) \dots (1-x^m)} \\
&= 1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)} \\
& \quad +\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\dots \infty \quad (2) \\
&= 1+x+x^2+x^3+2x^4+2x^5 \\
& \quad +3x^6+3x^7+\dots \infty \\
&= 1+\sum_{n=1}^{\infty} C'_1(n)x^n
\end{aligned}$$

where the coefficient  $C'_1(n)$  is the number of partitions of  $n$  into parts without repetitions or parts, whose minimal difference is 2.

From (1) and (2) we can establish the following Remark:

**Remark 1:**

$$C'_1(11)=C'(11) \quad (3)$$

i.e., the number of partitions of  $n$  with minimal difference 2 is equal to the number of partitions of  $n$  into parts of the forms  $5m+1$  and  $5m+4$ .

**Example 1:** For  $n=11$ , there are 7 partitions of 11 that are enumerated by  $C'_1(n)$  of above statement, which are given below [6]:

$$11, 10+1, 9+2, 8+3, 7+4, 7+3+1, 6+4+1,$$

$$\therefore C'_1(11)=7.$$

There are 7 partitions of 11 are enumerated by  $C'(n)$  of above statement, which are given below:

$$11, 9+1+1, 6+4+1, 6+1+1+1+1+1,$$

$$4+4+1+1+1, 4+1+1+1+1+1+1,$$

$$1+1+1+1+1+1+1+1+1+1,$$

$$\therefore C'(11)=7.$$

$$\text{Hence, } C'_1(11)=C'(11).$$

We can conclude that,  $C'_1(11)=C'(11)$ .

$$\begin{aligned}
& 1+\sum_{n=1}^{\infty} C'(n)x^n = 1+\sum_{n=1}^{\infty} C'_1(n)x^n. \\
& 1+\sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2) \dots (1-x^m)} \\
&= \sum_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})},
\end{aligned}$$

which will be proved later as identity 1, it is known as The Rogers-Ramanujan identity 1.

The generating function for  $C''(n)$  is of the form [1];

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})} \\
&= \frac{1}{(1-x^2)(1-x^3)(1-x^7)(1-x^8) \dots \infty} \quad (4) \\
&= 1+0.x+x^2+x^3+x^4+x^5+2x^6+2x^7+\dots \infty \\
&= 1+\sum_{n=1}^{\infty} C''(n)x^n
\end{aligned}$$

where the coefficient  $C''(n)$  is the number of partitions of  $n$  into parts of the forms  $5m+2$  and  $5m+3$ .

Now we consider a special function, which is of the form [1];

$$\begin{aligned}
& \frac{x^{m(m+1)}}{(1-x)(1-x^2) \dots (1-x^m)} \\
&= x^{m(m+1)} \sum_{n=m(m+1)}^{\infty} P_m(n-m(m+1))x^{n-m(m+1)} \\
&= \sum_{n=m(m+1)}^{\infty} P_m(n-m(m+1))x^n,
\end{aligned}$$

where the coefficient  $P_m(n-m(m+1))$  of  $x^n$  in the above expansion is the number of partitions of  $n-m(m+1)$  into  $m$  parts at most.

Another special function, which is defined as;

$$\begin{aligned}
 & 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\dots(1-x^m)} \\
 &= 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} \\
 & \quad + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \dots \infty \tag{5} \\
 &= 1 + x^2 + x^3 + x^4 + x^5 + 2x^6 \\
 & \quad + 2x^7 + 3x^8 + \dots \infty \\
 &= 1 + \sum_{n=1}^{\infty} C_1''(n) x^n,
 \end{aligned}$$

where the coefficient  $C_1''(n)$  is the number of partitions of  $n$  into parts not less than 2 and with minimal difference 2.

From (4) and (5) we can establish the following Remark:

**Remarks 2:**

$$C_1''(n) = C''(n), \tag{6}$$

i.e., the number of partitions of  $n$  into parts not less than 2 and with minimal difference 2 is equal to the number of partitions of  $n$  into parts of the forms  $5m + 2$  and  $5m + 3$ .

**Example 2:** If  $n = 11$ , the four partitions of 11 into parts not less than 2 and with minimal difference 2 are given below:

$$11, 9 + 2, 8 + 3, 7 + 4.$$

Hence,  $C_1''(11) = 4$ .

Again the four partitions of 11 into parts of the form  $5m + 2$  and  $5m + 3$  are given as;

$$8 + 3, 7 + 2 + 2, 3 + 3 + 3 + 2, 3 + 2 + 2 + 2 + 2.$$

Hence,  $C''(11) = 4$ .

$$\therefore C_1''(11) = C''(11).$$

We can conclude that,  $C_1''(n) = C''(n)$ .

$$\text{i.e., } 1 + \sum_{m=1}^{\infty} C_1''(n) x^n = 1 + \sum_{m=1}^{\infty} C''(n) x^n$$

$$\begin{aligned}
 & 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\dots(1-x^m)} \\
 &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})},
 \end{aligned}$$

which will be proved later as identity 2, it is known as The Rogers-Ramanujan identity 2.

Now we give two Corollaries, which are related to the Jacobi's triple product identity [3].

**Corollary 1:**

$$\begin{aligned}
 & \prod_{n=0}^{\infty} (1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5}) \\
 &= \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}}.
 \end{aligned}$$

**Proof:** From Jacobi's Theorem [2] we have;

$$\begin{aligned}
 & \prod_{n=0}^{\infty} \left\{ (1-x^{2n})(1+x^{2n+1}z)(1+x^{2n-1}z^{-1}) \right\} \\
 &= \sum_{n=-\infty}^{\infty} x^{n^2} z^n,
 \end{aligned}$$

for all  $z$  except  $z = 0$ , if  $|x| < 1$ .

If we write  $x^{5/2}$  for  $x$ ,  $-x^{3/2}$  for  $z$  and replace  $n$  by  $n + 1$  on the left hand side we obtain;

$$\begin{aligned}
 & \prod_{n=0}^{\infty} (1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5}) \\
 &= 1 - x - x^4 + x^7 + x^{13} - \dots \infty \\
 &= \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+3)}{2}}.
 \end{aligned}$$

Hence, the Corollary.

**Corollary 2:**

$$\begin{aligned}
 & \prod_{n=0}^{\infty} (1-x^{5n+2})(1-x^{5n+3})(1-x^{5n+5}) \\
 &= \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+1)}{2}}.
 \end{aligned}$$

**Proof:** From Jacobi's Theorem we have;

$$\begin{aligned}
 & \prod_{n=0}^{\infty} (1-x^{2n})(1+x^{2n+1}z)(1+x^{2n-1}z^{-1}) \\
 &= \sum_{n=-\infty}^{\infty} x^{n^2} z^n,
 \end{aligned}$$

for all  $z$  except  $z = 0$ , when  $|x| < 1$ .

If we write  $x^{5/2}$  for  $x$ ,  $-x^{1/2}$  for  $z$  and replace  $n$  by  $n + 1$  on the left hand side we obtain;

$$\begin{aligned}
 & \prod_{n=0}^{\infty} (1-x^{5n+2})(1-x^{5n+3})(1-x^{5n+5}) \\
 &= 1 - x^2 - x^3 + x^9 + x^{11} - \dots \infty \\
 &= \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(5n+1)}{2}}.
 \end{aligned}$$

Hence the Corollary.

### 4. The Rogers-Ramanujan Identities

First we transfer the following auxiliary function into another auxiliary function. Let us consider the auxiliary function [1, 2] with  $|x| < 1$  and  $|a| < 1$ .

$$G_k(a, x) = \prod_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn}) C_n \tag{7}$$

it is known as Ramanujan's device of the introduction of a second parameter  $a$ , where  $k$  is 0, 1 or 2 and  $C_0 = 1$ ,

$$C_n = \frac{(1-a)(1-ax)\dots(1-ax^{n-1})}{(1-x)(1-x^2)\dots(1-x^n)}.$$

Hence,

$$G_k(a, x) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn})}{(1-x)(1-x^2)\dots(1-x^n)} \right]$$

$$\begin{aligned} & \frac{G_k(a, x)}{(1-a)(1-ax)\dots} \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn})}{(1-x)(1-x^2)\dots(1-x^n)} \times \frac{1-a^k x^{2kn}}{(1-ax^n)(1-ax^{n+1})\dots} \right] \\ &= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn}) P_n Q_n(a) \end{aligned}$$

where  $P_n = \prod_{r=1}^n \frac{1}{1-x^r}$ ,

$$Q_n(a) = \prod_{r=n}^{\infty} \frac{1}{1-ax^r} = H_k(a, x) \tag{8}$$

which is another auxiliary function, and it is used in proving The Rogers-Ramanujan Identities [1].

But from (7) we can easily verify that with  $k = 1, 2$  and  $a = x$ .

$$G_1(x, x) = 1 - x - x^4 + x^7 + x^{13} - \dots$$

$$G_1(x, x) = \prod_{n=0}^{\infty} (1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5}) \tag{9}$$

(by Corollary 1).

$$G_2(x, x) = 1 - x^2 - x^3 + x^9 + x^{11} - \dots$$

$$G_2(x, x) = \prod_{m=0}^{\infty} (1-x^{5m+2})(1-x^{5m+3})(1-x^{5m+5}) \tag{10}$$

(by Corollary 2).

From (8) we can also find that, if  $k=1$  and  $a = x$ , then;

$$\begin{aligned} H_1(x, x) &= \frac{G_1(x, x)}{(1-x)(1-x^2)(1-x^3)\dots} \\ &= \frac{\prod_{m=0}^{\infty} (1-x^{5m+1})(1-x^{5m+4})(1-x^{5m+5})}{(1-x)(1-x^2)(1-x^3)\dots} \tag{11} \\ &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}. \end{aligned}$$

Again for  $k = 2$  and  $a = x$ , we get;

$$\begin{aligned} H_2(x, x) &= \frac{G_2(x, x)}{(1-x)(1-x^2)(1-x^3)\dots} \\ &= \frac{\prod_{m=0}^{\infty} (1-x^{5m+2})(1-x^{5m+3})(1-x^{5m+5})}{(1-x)(1-x^2)(1-x^3)\dots} \tag{12} \\ &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})}. \end{aligned}$$

Now we can consider the following Remark [2].

**Remark 3:**  $H_k - H_{k-1} = a^{k-1} \eta H_{3-k}$ , where the operator  $\eta$  is defined by  $\eta f(a) = f(ax)$ , and  $k = 1$  or  $2$ .

**Proof:** From (8) we have;

$$\begin{aligned} H_k &= H_k(a, x) \\ &= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn}) P_n Q_n(a), \end{aligned}$$

where  $P_n = \prod_{r=1}^n \frac{1}{1-x^r}$ , and  $Q_n(a) = \prod_{r=n}^{\infty} \frac{1}{1-ax^r}$ ,

It is convenient to define  $P_0 = 1, H_0 = 1$ . We have;

$$\begin{aligned} H_k - H_{k-1} &= \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n a^{2n} x^{\frac{n(5n+1)}{2}}}{\left[ \begin{array}{l} x^{-kn} - a^k x^{kn} - x^{(1-k)n} \\ + a^{k-1} x^{n(k-1)} \end{array} \right]} P_n Q_n \right\} \\ &= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)}{2}} \times \left[ \frac{a^{k-1} x^{n(k-1)} (1-ax^n)}{+x^{-kn} (1-x^n)} \right] P_n Q_n. \end{aligned}$$

Now we have,  $(1-ax^n) Q_n = Q_{n+1}$  and  $(1-x^n) P_n = P_{n-1}$ , hence,

$$\begin{aligned} H_k - H_{k-1} &= \sum_{n=0}^{\infty} (-1)^n a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} P_n Q_{n+1} \\ &\quad + \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} P_{n-1} Q_n. \end{aligned}$$

In the second sum on the right hand side of the Identity we change  $n$  into  $n + 1$ . Thus,

$$\begin{aligned} H_k - H_{k-1} &= \sum_{n=0}^{\infty} (-1)^n a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} P_n Q_{n+1} \\ &\quad - \sum_{n=0}^{\infty} (-1)^n a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} P_n Q_{n+1}. \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{matrix} a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} \\ -a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} \end{matrix} \right\} P_n Q_{n+1} \\
 &= \sum_{n=0}^{\infty} (-1)^n \left\{ \begin{matrix} a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} \\ \otimes \left( 1 - a^{3-k} x^{(2n+1)(3-k)} \right) \end{matrix} \right\} P_n Q_{n+1} \\
 &= \sum_{n=0}^{\infty} (-1)^n \left[ a^{k-1} \eta \left\{ \begin{matrix} a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \\ \times \left( 1 - a^{3-k} x^{2n(3-k)} \right) \end{matrix} \right\} \right] P_n Q_{n+1}.
 \end{aligned}$$

We have  $Q_{n+1} = \eta Q_n$  and so,

$$\begin{aligned}
 &H_k - H_{k-1} \\
 &= a^{k-1} \eta \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \left( 1 - a^{3-k} x^{2n(3-k)} \right) P_n Q_n \\
 &= a^{k-1} \eta H_{3-k}.
 \end{aligned}$$

Hence, the Remark.

**The Rogers-Ramanujan Identities**

**Identity 1 [4]:**

$$\begin{aligned}
 &1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\dots(1-x^m)} \\
 &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}.
 \end{aligned}$$

**Identity 2 [4]:**

$$\begin{aligned}
 &1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\dots(1-x^m)} \\
 &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}.
 \end{aligned}$$

**Proof:** From (8) we have;

$$H_k(a, x) = \frac{G_k(a, x)}{(1-a)(1-ax)\dots\infty} \tag{13}$$

where  $H_0 = 0$ .

From above Remark we have;

$$H_k - H_{k-1} = a^{k-1} \eta H_{3-k}$$

where the operator  $\eta$  is defined by  $\eta f(a) = f(ax)$ , and  $k = 1$  or  $2$ . In particular

$$H_1 = \eta H_2,$$

$$H_2 - H_1 = a\eta H_1. \tag{14}$$

So we have,

$$H_2 = \eta H_2 + a\eta^2 H_2. \tag{15}$$

Suppose now that;

$$H_2 = 1 + c_1 a + c_2 a^2 + \dots \infty. \tag{16}$$

where the coefficients depend on  $x$  only. Substituting this into (15), we obtain;

$$\begin{aligned}
 &1 + c_1 a + c_2 a^2 + \dots \infty \\
 &= 1 + c_1 ax + c_2 a^2 x^2 + \dots \infty + a(1 + c_1 ax^2 + c_2 a^2 x^4 + \dots \infty).
 \end{aligned}$$

Hence, equating the coefficients of various powers of  $a$  from both sides we get;

$$\begin{aligned}
 &c_1 = \frac{1}{1-x}, c_2 = \frac{x^2}{1-x^2} c_1, c_3 = \frac{x^4}{1-x^3} c_2, \dots, \\
 &c_n = \frac{x^{n(n-1)}}{(1-x)(1-x^2)\dots(1-x^n)}.
 \end{aligned}$$

From (13) and (16), we have for  $k = 2$ ;

$$\begin{aligned}
 &\frac{G_2(a, x)}{(1-a)(1-ax)\dots\infty} \\
 &= H_2(a, x) \\
 &= 1 + \frac{a}{1-x} + \frac{a^2 x^2}{(1-x)(1-x^2)} \\
 &\quad + \frac{a^3 x^6}{(1-x)(1-x^2)(1-x^3)} + \dots \infty.
 \end{aligned}$$

If  $a = x$ , then;

$$\begin{aligned}
 &1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} \\
 &\quad + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \dots \infty \\
 &= \frac{G_2(x, x)}{(1-x)(1-x^2)\dots\infty}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\dots(1-x^m)} \\
 &= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})}.
 \end{aligned}$$

Hence the Identity 1.

Again from (13), (14) and (16) we have with  $k = 1$ ,

$$\begin{aligned}
 &\frac{G_1(a, x)}{(1-a)(1-ax)\dots\infty} \\
 &= H_1(a, x) = \eta H_2(a, x) \\
 &= 1 + \frac{ax}{1-x} + \frac{a^2 x^4}{(1-x)(1-x^2)} \\
 &\quad + \frac{a^3 x^9}{(1-x)(1-x^2)(1-x^3)} + \dots \infty.
 \end{aligned}$$

If  $a = x$ , then we have;

$$\begin{aligned}
& 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} \\
& + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \dots \infty \\
& = \frac{G_1(x, x)}{(1-x)(1-x^3) \dots \infty}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2) \dots (1-x^m)} \\
& = \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}.
\end{aligned}$$

Hence the Identity 2.

## 5. Conclusion

In this study, we have shown  $C_1'(n) = C'(n)$  with the help of a numerical example when  $n=11$ , and also have shown  $C_1''(n) = C''(n)$  with the help of a numerical example when  $n=11$ . We have transferred the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter  $a$ , i.e.,

$$G_k(a, x) = \prod_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} (1-a^k x^{2kn}) C_n$$

to

$$G_2(x, x) = \sum_{m=0}^{\infty} (1-x^{5m+2})(1-x^{5m+3})(1-x^{5m+5}),$$

where  $k=2$ , and  $a=x$ , it is used in proving The Rogers-Ramanujan Identity 2. Finally we have proved The Rogers-Ramanujan Identities with the help of auxiliary function,

$$H_k(a, x) = \frac{G_k(a, x)}{(1-a)(1-ax) \dots \infty},$$

where  $H_0 = 0$ .

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