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Common Fixed Point Results for Generalized Symmetric Meir-Keeler Contraction

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Abstract We introduce the concept of generalized weakly compatibility for the pair {F,G} of mappings $F,G : X\times X \to X$ and also introduce the concept of common fixed point of the mappings $F,G : X\times X \to X$. We establish a common fixed point theorem for generalized weakly compatible pair of mappings $F,G : X\times X \to X$ without mixed monotone property of any mapping under generalized symmetric Meir-Keeler contraction on a non complete metric space, which is not partially ordered. An example supporting to our result has also been cited. We improve, extend and generalize several known results.

Keywords: *common fixed point, generalized symmetric meir-keeler contraction, generalized compatibility, generalized weakly compatibility, commuting mapping*

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1. Introduction and Preliminaries

The Banach contraction mapping principle has been generalized in several directions. One of these generalizations, known as the Meir-Keeler fixed point theorem [11], has been obtained by the following more general assumption: for all ε>0 there exists δ(ε) > 0 such that

$$x,y\in X, \varepsilon \le d(x,y) < \varepsilon+\delta(\varepsilon) \Rightarrow d(Tx,Ty) < \varepsilon. \quad (1)$$

Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point, mixed monotone mappings in the setting of single-valued mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces.

In [3], Bhaskar and Lakshmikantham introduced the following.

Definition 1. Let $(X,\preceq)$ be a partially ordered set and endow the product space $X\times X$ with the following partial order:

$$(u,v)\preceq(x,y) \Leftrightarrow x\succeq u \text{ and } y\preceq v, \quad \forall (u,v),(x,y)\in X\times X. \quad (2)$$

Definition 2. An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F : X\times X\to X$ if

$$F(x,y)=x \text{ and } F(y,x)=y. \quad (3)$$

Definition 3. Let $(X,\preceq)$ be a partially ordered set. Suppose $F : X\times X\to X$ be a given mapping. We say that F has the mixed monotone property if for all $x,y\in X$, we have

$$x_1,x_2\in X, x_1\preceq x_2 \Rightarrow F(x_1,y)\preceq F(x_2,y) \quad (4)$$

and

$$y_1,y_2\in X, y_1\preceq y_2 \Rightarrow F(x,y_1)\succeq F(x,y_2). \quad (5)$$

Lakshmikantham and Ciric [10] extended the notion of mixed monotone property to mixed g-monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Bhaskar and Lakshmikantham [3].

In [10], Lakshmikantham and Ciric introduced the following:

Definition 4. An element $(x,y)\in X\times X$ is called a coupled coincidence point of the mappings $F : X\times X\to X$ and $g : X\to X$ if

$$x=F(x,y)=g(x) \text{ and } F(y,x)= g(y). \quad (6)$$

Definition 5. an element $(x,y)\in X\times X$ is called a common coupled fixed point of the mappings $F : X\times X\to X$ and $g : X\to X$ if

$$x=F(x,y)=g(x) \text{ and } y=F(y,x)= g(y). \quad (7)$$

Definition 6. An element x 2 X is called a common fixed point of the mappings $F : X\times X\to X$ and $g : X\to X$ if

$$x=g(x)=F(x,x). \quad (8)$$

Definition 7. The mappings $F : X\times X\to X$ and $g : X\to X$ are said to be commutative if

$$g(F(x,y)) = F(g(x),g(y)),\ for\ all\ (x,y) \in X \times X. \quad (9)$$

Definition 8. Let $(X,\preceq)$ be a partially ordered set. Suppose $F: X \times X \to X$ and $g: X \to X$ are given mappings. We say that F has the mixed g-monotone property if for all $x, y \in X$; we have

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1,y) \preceq F(x_2,y) \quad (10)$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x,y_1) \succeq F(x,y_2) \quad (11)$$

If g is the identity mapping on X; then F satisfies the mixed monotone property.

These results used to study the existence and uniqueness of solution for periodic boundary value problems. Hussain et al. [9] introduced a new concept of generalized compatibility of a pair of mappings $F,G: X \times X \to X$ defined on a product space and proved some coupled coincidence point results.

In [9], Hussain et al. introduced the following:

Definition 9. An element $(x,y) \in X \times X$ is called a coupled coincidence point of mappings $F,G: X \times X \to X$ if

$$F(x,y) = G(x,y)\ and\ F(y,x) = G(y,x). \quad (12)$$

Example 10. Let $F,G: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by F(x,y) = xy and G(x,y) = 2/3 (x + y) for all $(x,y) \in X \times X$. Note that (0,0), (1,2) and (2,1) are coupled coincidence points of F and G.

Definition 11. Let $F,G: X \times X \to X$ be two mappings. We say that the pair {F,G} is commuting if

$$F(G(x,y),G(y,x)) = G(F(x,y),F(y,x)),\quad for\ all\ x,y \in X. \quad (13)$$

Definition 12. Let $F,G: X \times X \to X$. We say that the pair {F,G} is generalized compatible if

$$\lim_{n\to\infty} d\begin{pmatrix} F(G(x_n,y_n),G(y_n,x_n)), \\ G(F(x_n,y_n),F(y_n,x_n)) \end{pmatrix} = 0,$$

$$\lim_{n\to\infty} d\begin{pmatrix} F(G(y_n,x_n),G(x_n,y_n)), \\ G(F(y_n,x_n),F(x_n,y_n)) \end{pmatrix} = 0,$$

whenever (x_n) and (y_n) are sequences in X such that

$$\lim_{n\to\infty} G(x_n,y_n) = \lim_{n\to\infty} F(x_n,y_n) = x,$$
$$\lim_{n\to\infty} G(y_n,x_n) = \lim_{n\to\infty} F(y_n,x_n) = y.$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Coupled fixed point theory have developed literature, some of the instances of these works are [1,2,4,5,6,7,8,11,12,13,15]. Recently Samet et al. [14] claimed that most of the coupled fixed point theorems in the setting of single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

In [13], Samet established the coupled fixed points of mixed strict monotone generalized Meir-Keeler operators and also established the existence and uniqueness results for coupled fixed point. Berinde and Pecurar [2] obtained more general coupled fixed point theorems for mixed monotone operators $F: X \times X \to X$ satisfying a generalized symmetric Meir-Keeler contractive condition.

In this paper, we introduce the concept of generalized weakly compatibility for the pair {F,G} of mappings $F,G: X \times X \to X$ and also introduce the concept of common fixed point of the mappings $F,G: X \times X \to X$. We establish a common fixed point theorem for generalized weakly compatible pair of mappings $F,G: X \times X \to X$ without mixed monotone property of any mapping under generalized symmetric Meir-Keeler contraction on a non complete metric space, which is not partially ordered. We also give an example to support our result presented here. We extend and generalize the results of Berinde and Pecurar [2], Bhaskar and Lakshmikantham [3], Meir and Keeler [11], Samet [13] and many other results in the existing literature.

2. Main Results

First, we introduce the following:

Definition 13. An element $x \in X$ is called a common fixed point of the mappings $F,G: X \times X \to X$ if

$$x = F(x,x) = G(x,x).$$

Definition 14. Let X be a non-empty set. The mappings $F,G: X \times X \to X$ are called generalized weakly compatible mappings if F(x, y) = G(x, y), F(y, x) = G(y, x) implies that G(F(x, y), F(y, x)) = F(G(x, y), G(y, x)), G(F(y, x), F(x, y)) = F(G(y, x), G(x, y)), for all $(x,y) \in X$. Obviously, a generalized compatible pair is generalized weakly compatible but converse is not true in general.

Example 15. Let (X, d) be a usual metric space where $X = \left\{0,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\right\}$. Define $F,G: X \times X \to X$ by

$$F(x,y) = \begin{cases} \dfrac{1}{(2n+1)^4}, (x,y) = \left(\dfrac{1}{2n},\dfrac{1}{2n}\right) \\ 0,\ otherwise \end{cases}$$

and

$$G(x,y) = \begin{cases} 1, (x,y) = \left(\dfrac{1}{2n+1},\dfrac{1}{2n+1}\right) \\ \dfrac{1}{2n+1}, (x,y) = \left(\dfrac{1}{2n},\dfrac{1}{2n}\right) \\ 0,\ otherwise \end{cases}$$

Let $x_n = y_n = \frac{1}{2n}$. Then, we have

$$G(x_n,y_n) = \frac{1}{2n+1} \to 0, F(x_n,y_n) = \frac{1}{(2n+1)^4} \to 0$$

$as\ n \to \infty, but$

$$\lim_{n\to\infty} d\begin{pmatrix} F(G(x_n,y_n),G(y_n,x_n)), \\ G(F(x_n,y_n),F(y_n,x_n)) \end{pmatrix} = d(0,1) \not\to 0.$$

So F and G are not generalized compatible. From F(x, y) = G(x, y), F(y, x) =G(y, x), we can get (x, y) = (0, 0) and we have G(F(0, 0), F(0, 0)) = F(G(0,0), G(0, 0)), G(F(0, 0), F(0, 0)) = F(G(0, 0), G(0, 0)), which implies that F and G are generalized weakly compatible.

Theorem 16. Let (X, d) be a metric space. Assume $F,G : X\times X \to X$ be two generalized weakly compatible mappings and for each ε> 0, there exists δ(ε) > 0 such that

$$\varepsilon \le \frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2} \le \varepsilon+\delta(\varepsilon)$$

implies

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2} \le \varepsilon \quad (14)$$

for all $x,y,u,v\in X$. Suppose that for any $x,y\in X$, there exist $u,v\in X$ such that

$$F(x,y)=G(u,v)\ and\ F(y,x)=G(v,u) \quad (15)$$

Suppose that $F(X\times X)\ or\ G(X\times X)$ is complete. Then there exists a unique $x\in X$ such that x = G(x,x) = F(x,x).

Proof. Let x_0, y_0 be two arbitrary points in X. From (15); we can choose $x_1, y_1 \in X$ such that

$$G(x_1,y_1)=F(x_0,y_0)\quad and \quad G(y_1,x_1)=F(y_0,x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}&G(x_{n+1},y_{n+1})=F(x_n,y_n)\\ &and\\ &G(y_{n+1},x_{n+1})=F(y_n,x_n),\\ &for\ all\ n\ge 0.\end{aligned} \quad (16)$$

The proof is divided into 4 steps.

Step 1. Prove that $\{G(x_n,y_n)\}$ and $\{G(y_n,x_n)\}$ are Cauchy sequences.

Now, by (14), for each ε> 0, there exists δ(ε) > 0 such that

$$\varepsilon \le \frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2} \le \varepsilon+\delta(\varepsilon)$$

implies

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2} \le \varepsilon \quad (17)$$

Condition (17) implies the strict contractive condition

$$\begin{aligned}&\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2}\\ &<\frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2},\end{aligned} \quad (18)$$

for $G(x,y)\le G(u,v)\ and\ G(y,x)\ge G(v,u)$. Thus, by (18), we have

$$\begin{aligned}&\frac{d\left(G(x_{n+1},y_{n+1}),G(x_n,y_n)\right)+d\left(G(y_{n+1},x_{n+1}),G(y_n,x_n)\right)}{2}\\ &=\frac{d\left(F(x_n,y_n),F(x_{n-1},y_{n-1})\right)+d\left(F(y_n,x_n),F(y_{n-1},x_{n-1})\right)}{2}\\ &<\frac{d\left(G(x_n,y_n),G(x_{n-1},y_{n-1})\right)+d\left(G(y_n,x_n),G(y_{n-1},x_{n-1})\right)}{2}\end{aligned}$$

which shows that the sequence of nonnegative numbers $\{\Delta_n\}_{n=0}^{\infty}$ given by

$$\Delta_n=\frac{d\left(G(x_n,y_n),G(x_{n-1},y_{n-1})\right)+d\left(G(y_n,x_n),G(y_{n-1},x_{n-1})\right)}{2}, \quad (19)$$

is non-increasing, Therefore, there exists some ε≥0 such that

$$\lim_{n\to\infty}\Delta_n=\lim_{n\to\infty}\frac{1}{2}\left[\begin{aligned}&d\left(G(x_n,y_n),G(x_{n-1},y_{n-1})\right)\\ &+d\left(G(y_n,x_n),G(y_{n-1},x_{n-1})\right)\end{aligned}\right]=\varepsilon.$$

We shall prove that ε= 0. Suppose, to the contrary, that ε> 0. Then there exists a positive integer p such that

$$\varepsilon<\Delta_p<\varepsilon+\delta(\varepsilon),$$

which, by (17); implies

$$\frac{d\left(F(x_p,y_p),F(x_{p-1},y_{p-1})\right)+d\left(F(y_p,x_p),F(y_{p-1},x_{p-1})\right)}{2}<\varepsilon$$

it follows, by (16) and (19); that

$$\Delta_{p+1}=\frac{d\left(G(x_{p+1},y_{p+1}),G(x_p,y_p)\right)+d\left(G(y_{p+1},x_{p+1}),G(y_p,x_p)\right)}{2}$$

which is a contradiction. Thus ε = 0 and hence

$$\lim_{n\to\infty}\Delta_n=\lim_{n\to\infty}\frac{1}{2}\left[\begin{aligned}&d\left(G(x_n,y_n),G(x_{n-1},y_{n-1})\right)\\ &+d\left(G(y_n,x_n),G(y_{n-1},x_{n-1})\right)\end{aligned}\right]=0. \quad (20)$$

Let now ε> 0 be arbitrary and δ(ε) the corresponding value from the hypothesis of our theorem. By (20), there exists a positive integer k such that

$$\Delta_{k+1}=\frac{1}{2}\left[\begin{aligned}&d\left(G(x_{k+1},y_{k+1}),G(x_k,y_k)\right)\\ &+d\left(G(y_{k+1},x_{k+1}),G(y_k,x_k)\right)\end{aligned}\right]<\delta(\varepsilon). \quad (21)$$

For this fixed number k, consider now the set A_k = {(G(x, y), G(y, x)): $G(x_k, y_k) \le G(x, y)$, $G(y, x) \ge G(y_k, x_k)$, ½ [d(G(x_k, y_k), G(x, y))+d(G(y_k, x_k), G(y,x))] < ε + δ(ε). By (21), $A_k \ne ø$. We claim that

$$\left(G(x,y),G(y,x)\right)\in A_k \Rightarrow \left(F(x,y),F(y,x)\right)\in A_k. \quad (22)$$

Let $\left(G(x, y),\ G(y, x)\right) \in A_k$. Then

$$\frac{d\left(G\left(x_k, y_k\right), G(x, y)\right)+d\left(G\left(y_k, x_k\right), G(y, x)\right)}{2}<\varepsilon. \quad (23)$$

which, by (14), implies

$$\frac{d\left(F\left(x_k, y_k\right), F(x, y)\right)+d\left(F\left(y_k, x_k\right), F(y, x)\right)}{2}<\varepsilon. \quad (24)$$

Now, by (21) and (24), we have

$$\begin{aligned}
&\frac{d\left(G\left(x_k, y_k\right), G(x, y)\right)+d\left(G\left(y_k, x_k\right), G(y, x)\right)}{2} \\
&\leq \frac{d\left(G\left(x_k, y_k\right), G\left(x_k, y_k\right)\right)+d\left(G\left(y_k, x_k\right), G\left(y_k, x_k\right)\right)}{2} \\
&+\frac{d\left(F\left(x_k, y_k\right), F(x, y)\right)+d\left(F\left(y_k, x_k\right), F(y, x)\right)}{2} \\
&\leq \frac{d\left(G\left(x_k, y_k\right), G\left(x_{k+1}, y_{k+1}\right)\right)+d\left(G\left(y_k, x_k\right), G\left(y_{k+1}, x_{k+1}\right)\right)}{2} \\
&+\frac{d\left(F\left(x_k, y_k\right), F(x, y)\right)+d\left(F\left(y_k, x_k\right), F(y, x)\right)}{2} \\
&<\varepsilon+\delta(\varepsilon).
\end{aligned}$$

Thus $\left(F(x, y),\ F(y, x)\right) \in A_k$. Again

$$\begin{aligned}
&\frac{d\left(G\left(x_k, y_k\right), G\left(x_{k+1}, y_{k+1}\right)\right)+d\left(G\left(y_k, x_k\right), G\left(y_{k+1}, x_{k+1}\right)\right)}{2} \\
&\leq \frac{d\left(G\left(x_k, y_k\right), G(x, y)\right)+d\left(G\left(y_k, x_k\right), G(y, x)\right)}{2} \\
&+\frac{d\left(F(x, y), F\left(x_{k+1}, y_{k+1}\right)\right)+d\left(F(y, x), F\left(y_{k+1}, x_{k+1}\right)\right)}{2} \\
&<2(\varepsilon+\delta(\varepsilon)).
\end{aligned}$$

Thus $\left(G\left(x_{k+1}, y_{k+1}\right), G\left(y_{k+1}, x_{k+1}\right)\right) \in A_k$ and by induction,

$$\left(G\left(x_{k+1}, y_{k+1}\right), G\left(y_{k+1}, x_{k+1}\right)\right) \in A_k, \quad \text{for all } n>k.$$

This implies that for all n, m > k, we have

$$\begin{aligned}
&\frac{d\left(G\left(x_n, y_n\right), G\left(x_m, y_m\right)\right)+d\left(G\left(y_n, x_n\right), G\left(y_m, x_m\right)\right)}{2} \\
&\leq \frac{d\left(G\left(x_n, y_n\right), G\left(x_k, y_k\right)\right)+d\left(G\left(y_n, x_n\right), G\left(y_k, x_k\right)\right)}{2} \\
&+\frac{d\left(G\left(x_k, y_k\right), G\left(x_m, y_m\right)\right)+d\left(G\left(y_k, x_k\right), G\left(y_m, x_m\right)\right)}{2} \\
&<2(\varepsilon+\delta(\varepsilon))=4\varepsilon.
\end{aligned}$$

This shows that $\left\{G\left(x_n, y_n\right)\right\}_{n=0}^{\infty}$ and $\left\{G\left(y_n, x_n\right)\right\}_{n=0}^{\infty}$ are Cauchy sequences in X.

Step 2. Prove that G and F have a coupled coincidence point.

Since $G(X \times X)$ is complete, then there exist $x, y \in G(X \times X)$ and $(a, b) \in X \times X$ such that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} G\left(x_n, y_n\right)=\lim_{n \rightarrow \infty} F\left(x_n, y_n\right)=G(a, b)=x, \\
&\lim_{n \rightarrow \infty} G\left(y_n, x_n\right)=\lim_{n \rightarrow \infty} F\left(y_n, x_n\right)=G(b, a)=y.
\end{aligned} \quad (25)$$

Now, by (18), we have

$$\begin{aligned}
&\frac{d\left(F\left(x_n, y_n\right), F(a, b)\right)+d\left(F\left(y_n, x_n\right), F(b, a)\right)}{2} \\
&<\frac{d\left(G\left(x_n, y_n\right), G(a, b)\right)+d\left(G\left(y_n, x_n\right), G(b, a)\right)}{2}.
\end{aligned}$$

Taking limit as n→ 1 in the above inequality and using (25), we have

$$d(G(a, b), F(a, b))=0 \text{ and } d(G(b, a), F(b, a))=0,$$

which implies that

$$F(a, b)=G(a, b)=x \text{ and } F(b, a)=G(b, a)=y.$$

Since F and G are generalized weakly compatible, we get that

$$\begin{aligned}
&G(F(a,\ b),\ F(b,\ a))=F(G(a,\ b),\ G(b,\ a)), \\
&G(F(b,\ a),\ F(a,\ b))=F(G(b,\ a),\ G(a,\ b)),
\end{aligned}$$

which implies that

$$G(x,\ y)=F(x,\ y) \text{ and } G(y,\ x)=F(y,\ x),$$

that is, (x, y) is a coupled coincidence point of F and G.

Step 3. Prove that G(x, y) = y and G(y, x) = x.

If, not. Then by (18), we have

$$\begin{aligned}
&\frac{d\left(F(x, y), F\left(y_n, x_n\right)\right)+d\left(F(y, x), F\left(x_n, y_n\right)\right)}{2} \\
&<\frac{d\left(G(x, y), G\left(y_n, x_n\right)\right)+d\left(G(y, x), G\left(x_n, y_n\right)\right)}{2}.
\end{aligned}$$

Letting n→∞ in the above inequality and using (25), we have

$$\begin{aligned}
&\frac{d(G(x, y), y)+d(G(y, x), x)}{2} \\
&<\frac{d(G(x, y), y)+d(G(y, x), x)}{2},
\end{aligned}$$

which is a contradiction. Thus we must have G(x, y) = y and G(y, x) = x.

Step 4. Prove that x = y.

If, not. Then by (18), we have

$$\begin{aligned}
&\frac{d\left(F\left(x_n, y_n\right), F\left(y_n, x_n\right)\right)+d\left(F\left(y_n, x_n\right), F\left(x_n, y_n\right)\right)}{2} \\
&<\frac{d\left(G\left(x_n, y_n\right), G\left(y_n, x_n\right)\right)+d\left(G\left(y_n, x_n\right), G\left(x_n, y_n\right)\right)}{2}.
\end{aligned}$$

Letting n→∞ in the above inequality and using (25), we get

$$\frac{d(x, y)+d(y, x)}{2}<\frac{d(x, y)+d(y, x)}{2},$$

which is a contradiction. Thus x = y.

Example 17. Suppose that X = $\mathbb{R}$, equipped with the usual metric $d: X\times X \to [0,+\infty)$. Let $F,G: X\times X\to X$ be defined as

$$F(x,y)=\begin{cases}\dfrac{x^2-y^2}{3}, \textit{if } x\ge y,\\ 0, \textit{if } x<y,\end{cases}$$

and

$$G(x,y)=\begin{cases}x^2-y^2, \textit{if } x\ge y,\\ 0, \textit{if } x<y.\end{cases}$$

From F(x, y) = G(x, y), F(y, x) = G(y, x), we can get (x, y) = (0, 0) and we have G(F(0, 0), F(0, 0)) = F(G(0, 0), G(0, 0)), G(F(0, 0), F(0, 0)) = F(G(0, 0), G(0, 0)), which implies that F and G are generalized weakly compatible.

Now, we prove that for any $x, y\in X$, there exist $u, v\in X$ such that

$$F(x,y)=G(u,v) \text{ and } F(y,x)=G(v,u).$$

Let $(x,y)(u,v)\in X\times X$ be fixed. We consider the following cases:

Case 1: If x = y, then we have F(x, y) = 0 = G(x, y) and F(y, x) = 0 =G(y, x).

Case 2: If x > y, then we have $F(x,y)=\frac{x^2-y^2}{3}=G\left(\frac{x}{\sqrt{3}},\frac{y}{\sqrt{3}}\right)$ and $F(y,x)=0=G\left(\frac{y}{\sqrt{3}},\frac{x}{\sqrt{3}}\right)$.

Case 3: If x < y, then we have $F(x,y)=0=G\left(\frac{x}{\sqrt{3}},\frac{y}{\sqrt{3}}\right)$ and $F(y,x)=\frac{y^2-x^2}{3}=G\left(\frac{y}{\sqrt{3}},\frac{x}{\sqrt{3}}\right)$. Now, we shall show that the mappings F and G satisfy the condition (14): For each $x,y,u,v\in X\times X$, we have

$$\varepsilon\le\frac{d\left(G(,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2}$$
$$\le\varepsilon+\delta(\varepsilon).$$

Then

$$\frac{d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right)}{2}$$
$$=\frac{1}{2}\left[\left|\frac{x^2-y^2}{3}-\frac{u^2-v^2}{3}\right|+\left|\frac{y^2-x^2}{3}-\frac{v^2-u^2}{3}\right|\right]$$
$$=\frac{1}{6}\left[\left|G(x,y)-G(u,v)\right|+\left|G(y,x)-G(v,u)\right|\right]$$
$$=\frac{1}{3}\left[\frac{d\left(G(x,y),G(u,v)\right)+d\left(G(y,x),G(v,u)\right)}{2}\right]$$
$$=\frac{1}{3}\left(\varepsilon+\delta(\varepsilon)\right)<\varepsilon.$$

Thus the contractive condition (14) is satisfied for all $x,y,u,v\in X$. In addition, all the other conditions of Theorem 16 are satisfied and 0 is a unique common fixed point of F and G.

Corollary 18. Let (X, d) be a metric space. Assume $F,G: X\times X\to X$ be two generalized compatible mappings satisfying (14), (15) and $F(X\times X)$ *or* $G(X\times X)$ is complete. Then there exists a unique $x\in X$ such that x = G(x, x) = F(x, x).

Corollary 19. Let (X, d) be a metric space. Assume $F,G: X\times X\to X$ be two commuting mappings satisfying (14), (15) and $F(X\times X)$ *or* $G(X\times X)$ is complete. Then there exists a unique $x\in X$ such that x = G(x, x) = F(x, x).

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