Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 6, 230-232
Available online at http://pubs.sciepub.com/tjant/2/6/8

DOI:10.12691/tjant-2-6-8

On the Error Term for the Number of Integral Ideals in Galois Extensions

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Received November 05, 2014; Revised December 05, 2014; Accepted December 18, 2014

Abstract Suppose that E is an algebraic number field over the rational field $\mathbb{Q}$. Let a(n) be the number of integral ideals in E with norm n and Δ(x) denote the remainder term in the asymptotic formula of the l-th integral power sum of a(n). In this paper the bound of the average behavior of Δ(x) is given. This result constitutes an improvement upon that of Lü and Wang for the error terms in mean value.

Keywords: *dedekind zeta-function, dirichlet series, mean value*

Cite This Article: Sanying Shi, "On the Error Term for the Number of Integral Ideals in Galois Extensions." *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 6 (2014): 230-232. doi: 10.12691/tjant-2-6-8.

1. Introduction and the Result

Let E be an algebraic number field of degree d over the rational field $\mathbb{Q}$, and $\zeta(s,E)$ be its Dedekind zeta-function. Thus

$$\zeta(s,E)=\sum_{\mathfrak{a}}\frac{1}{(N\mathfrak{a})^s},\quad (\Re\mathfrak{e}(s)>1),$$

where $\mathfrak{a}$ runs over all integral ideals of the field E, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. If $a(n)$ denotes the number of integral ideals in E with norm n, then we have

$$\zeta(s,E)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},\quad s=\sigma+it,\quad \sigma>1.$$

It is known that $a(n)$ is a multiplicative function and satisfies

$$a(n)\ll\tau(n)^d, \tag{1}$$

where $\tau(n)$ is the divisor function.

It is an important problem to study the function $a(n)$. In 1927, Landau [7] first proved that

$$\sum_{n\le x}a(n)=\alpha x+O\left(x^{1-\frac{2}{d+1}+\epsilon}\right),$$

for any arbitrary algebraic number field of degree $d\ge 2$, where α is the residue of $\zeta(s,E)$ at its simple pole $s=1$.

It is hard to refine Landau's result. Later, Huxley and Watt [3] and Müller [9] improved the results for the quadratic and cubic fields, respectively.

Until 1993, Nowak [10] obtained the best result

$$\sum_{n\le x}a(n)=\alpha x+\begin{cases}O\left(x^{1-\frac{2}{d}+\frac{8}{d(5d+2)}}(\log x)^{\frac{10}{5d+2}}\right), & for\ 3\le d\le 6\\ O\left(x^{1-\frac{2}{d}+\frac{3}{2d^2}}(\log x)^{\frac{2}{d}}\right), & for\ d\ge 7\end{cases}$$

for any arbitrary algebraic number field of degree $d\ge 3$.

In [1], Chandraseknaran and Good studied the l-th integral power sum of $a(n)$ in some Galois fields, and they showed that

Theorem 1.0. If E is a Galois extension of $\mathbb{Q}$ of degree d, then for any $\epsilon>0$ and any integer $d\ge 2$, we have

$$\sum_{n\le x}a(n)^l=xQ_m(\log x)+O\left(x^{1-\frac{2}{md}+\epsilon}\right),$$

where $m=d^{l-1}$, and $Q_l(t)$ is a suitable polynomial in t of degree $m-1$.

Recently, Lü and Wang [8] improved the classical result of [1] by replacing $\frac{2}{md}$ with $\frac{3}{md+6}$.

Motivated by [2,4,5], the purpose of this paper is to study the remainder term in mean square, and we shall prove the following result.

Theorem 1.1 Subject to assumptions in Theorem 1.0, and define

$$\Delta(x):=\sum_{n\le x}a(n)^l-xQ_m(\log x). \tag{2}$$

Then we have

$$\int_1^X\Delta^2(x)dx\ll_\epsilon X^{3-\frac{6}{md+3}+\epsilon}$$

for any given $\epsilon > 0$.

Notations. As usual, the Vinogradov symbol $A \ll B$ means that B is positive and the ratio A/B is bounded. The letter ϵ denotes an arbitrary small positive number, not the same at each occurrence.

2. Proof of Theorem 1.1

To prove our Theorem, we need the following lemmas.

Lemma 2.1 Let $E/\mathbb{Q}$ be a Galois extension of degree d, and $a(n)$ be defined in (1). Define

$$N_l(s) = \sum_{n=1}^{\infty} \frac{a(n)^l}{n^s}, (\Re\mathfrak{e} s > 1). \quad (3)$$

Then we have

$$N_l(s) = \zeta^m(s, E) \cdot A_1(s),$$

for any integer $l \geq 1$, where $m = d^{l-1}$, and $A_1(s)$ denotes a Dirichlet series, which is absolutely and uniformly convergent for $\Re\mathfrak{e}(s) > 1/2$.

Proof. This is Lemma 2.1 in [8].

Lemma 2.2. Let E be an algebraic number field of degree d, then

$$\zeta(\sigma + it, E) \ll (1 + |t|)^{\frac{d}{3}(1-\sigma)+\epsilon},$$

for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and any fixed $\epsilon > 0$.

Proof. By Lemma 2.2 in [8] and the Phragmen-Lindelöf principle for a strip (see, e.g. Theorem 5.53 in [6]), Lemma 2.2 follows immediately.

Now we begin to prove our theorem.

Let E be a Galois extension of $\mathbb{Q}$ of degree d.

Recall $a(n)$ denotes the number of integral ideals in E with norm n, and

$$\zeta(s, E) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1.$$

Let

$$T = X^{\frac{3}{d^l + 3}}.$$

From (1), (3) and Perron's formula (see Proposition 5.54 in [6], we get

$$\sum_{n \leq x} a(n)^l = \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} N_l(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right).$$

By the property $N_l(s)$ only has a simple pole at $s = 1$ for $\Re\mathfrak{e}(s) > \frac{1}{2}$ and Cauchy's residue theorem, we have

$$\sum_{n \leq x} a(n^2)^l = \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} + \int_{\frac{1}{2}+\epsilon+iT}^{1+\epsilon+iT} + \int_{1+\epsilon-iT}^{\frac{1}{2}+\epsilon-iT} \right\} N_l(s) \frac{x^s}{s} ds$$
$$+ \operatorname*{\mathbf{Res}}_{s=1} N_l(s) x + O\left(\frac{x^{1+\epsilon}}{T}\right)$$
$$= x Q_m(\log x) + J_1(x) + J_2(x) + J_3(x) + O\left(x^{1+\epsilon} T^{-1}\right).$$

where $m = d^{l-1}$, and $Q_m(t)$ is a suitable polynomial in t of degree $m - 1$.

From the definition of $\Delta(x)$ in (2), we have

$$\Delta(x) = J_1(x) + J_2(x) + J_3(x) + O\left(x^{1+\epsilon} T^{-1}\right).$$

Therefore to prove Theorem 1.1, we shall prove the following results.

$$\int_1^X J_i^2(x) dx \ll_\epsilon X^{3 - \frac{6}{md+3} + \epsilon}, i = 1, 2, 3 \quad (4)$$

and

$$\int_1^X \left(O\left(\frac{x^{1+\epsilon}}{T} \right) \right)^2 dx \ll_\epsilon X^{3 - \frac{6}{md+3} + \epsilon}. \quad (5)$$

It is easy to get

$$\int_1^X \left(O\left(\frac{x^{1+\epsilon}}{T} \right) \right)^2 dx \ll \frac{X^{3+\epsilon}}{T^2} \ll X^{3 - \frac{6}{md+3} + \epsilon}. \quad (6)$$

Now we consider the integral $J_1(x)$. We have

$$J_1(x) = \frac{1}{2\pi} \int_{-T}^{T} N_l\left(\frac{1}{2} + \epsilon + it\right) \frac{x^{\frac{1}{2}+\epsilon+it}}{\frac{1}{2} + \epsilon + it} dt.$$

Then

$$\int_1^X J_1^2(x) dx = \frac{1}{4\pi^2} \int_1^X \left(\int_{-T}^{T} N_l\left(\frac{1}{2} + \epsilon + it_1\right) \frac{x^{\frac{1}{2}+\epsilon+it_1}}{\frac{1}{2} + \epsilon + it_1} dt_1 \right.$$
$$\left. \times \int_{-T}^{T} \overline{N_l\left(\frac{1}{2} + \epsilon + it_2\right)} \frac{x^{\frac{1}{2}+\epsilon-it_2}}{\frac{1}{2} + \epsilon - it_2} dt_2 \right) dx$$
$$= \frac{1}{4\pi^2} \int_{-T}^{T} \int_{-T}^{T} \frac{N_l\left(\frac{1}{2} + \epsilon + it_1\right) \overline{N_l\left(\frac{1}{2} + \epsilon + it_2\right)}}{(\frac{1}{2} + \epsilon + it_1)(\frac{1}{2} + \epsilon - it_2)}$$
$$\times \left(\int_1^X x^{1+2\epsilon+i(t_1-t_2)} \right) dx dt_1 dt_2$$
$$\ll X^{2+2\epsilon} \int_{-T}^{T} dt_1 \int_{-T}^{T} \frac{|N_l\left(\frac{1}{2} + \epsilon + it_1\right)| |N_l\left(\frac{1}{2} + \epsilon + it_2\right)|}{(1 + |t_1|)(1 + |t_2|)(1 + |t_1 - t_2|)} dt_2$$

$$\ll X^{2+2\epsilon}\int_{-T}^{T}dt_1\int_{-T}^{T}\left(\frac{|N_l\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2}+\frac{|N_l\left(\frac{1}{2}+\epsilon+it_2\right)|^2}{(1+|t_2|)^2}\right)\frac{dt_2}{1+|t_1-t_2|} \tag{7}$$

$$\ll X^{2+2\epsilon}\int_{-T}^{T}\frac{|N_l\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2}dt_1\int_{-T}^{T}\frac{dt_2}{1+|t_1-t_2|}.$$

To go further, we get

$$\begin{aligned}\int_{-T}^{T}\frac{dt_2}{1+|t_1-t_2|}&\ll\int_{t_1-1}^{t_1+1}dt_2+\left(\int_{t_1+1}^{T}+\int_{-T}^{t_1-1}\right)\frac{dt_2}{|t_1-t_2|}\\&\ll 1+\int_{t_1+1}^{T}\frac{dt_2}{|t_1-t_2|}\\&\ll\int_1^{T+|t_1|}\frac{dt}{t}\ll\log 2T.\end{aligned}\tag{8}$$

By (7) and (8)

$$\int_1^X J_1^2(x)dx\ll X^{2+3\epsilon}\int_{-T}^{T}\frac{|N_l\left(\frac{1}{2}+\epsilon+it_1\right)|^2}{(1+|t_1|)^2}dt_1. \tag{9}$$

From (9), Lemma 2.1 and 2.3, we have (for $d\ge 3$)

$$\begin{aligned}\int_1^X J_1^2(x)dx&\ll X^{2+3\epsilon}+X^{2+3\epsilon}\int_1^T\left|\zeta^{d^{l-1}}\left(\frac{1}{2}+\epsilon+it,E\right)\times A_1\left(\frac{1}{2}+\epsilon+it_1\right)\right|^2t^{-2}dt\\&\ll X^{2+3\epsilon}+X^{2+3\epsilon}\int_1^T\left|\zeta^m\left(\frac{1}{2}+\epsilon+it,E\right)\right|^2t^{-2}dt\\&\ll X^{2+3\epsilon}+X^{2+3\epsilon}\int_1^T\left(t^{\frac{md}{6}+\epsilon}\right)^2t^{-2}dt\\&\ll X^{2+3\epsilon}+X^{2+4\epsilon}T^{\frac{md}{3}-1}\\&\ll X^{3-\frac{6}{md+3}+\epsilon}.\end{aligned}\tag{10}$$

Finally we estimate trivial bounds of the integrals $J_2(x)$, $J_3(x)$. By Lemma 2.2, we get

$$\begin{aligned}J_2(x)+J_3(x)&\ll\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}x^{\sigma}|\zeta^m(\sigma+iT,E)|T^{-1}d\sigma\\&\ll\max_{1/2+\epsilon\le\sigma\le 1+\epsilon}x^{\sigma}T^{\frac{md}{3}(1-\sigma)+\epsilon}T^{-1}\\&=\max_{\frac{1}{2}+\epsilon\le\sigma\le 1+\epsilon}\left(\frac{x}{T^{md/3}}\right)^{\sigma}T^{\frac{md}{3}-1+\epsilon}\\&\ll\frac{x^{1+\epsilon}}{T}+x^{\frac{1}{2}+\epsilon}T^{\frac{d^l}{6}-1+\epsilon},\end{aligned}$$

which yields

$$\begin{aligned}\int_1^X(J_2(x)+J_3(x))^2dx&\ll\int_1^X\left(\frac{x^{1+\epsilon}}{T}+x^{\frac{1}{2}+\epsilon}T^{\frac{md}{6}-1+\epsilon}\right)^2dx\\&\ll\int_1^X\left(\frac{x^{1+\epsilon}}{T}\right)^2\\&\ll\frac{X^{3+\epsilon}}{T^2}+X^{2+2\epsilon}T^{\frac{md}{3}-2+2\epsilon}\\&\ll X^{3-\frac{6}{md+3}+\epsilon}.\end{aligned}\tag{11}$$

The inequalities (4), (5) immediately follow from (6), (10) and (11). That is,

$$\int_1^X\Delta^2(x)dx\ll_{\epsilon}X^{3-\frac{6}{md+3}+\epsilon}.$$

Then this completes the proof of Theorem 1.1.

Acknowledgement

This work is supported by The National Science Foundation of China (grant no. 11201107 and 11071186) and by Natural Science Foundation of Anhui province (Grant No. 1208085QA01).

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