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Identities of Generalized Fibonacci-Like Sequence

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Abstract The Fibonacci and Lucas sequences are well-known examples of second order recurrence sequences. The Fibonacci sequence, Lucas numbers and their generalization have many interesting properties and applications to almost every field. Fibonacci sequence is defined by the recurrence formula $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0, F_1 = 1$, where F_n is a n^{th} number of sequence. Many authors have defined Fibonacci pattern based sequences which are popularized and known as Fibonacci-Like sequences. In this paper, Generalized Fibonacci-Like sequence is introduced and defined by the recurrence relation $M_n = M_{n-1} + M_{n-2}$, $n \geq 2$ with $M_0 = 2, M_1 = s+1$, where s being a fixed integers. Some identities of Generalized Fibonacci-Like sequence are presented by Binet's formula. Also some determinant identities are discussed.

Keywords: *Fibonacci sequence, Lucas Sequence, Generalized Fibonacci-Like Sequence, Binet's Formula*

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1. Introduction

Fibonacci numbers Fn and Lucas numbers Ln have delighted mathematicians and amateurs alike for centuries with their beauty and their propensity to pop up in quite unexpected places [3], [13] and [14]. It is well known that Fibonacci and Lucas numbers play an important role in many subjects such as algebra, geometry, and number theory. Their various elegant properties and wide applications have been studied by many authors.

The Fibonacci and Lucas sequences are examples of second order recursive sequences. The Fibonacci sequence [4] is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2, \text{ with } F_0 = 0,\ F_1 = 1. \quad (1.1)$$

The similar interpretation also exists for Lucas sequence. Lucas sequence [4] is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \text{ with } L_0 = 2,\ L_1 = 1. \quad (1.2)$$

Authors [1], [2] and [8] to [14] have been generalized second order recurrence sequences by preserving the recurrence relation and altering the first two terms of the sequence or preserving the first two terms of sequence and altering the recurrence relation slightly.

Horadam [1] introduced and studied properties of a generalized Fibonacci sequence $\{H_n\}$ and defined generalized Fibonacci sequence $\{H_n\}$ by the recurrence relation:

$$H_{n+2} = H_{n+1} + H_n,\ H_0 = q \text{ and } H_1 = p,\ n \geq 0, \quad (1.3)$$

where p, q are arbitrary integers.

Horadam [2] introduced and studied properties of another generalized Fibonacci sequence $\{w_n\}$ and defined generalized Fibonacci sequence $\{w_n\}$ by the recurrence relation:

$$\{w_n\} = \{w_n(a, b; p, q)\}: \ w_0 = a,\ w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}\ (n \geq 2), \quad (1.4)$$

where a, b, p and q are arbitrary integers.

Waddill and Sacks [10] extended the Fibonacci numbers recurrence relation and defined the sequence $\{P_n\}$ by recurrence relation:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3},\ n \geq 3, \quad (1.5)$$

where P_0, P_1 and P_2 are not all zero given arbitrary algebraic integers.

Jaiswal [8] introduced and studied properties of generalized Fibonacci sequence $\{T_n\}$ and defined it by

$$T_{n+1} = T_n + T_{n-1},\ T_1 = a \text{ and } T_2 = b,\ n \geq 1. \quad (1.6)$$

Falcon and Plaza [12] introduced k^{th} Fibonacci sequence $\{F_{k,n}\}_{n\in N}$ and studied its properties. For any positive integer $k \geq 1$, k^{th} Fibonacci sequence is defined by

$$F_{k,0}=0, F_{k,1}=1$$
$$\textit{and } F_{k,n+1},=kF_{k,n}+F_{k,n-1}, n\geq 1. \quad (1.7)$$

Many authors have been defined Fibonacci pattern based sequences which are known as Fibonacci-like sequences. The Fibonacci-Like sequence [4] is defined by recurrence relation,

$$S_n=S_{n-1}+S_{n-2},\ n\geq 2 \text{ with } S_0=2,\ S_1=2. \quad (1.8)$$

The associated initial conditions S_0 and S_1 are the sum of initial conditions of Fibonacci and Lucas sequence respectively.

i.e. $S_0=F_0+L_0$ and $S_1=F_1+L_1$.

Fibonacci-Like sequence [6] is defined by the recurrence relation,

$$H_n=2H_{n-1}+H_{n-2} \text{ for } n\geq 2 \text{ with } H_0=2, H_1=1. \quad (1.9)$$

In this paper, Generalized Fibonacci-Like sequence is introduced. The Binet's formula is presented and established some identities of Generalized Fibonacci-Like sequence. Also determinants identities are discussed.

2. Generalized Fibonacci-Like Sequence

Generalized Fibonacci-Like sequence is introduced and defined by the recurrence relation

$$M_n=M_{n-1}+M_{n-2}, n\geq 2 \textit{ with } M_0=2, M_1=s+1, \quad (2.1)$$

where s being a fixed integers. The first few terms are as follows:

$$M_0=2,$$
$$M_1=s+1,$$
$$M_2=s+3,$$
$$M_3=2s+4,$$
$$M_4=3s+7,$$
$$M_5=5s+11,$$
$$M_6=8s+18,$$
$$B_7=13s+29 \textit{ and so on.}$$

The characteristic equation of recurrence relation (2.1) is $t^2-t-1=0.$ which has two real roots

$$\alpha=\frac{1+\sqrt{5}}{2} \text{ and } \beta=\frac{1-\sqrt{5}}{2}. \quad (2.2)$$

Also, $\alpha\beta=-1,\ \alpha+\beta=1,\ \alpha-\beta=\sqrt{5},\ \alpha^2+\beta^2=3.$

Generating function of generalized Fibonacci-Like sequence is

$$\sum_{n=0}^{\infty} M_n t^n = M(t)=\frac{2+(s-1)t}{1-t-t^2}. \quad (2.3)$$

Binet's formula of Generalized Fibonacci-Like sequence is defined by

$$M_n=C_1\alpha^n+C_2\beta^n=C_1\left(\frac{1+\sqrt{5}}{2}\right)^n+C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (2.4)$$

Here, $C_1=\frac{s+\sqrt{5}}{\sqrt{5}}$ and $C_2=\frac{\sqrt{5}-s}{\sqrt{5}}$. Also,

$$C_1C_2=\frac{5-s^2}{(\alpha-\beta)^2}=\frac{5-s^2}{5},\ C_1\beta+C_2\alpha=-s+1,$$
$$C_1\alpha+C_2\beta=s+1, C_1\beta^2+C_2\alpha^2=3, \quad (2.5)$$
$$C_1+C_2=M_0=2.$$

3. Identities of Generalized Fibonacci-Like Sequence

Now some identities of Generalized Fibonacci-Like sequence are present using generating function and Binet's formula. Authors [6,7] have been described such type identities.

Theorem (3.1). (Explicit Sum Formula) Let M_n be the n^{th} term of generalized Fibonacci-Like sequence. Then

$$M_n=2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}+(s-1)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}. \quad (3.1)$$

Proof. By generating function (2.3), we have

$$\sum_{n=0}^{\infty} M_n t^n = M(t)=\frac{2+(s-1)t}{1-t-t^2}=\{2+(s-1)t\}\left(1-t-t^2\right)^{-1}$$
$$=\{2+(s-1)t\}\left[1-(t+t^2)\right]^{-1}$$
$$=\{2+(s-1)t\}.\sum_{n=0}^{\infty}\left(t+t^2\right)^n=\{2+(s-1)t\}.\sum_{n=0}^{\infty}t^n(1+t)^n$$
$$=\{2+(s-1)t\}.\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}t^k$$
$$=\{2+(s-1)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\lfloor n}{\lfloor k\lfloor n-k}t^{n+k}$$

(Replace n by n + k)

$$=\{2+(s-1)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lfloor n+k}{\lfloor k\lfloor n}t^{n+2k}$$
$$=\{2+(s-1)t\}\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}t^{n}$$

(Replace n by n − 2k)

$$=2\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}\right\}t^n+(s-1)\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\lfloor n-k}{\lfloor k\lfloor n-2k}\right\}t^{n+1}.$$

Equating the coefficient of t^n we obtain

$$M_n=2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}+(s-1)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}.$$

For s=1 in above identity, explicit formulas can be obtained for Fibonacci sequence.

Theorem (3.2). (Sum of First n terms) Sum of first n terms of Generalized Fibonacci-Like sequence is

$$\sum_{k=0}^{n} M_k = M_{n+2} - s - 1. \tag{3.2}$$

Proof. By Binet's formula (2.4), we have

$$\sum_{k=0}^{n} M_k = \sum_{k=0}^{n}\left[C_1\alpha^k + C_2\beta^k\right]$$

$$= C_1\left[\frac{1-\alpha^{n+1}}{1-\alpha}\right] + C_2\left[\frac{1-\beta^{n+1}}{1-\beta}\right]$$

$$= \frac{\begin{bmatrix}(C_1+C_2)-(C_1\beta+C_2\alpha) \\ -(C_1\alpha^n+C_1\beta^n)+\alpha\beta(C_1\alpha^{n-1}+C_1\beta^{n-1})\end{bmatrix}}{1-(\alpha+\beta)+\alpha\beta}$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n} M_k = M_{n+1} + M_n - s - 1 = M_{n+2} - s - 1..$$

Theorem (3.3). (Sum of First n terms with odd indices): Sum of first n terms (with odd indices) of Generalized Fibonacci-Like sequence is

$$\sum_{k=1}^{n} M_{2k-1} = M_{2n+1} - M_{2n-1} - 2. \tag{3.3}$$

Proof. By Binet's formula (2.4), we have

$$\sum_{k=1}^{n} B_{2k-1} = \sum_{k=1}^{n}\left[C_1\alpha^{2k-1} + C_2\beta^{2k-1}\right]$$

$$= C_1\left[\alpha+\alpha^3+\alpha^5+....+\alpha^{2n-1}\right] + C_2\left[\beta+\beta^3+\beta^5+....+\beta^{2n-1}\right]$$

$$= -C_1\alpha\left[\frac{1-\alpha^{2n}}{1-\alpha^2}\right] + C_2\beta\left[\frac{1-\beta^{2n}}{1-\beta^2}\right]$$

$$= \frac{\begin{bmatrix}(C_1\alpha^{2n+1}+C_2\beta^{2n+1})-(C_1\alpha+C_2\beta) \\ +\alpha\beta(C_1\beta+C_2\alpha)-\alpha^2\beta^2(C_1\alpha^{2n-1}+C_1\beta^{2n-1})\end{bmatrix}}{\alpha^2+\beta^2-\alpha^2\beta^2-1}.$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=1}^{n} M_{2k-1} = M_{2n+1} - M_{2n-1} - 2.$$

Theorem (3.4). (Sum of First n terms with even indices) Sum of first n terms (with even indices) of generalized Fibonacci-Like sequence is given by

$$\sum_{k=0}^{n} M_{2k} = M_{2n} - M_{2n-2} + 1. \tag{3.4}$$

Proof. By Binet's formula (2.4), we have

$$\sum_{k=0}^{n} M_{2k} = \sum_{k=0}^{n}\left[C_1\alpha^{2k} + C_2\beta^{2k}\right]$$

$$= -C_1\left[\frac{1-\alpha^{2n}}{1-\alpha^2}\right] + C_2\left[\frac{1-\beta^{2n}}{1-\beta^2}\right]$$

$$= \frac{\begin{bmatrix}(C_1\alpha^{2n}+C_2\beta^{2n})-(C_1+C_2) \\ +(C_1\beta^2+C_2\alpha^2)-\alpha^2\beta^2(C_1\alpha^{2n-2}+C_1\beta^{2n-2})\end{bmatrix}}{\alpha^2+\beta^2-\alpha^2\beta^2-1}.$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n} M_{2k} = M_{2n} - M_{2n-2} + 1.$$

Theorem (3.5). (Catalan's Identity) Let M_n be the n^{th} term of Generalized Fibonacci-Like sequence. Then

$$M_n^2 - M_{n+r}M_{n-r} = \frac{(-1)^{n-r}}{s^2-5}\left[(s+1)M_r - 2M_{r+1}\right]^2,\ n > r \geq 1. \tag{3.5}$$

Proof. By Binet's formula (2.4), we have

$$M_n^2 - M_{n+r}M_{n-r}$$
$$= (C_1\alpha^n + C_2\beta^n)^2 - (C_1\alpha^{n+r}+C_2\beta^{n+r})(C_1\alpha^{n-r}+C_2\beta^{n-r})$$

$$= C_1C_2(\alpha\beta)^n(2-\alpha^r\beta^{-r}-\alpha^{-r}\beta^r)$$
$$= C_1C_2(\alpha\beta)^{n-r}(2\alpha^r\beta^r-\alpha^{2r}-\beta^{2r})$$
$$= \text{-}C_1C_2(\alpha\beta)^{n-r}(\alpha^r-\beta^r)^2.$$

Using subsequent results of Binet's formula, we get

$$M_n^2 - M_{n+r}M_{n-r} = \left(s^2-5\right)(-1)^{n-r}\frac{(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2}.$$

Since $\dfrac{\alpha^r-\beta^r}{\alpha-\beta} = \dfrac{(s+1)M_r - 2M_{r+1}}{(s+1)^2-2(s+1)-4}$

$$= \frac{(s+1)M_r - 2M_{r+1}}{s^2-5},$$

we obtain

$$M_n^2 - M_{n+r}M_{n-r} = \frac{(-1)^{n-r}}{s^2-5}\left[(s+1)M_r - 2M_{r+1}\right]^2,\ n > r \geq 1.$$

Corollary (3.5.1). (Cassini's Identity) Let M_n be the n^{th} term of Generalized Fibonacci-Like sequence. Then

$$M_n^2 - M_{n+1}M_{n-1} = (-1)^{n-1}(s^2-5),\ n \geq 1. \tag{3.6}$$

Taking $r = 1$ in the Catalan's identity (3.5), the required identity is obtained.

Theorem (3.6). (d'Ocagne's Identity) Let M_n be the n^{th} term of generalized Fibonacci-Like sequence. Then

$$M_m M_{n+1} - M_{m+1} M_n = (-1)^n \left[(s+1) M_{m-n} - 2M_{m-n+1} \right], \quad m > n \geq 0. \tag{3.7}$$

Proof. By Binet's formula (2.4), we have

$$M_m M_{n+1} - M_{m+1} M_n = (C_1\alpha^m + C_2\beta^m)(C_1\alpha^{n+1} + C_2\beta^{n+1}) - (C_1\alpha^{m+1} + C_2\beta^{m+1})(C_1\alpha^n + C_2\beta^n) = C_1C_2(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n)$$

$$M_m M_{n+1} - M_{m+1} M_n = C_1C_2(\alpha\beta)^n \left[\beta(\alpha^{m-n} - \beta^{m-n}) - \alpha(\alpha^{m-n} - \beta^{m-n}) \right] = -C_1C_2(\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n} - \beta^{m-n}).$$

Using subsequent results of Binet's formula, we get

$$M_m M_{n+1} - M_{m+1} M_n = (-1)^n \left(s^2-5\right) \frac{\left(\alpha^{m-n} - \beta^{m-n}\right)}{\alpha - \beta}.$$

Since $\dfrac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta} = \dfrac{(s+1)M_{m-n} - 2M_{m-n+1}}{\left(s^2-5\right)}$,

We get

$$M_m M_{n+1} - M_{m+1} M_n = (-1)^n \left[(s+1) M_{m-n} - 2M_{m-n+1} \right], \quad m > n \geq 0.$$

Theorem (3.7). (Generalized Identity) Let M_n be the n^{th} term of Generalized Fibonacci-Like sequence. Then

$$M_m M_n - M_{m-r} M_{n+r} = \frac{(-1)^{m-r}}{\left(s^2-5\right)} \left\{ (s+1)M_r - 2M_{r+1} \right\} \left\{ \begin{matrix} (s+1)M_{n-m+r} \\ -2M_{n-m+r+1} \end{matrix} \right\}, \tag{3.8}$$

$n > m \geq r \geq 1.$

Proof. By Binet's formula (2.4), we have

$$M_m M_n - M_{m-r} M_{n+r} = (C_1\alpha^m + C_2\beta^m)(C_1\alpha^n + C_2\beta^n) - (C_1\alpha^{m-r} + C_2\beta^{m-r})(C_1\alpha^{n+r} + C_2\beta^{n+r})$$

$$= C_1C_2(\alpha^r - \beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r} - \frac{\alpha^n\beta^m}{\beta^r} \right]$$

$$= C_1C_2(-1)^{-r}(\alpha^r - \beta^r)(\alpha^m\beta^{n+r} - \alpha^{n+r}\beta^m)$$

$$= C_1C_2(-1)^{-r}\alpha^m\beta^m(\alpha^r - \beta^r)(\beta^{n-m+r} - \alpha^{n-m+r})$$

$$= -C_1C_2(-1)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r}).$$

Using subsequent results of Binet's formula, we get

$$M_m M_n - M_{m-r} M_{n+r} = \frac{(s^2-5)}{(\alpha-\beta)^2}(-1)^{m-r}(\alpha^r - \beta^r)(\alpha^{n-m+r} - \beta^{n-m+r}).$$

Since $\dfrac{\alpha^r - \beta^r}{\alpha - \beta} = \dfrac{1}{s^2-5}\left\{ (s+1)M_r - 2M_{r+1} \right\}$ *and*

$$\frac{\alpha^{n-m+r} - \beta^{n-m+r}}{\alpha - \beta} = \frac{1}{s^2-5}\left\{ (s+1)M_{n-m+r} - 2M_{n-m+r+1} \right\},$$

we obtain

$$M_m M_n - M_{m-r} M_{n+r} = \frac{(-1)^{m-r}}{\left(s^2-5\right)} \left\{ (s+1)M_r - 2M_{r+1} \right\} \left\{ \begin{matrix} (s+1)M_{n-m+r} \\ -2M_{n-m+r+1} \end{matrix} \right\},$$

$n > m \geq r \geq 1.$

The identity (3.8) provides Catalan's, Cassini's and d'Ocagne's and other identities:

(i) If m=n, the Catalan's identity (3.5) is obtained.

(ii) If m=n and $r=1$ in identity (3.8), the Cassini's identity (5.1) is obtained.

(iii) n=m, m= $n+1$ and $r=1$ in identity (3.8), the d'Ocagne's identity (3.6) is obtained.

4. Determinant Identities

There is a long tradition of using matrices and determinants to study Fibonacci numbers. Problems on determinants of Fibonacci sequence and Lucas sequence are appeared in various issues of Fibonacci Quarterly. T. Koshy [13] explained two chapters on the use of matrices and determinants. Many determinant identities of generalized Fibonacci sequence are discussed in [4], [6] and [11]. In this section some determinant identities of Generalized Fibonacci-Like sequence are presented. Entries of determinants are satisfying the recurrence relation of Generalized Fibonacci-Like sequence and other sequences.

Theorem(4.1). For any integers $n \geq 0$, prove that

$$\begin{vmatrix} M_{n+1} & M_{n+2} & M_{n+3} \\ M_{n+4} & M_{n+5} & M_{n+6} \\ M_{n+7} & M_{n+8} & M_{n+9} \end{vmatrix} = 0. \tag{4.1}$$

Proof.

$$\text{Let } \Delta = \begin{vmatrix} M_{n+1} & M_{n+2} & M_{n+3} \\ M_{n+4} & M_{n+5} & M_{n+6} \\ M_{n+7} & M_{n+8} & M_{n+9} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\text{Let } \Delta = \begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+3} \\ M_{n+6} & M_{n+5} & M_{n+6} \\ M_{n+9} & M_{n+8} & M_{n+9} \end{vmatrix}$$

Since two columns are identical, thus we obtained required result.

Theorem (4.2). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n - M_{n+1} & M_{n+1} - M_{n+2} & M_{n+2} - M_n \\ M_{n+1} - M_{n+2} & M_{n+2} - M_n & M_n - M_{n+1} \\ M_{n+2} - M_n & M_n - M_{n+1} & M_{n+1} - M_{n+2} \end{vmatrix} = 0. \tag{4.2}$$

Proof.

Let $\Delta = \begin{vmatrix} M_n - M_{n+1} & M_{n+1} - M_{n+2} & M_{n+2} - M_n \\ M_{n+1} - M_{n+2} & M_{n+2} - M_n & M_n - M_{n+1} \\ M_{n+2} - M_n & M_n - M_{n+1} & M_{n+1} - M_{n+2} \end{vmatrix}$.

By applying $C_1 \to C_1 + C_2 + C_3$ and expanding along first row, we obtained required result.

Theorem (4.3). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ M_{n+1} + M_{n+2} & M_n + M_{n+2} & M_n + M_{n+1} \end{vmatrix} = 0. \quad (4.3)$$

Proof.

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ M_{n+1} + M_{n+2} & M_n + M_{n+2} & M_n + M_{n+1} \end{vmatrix}$.

Applying $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ 2M_{n+2} & 2M_{n+2} & 2M_{n+2} \end{vmatrix}.$$

Taking common out $2M_{n+2}$ from third row,

$$\Delta = 2M_{n+2} \begin{vmatrix} 1 & 1 & 1 \\ M_n & M_{n+1} & M_{n+2} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, thus we obtained required result.

Theorem (4.4). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 2M_n & 2M_n + 3M_{n+1} & 2M_n + 3M_{n+1} + 4M_{n+2} \\ 3M_n & 3M_n + 6M_{n+1} & 3M_n + 6M_{n+1} + 12M_{n+2} \end{vmatrix} \quad (4.4)$$
$$= 3M_n M_{n+1} M_{n+2}.$$

Proof. Let

$$\Delta = \begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 2M_n & 2M_n + 3M_{n+1} & 2M_n + 3M_{n+1} + 4M_{n+2} \\ 3M_n & 3M_n + 6M_{n+1} & 3M_n + 6M_{n+1} + 12M_{n+2} \end{vmatrix}.$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} M_n & M_n + M_{n+1} & M_n + M_{n+1} + M_{n+2} \\ 0 & M_{n+1} & M_{n+1} + 2M_{n+2} \\ 0 & 3M_{n+1} & 3M_{n+1} + 9M_{n+2} \end{vmatrix}.$$

Applying $R_3 \to R_3 - 3R_2$ and expanding along first row, we obtained required result.

Theorem (4.5). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 0 & M_n M_{n+1}^2 & M_n M_{n+2}^2 \\ M_n^2 M_{n+1} & 0 & M_{n+1} M_{n+2}^2 \\ M_n^2 M_{n+2} & M_{n+2} M_{n+1}^2 & 0 \end{vmatrix} \quad (4.5)$$
$$= 2M_n^3 M_{n+1}^3 M_{n+2}^3.$$

Proof. Let

$$\Delta = \begin{vmatrix} 0 & M_n M_{n+1}^2 & M_n M_{n+2}^2 \\ M_n^2 M_{n+1} & 0 & M_{n+1} M_{n+2}^2 \\ M_n^2 M_{n+2} & M_{n+2} M_{n+1}^2 & 0 \end{vmatrix}.$$

Taking common out M_n^2, M_{n+1}^2, M_{n+2}^2 from C_1, C_2, C_3 respectively, we get

$$\Delta = M_n^2 M_{n+1}^2 M_{n+2}^2 \begin{vmatrix} 0 & M_n & M_n \\ M_{n+1} & 0 & M_{n+1} \\ M_{n+2} & M_{n+2} & 0 \end{vmatrix}.$$

Taking common out M_n, M_{n+1}, M_{n+2} from R_1, R_2, R_3 respectively and expanding along first row, we obtained required result.

Theorem (4.6). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix} = \left[F_n M_{n+1} - M_n F_{n+1}\right]. \quad (4.6)$$

Proof: Let $\Delta = \begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix}$.

Assume $M_n = a$, $M_{n+1} = b$, $F_n = p$, $F_{n+1} = q$ then $M_{n+2} = a + b$ *and* $F_{n+2} = p + q$

Now substituting the above values in determinant, we get

$$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix} = (pb - aq).$$

Substituting the values of a, b, p and q, we get required result.

Similarly following identities can be derived:

Theorem (4.7). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & M_{n+1} & M_{n+2} \\ M_{n+2} & M_n & M_{n+1} \\ M_{n+1} & M_{n+2} & M_n \end{vmatrix} = 2(M_n^3 + M_{n+1}^3). \quad (4.7)$$

Theorem (4.8). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n & L_n & 1 \\ M_{n+1} & L_{n+1} & 1 \\ M_{n+2} & L_{n+2} & 1 \end{vmatrix} = 2(L_n M_{n+1} - M_n L_{n+1}). \quad (4.8)$$

Theorem (4.9). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} M_n + M_{n+1} & M_{n+1} + M_{n+2} & M_{n+2} + M_n \\ M_{n+2} & M_n & M_{n+1} \\ 1 & 1 & 1 \end{vmatrix} = 0. \quad (4.9)$$

Theorem 4.(10). For any integer $n \geq 0$, prove that

$$\begin{vmatrix} 1+M_n & M_{n+1} & M_{n+2} \\ M_n & 1+M_{n+1} & M_{n+2} \\ M_n & M_{n+1} & 1+M_{n+2} \end{vmatrix} = 1 + M_n + M_{n+1} + M_{n+2}. \quad (4.10)$$

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