Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 4, 125-129
Available online at http://pubs.sciepub.com/tjant/2/4/4

DOI:10.12691/tjant-2-4-4

Generating Function for M(m,n)

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Received April 04, 2014; Revised June 14, 2014; Accepted July 31, 2014

Abstract This paper shows that the coefficient of x in the right hand side of the equation $\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)\, z^m x^n = \prod_{n=1}^{\infty}\frac{\left(1-x^n\right)}{\left(1-zx^n\right)\left(1-z^{-1}x^n\right)}$ is an algebraic relation in terms of z. The exponent of z represents the crank of partitions of a positive integral value of n and also shows that the sum of weights of corresponding partitions of n is the sum of ordinary partitions of n and it is equal to the number of partitions of n with crank m. This paper shows how to prove the Theorem "The number of partitions π of n with crank $C(\pi)=m$ is $M(m,n)$ for all $n>1$."

Keywords: *crank, j-times, vector partitions, weight, exponent*

Cite This Article: Sabuj Das, and Haradhan Kumar Mohajan, "Generating Function for M(m,n)." *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 4 (2014): 125-129. doi: 10.12691/tjant-2-4-4.

1. Introduction

First we give definitions of $P(n)$, the crank of partitions, $(x)_\infty$, $(zx)_\infty$, $\left(x^2;x\right)_\infty$ and $M(m,n)$. We generate some generating functions related to the crank and show the coefficient of x is the algebraic relations in terms of various powers of z, the exponent of z represent the crank of partitions of n (for all $n>1$). We show the results with the help of examples when n = 5 and 6 respectively. We introduce the special term weight $\omega(\vec{\pi})$ related to the vector partitions V and show the relations in terms of $M(m,n)$, weight $\omega(\vec{\pi})$ and crank $(\vec{\pi})$. We prove the Theorem "The number of partitions π of n with crank $C(\pi)=m$ is $M(m,n)$ for all $n>1$."

2. Definitions

Now we give some definitions following ([3,4,5]).

$P(n)$: Number of partitions of n, like 4, 3+1, 2+2, 2+1+1, 1+1+1+1. Therefore, $P(4)=5$ and similarly $P(5)=7$ etc.

Crank of partitions [2]: For a partition π, let $l(\pi)$ denotes the largest part of π, $\omega(\pi)$ denote the number of 1's in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$, the crank $c(\pi)$ is given by;

$$c(\pi)=\begin{cases} l(\pi)\ ; & \text{if } \omega(\pi)>0 \\ \mu(\pi)-\omega(\pi)\ ; & \text{if } \omega(\pi)>0. \end{cases}$$

$$(x)_\infty = (1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots$$

$$(zx)_\infty = (1-zx)\left(1-zx^2\right)\left(1-zx^3\right)\ldots$$

$$\left(x^2;x\right)_\infty = \left(1-x^2\right)\left(1-x^3\right)\left(1-x^4\right)\ldots$$

$M(m,n)$: The number of partitions of n with crank m.

2.1. Notations

For all integers $n\geq 0$ and all integers m, the number of n with crank equal to m is $M(-1,1)=1$, like;

Partitions of 1 (π)	Largest part $l(\pi)$	Number of 1's $\omega(\pi)$	Number of parts larger than $\omega(\pi)$ $\mu(\pi)$	Crank $c(\pi)$
1	1	1	0	–1

$$M(-1,1)=1.$$

But we see that;

$$M(-1,1)=M(1,1)=-M(0,1)=1.$$

Since, the coefficient of x in the right hand side of the equation;

$$\sum_{m-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)\, z^m x^n = \frac{(x)_\infty}{(zx)_\infty\left(z^{-1}x\right)_\infty}$$

is $z^{-1}+z-1$ i.e., $z^{-1}+z^{1}-z^{0}$ the exponent of z being the crank of partition.

Therefore, $M(-1,1)=M(1,1)=-M(0,1)=1$.

3. The Generating Function for $M(m,n)$

The generating function for $M(m,n)$ is given by [2];

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)\, z^{m}x^{n}=\prod_{n=1}^{\infty}\frac{(1-x^{n})}{(1-zx^{n})\,(1-z^{-1}x^{n})}$$

$$=\frac{(1-x)\,(1-x^{2})\,(1-x^{3})\ldots}{(1-zx)\,(1-zx^{2})\ldots(1-z^{-1}x)\,(1-z^{-1}x^{2})\ldots}$$

$$=\frac{(1-x)\,(1-x^{2})\,(1-x^{3})\ldots}{(1-zx)\,(1-zx^{2})\ldots(1-z^{-1}x)\,(1-z^{-1}x^{2})\ldots}$$

$$=\frac{(1-x)}{(zx)_{\infty}}\left\{\frac{(1-x^{2})\,(1-x^{3})\ldots}{(1-z^{-1}x)\,(1-z^{-1}x^{2})\ldots}\right\}$$

$$=\frac{(1-x)}{(zx)_{\infty}}\left\{\sum_{j=0}^{\infty}\frac{(zx)_{j}\,(xz^{-1})^{j}}{(x)_{j}}\right\}, \text{ by Andrews } [1],$$

$$=\frac{(1-x)}{(zx)_{\infty}}\left\{\begin{array}{l}1+\frac{(zx)_{1}\,(xz^{-1})^{1}}{(x)_{1}}+\frac{(zx)_{2}\,(xz^{-1})^{2}}{(x)_{2}}\\ +\frac{(zx)_{3}\,(xz^{-1})^{3}}{(x)_{3}}+\ldots\end{array}\right\}$$

$$=\frac{(1-x)}{(zx)_{\infty}}\left\{1+\frac{(1-zx)\,xz^{-1}}{(1-x)}+\frac{(1-zx)\,(1-zx^{2})\,x^{2}z^{-2}}{(1-x)\,(1-x^{2})}+\right\}$$

$$=\frac{(1-x)}{(1-zx)\,(1-zx^{2})\ldots}+\frac{xz^{-1}}{(1-zx^{2})\ldots}+$$

$$\frac{x^{2}z^{-2}}{(1-x^{2})\,(1-zx^{3})\ldots}+\frac{(1-zx)\,(1-zx^{2})\,(1-zx^{3})\,x^{3}z^{-3}}{(1-x)\,(1-x^{2})\,(1-x^{3})}$$

$$+\ldots+\frac{x^{3}z^{-3}}{(1-x^{2})\,(1-x^{3})\,(1-zx^{4})\ldots}+\ldots$$

$$=\frac{(1-x)}{(1-zx)\,(1-zx^{2})\ldots}+\sum_{j=1}^{\infty}\frac{x^{j}z^{-j}}{(x^{2};x)_{j-1}\,(zx^{j+1})_{\infty}}$$

$$=1+(z^{-1}+z-1)\,x+(z^{-2}+z^{2})\,x^{2}+(z^{-3}+z^{3}+1)\,x^{3}$$

$$+\begin{pmatrix}1+z^{-2}+z^{2}\\ +z^{-4}+z^{4}\end{pmatrix}x^{4}+\begin{pmatrix}1+z+z^{-1}+z^{3}\\ +z^{-3}+z^{5}+z^{-5}\end{pmatrix}x^{5} \qquad (1)$$

$$+\begin{pmatrix}1+z+z^{-1}+z^{2}+z^{-2}+z^{3}\\ +z^{-3}+z^{4}+z^{-4}+z^{6}+z^{-6}\end{pmatrix}x^{6}+\ldots$$

We see that the exponent of z represents the crank of partitions of n (for $n>1$). As for examples when n = 5 and 6,

For n = 5,

Partitions of 5 (π)	Largest part $l(\pi)$	Number of 1's $\omega(\pi)$	Number of parts larger than $\omega(\pi)$ $\mu(\pi)$	Crank $c(\pi)$
5	5	0	1	5
4+1	4	1	1	0
3+2	3	0	2	3
3+1+1	3	2	1	–1
2+2+1	2	1	2	1
2+1+1+1	2	3	0	–3
1+1+1+1+1	1	5	0	–5

For n = 6,

Partitions π	Largest part $l(\pi)$	Numbers of ones $\omega(\pi)$	Number of parts larger than $\omega(\pi)$ $\mu(\pi)$	Crank $c(\pi)$
6	6	0	1	6
5+1	5	1	1	0
4+2	4	0	2	4
4+1+1	4	2	1	–1
3+3	3	0	2	3
3+2+1	3	1	2	1
3+1+1+1	3	3	0	–3
2+2+2	2	0	3	2
2+2+1+1	2	2	0	–2
2+1+1+1+1	2	4	0	–4
1+1+1+1+1+1	1	6	0	–6

4. Vector Partitions of *n*

Let, $V=D\times P\times P$, where D denotes the set of partitions into distinct parts and P denotes the set of partitions. The set of vector partitions V is defined by the Cartesian product, $V=D\times P\times P$.

For $\vec{\pi}=(\pi_1,\pi_2,\pi_3)\in V$, where $|\vec{\pi}|=|\pi_1|+|\pi_2|+|\pi_3|$ weight = $\omega(\vec{\pi})=(-1)^{\#(\pi_1)}$, the crank $(\vec{\pi})=\#(\pi_2)\ -\#(\pi_3)$.

We have 41 vector partitions of 4 are given in the following table:

Vector partitions of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1=(\varphi,\varphi,4)$	+1	–1
$\vec{\pi}_2=(\varphi,\varphi,3+1)$	+1	–2
$\vec{\pi}_3=(\varphi,\varphi,2+2)$	+1	–2
$\vec{\pi}_4=(\varphi,\varphi,2+1+1)$	+1	–3
$\vec{\pi}_5=(\varphi,\varphi,1+1+1+1)$	+1	–4
$\vec{\pi}_6=(\varphi,1,3)$	+1	0
$\vec{\pi}_7=(\varphi,1,2+1)$	+1	–1
$\vec{\pi}_8=(\varphi,1+1+1+1)$	+1	–2
$\vec{\pi}_9=(\varphi,2+2)$	+1	0
$\vec{\pi}_{10}=(\varphi,2,1+1)$	+1	–1

$\vec{\pi}_{11}=(\varphi,1+1,2)$	+1	1
$\vec{\pi}_{12}=(\varphi,1+1,1+1)$	+1	0
$\vec{\pi}_{13}=(\varphi,3,1)$	+1	0
$\vec{\pi}_{14}=(\varphi,2+1,1)$	+1	1
$\vec{\pi}_{15}=(\varphi,1+1+1,1)$	+1	2
$\vec{\pi}_{16}=(\varphi,4,\varphi)$	+1	1
$\vec{\pi}_{17}=(\varphi,3+1,\varphi)$	+1	2
$\vec{\pi}_{18}=(\varphi,2+2,\varphi)$	+1	2
$\vec{\pi}_{19}=(\varphi,2+1+1,\varphi)$	+1	3
$\vec{\pi}_{20}=(\varphi,1+1+1+1,\varphi)$	+1	4
$\vec{\pi}_{21}=(1,\varphi,3)$	−1	−1
$\vec{\pi}_{22}=(1,\varphi,2+1)$	−1	−2
$\vec{\pi}_{23}=(1,\varphi,1+1+1)$	−1	−3
$\vec{\pi}_{24}=(1,1,2)$	−1	0
$\vec{\pi}_{25}=(1,1,1+1)$	−1	−1
$\vec{\pi}_{26}=(1,2,1)$	−1	0
$\vec{\pi}_{27}=(1+1,1,1)$	−1	1
$\vec{\pi}_{28}=(1,3,\varphi)$	−1	1
$\vec{\pi}_{29}=(1,2+1,\varphi)$	−1	2
$\vec{\pi}_{30}=(1,1+1+1,\varphi)$	−1	3
$\vec{\pi}_{31}=(2,\varphi,2)$	−1	−1
$\vec{\pi}_{32}=(2,\varphi,1+1)$	−1	−2
$\vec{\pi}_{33}=(2,1,1)$	−1	0
$\vec{\pi}_{34}=(2,2,\varphi)$	−1	1
$\vec{\pi}_{35}=(2,1+1,\varphi)$	−1	2
$\vec{\pi}_{36}=(3,\varphi,1)$	−1	−1
$\vec{\pi}_{37}=(2+1,\varphi,1)$	+1	−1
$\vec{\pi}_{38}=(3,1,\varphi)$	−1	1
$\vec{\pi}_{39}=(2+1,1,\varphi)$	+1	1
$\vec{\pi}_{40}=(4,\varphi,\varphi)$	−1	0
$\vec{\pi}_{41}=(3+1,\varphi,\varphi)$	+1	0

From the above table we have,

$$M(0,4)=\omega(\vec{\pi}_6)+\omega(\vec{\pi}_9)+\omega(\vec{\pi}_{12})+\omega(\vec{\pi}_{13})+\omega(\vec{\pi}_{24})$$
$$+\omega(\vec{\pi}_{26})+\omega(\vec{\pi}_{33})+\omega(\vec{\pi}_{40})+\omega(\vec{\pi}_{41})$$
$$=1+1+1+1-1-1-1-1+1=1$$

$$M(1,4)=\omega(\vec{\pi}_{11})+\omega(\vec{\pi}_{14})+\ldots+\omega(\vec{\pi}_{39})$$
$$=1+1+1-1-1-1-1+1=0.$$

and

$$M(-1,4)=\omega(\vec{\pi}_1)+\omega(\vec{\pi}_7)+\ldots+\omega(\vec{\pi}_{37})$$
$$=1+1+1-1-1-1-1+1=0$$

$$M(2,4)=1+1+1-1-1=1.$$

$$M(-2,4)=1+1+1-1-1=1.$$

$$M(3,4)=1-1=0$$

$$M(-3,4)=1-1=0$$

$$M(4,4)=1$$

$$M(-4,4)=1$$

$$\sum M(m,4)=\sum\omega(\vec{\pi});$$

i.e., $$\sum_{m=-\infty}^{\infty}M(m,4)=\sum_{\substack{\vec{\pi}\in V\\|\pi|=4\\\text{crank}(\vec{\pi})=m}}\omega(\vec{\pi})=5=$$

i.e., $$\sum_{m=-\infty}^{\infty}M(m,4)=\sum_{\substack{\vec{\pi}\in V\\|\pi|=4\\\text{crank}(\vec{\pi})=m}}\omega(\vec{\pi})=P(4).$$

Again we have 83 vector partitions of 5 are given in the following table:

Vector partitions of 5	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1=(\varphi,\varphi,5)$	+1	−1
$\vec{\pi}_2=(\varphi,\varphi,4+1)$	+1	−2
$\vec{\pi}_3=(\varphi,\varphi,3+2)$	+1	−2
$\vec{\pi}_4=(\varphi,\varphi,3+1+1)$	+1	−3
$\vec{\pi}_5=(\varphi,\varphi,2+2+1)$	+1	−3
$\vec{\pi}_6=(\varphi,\varphi,2+1+1+1)$	+1	−4
$\vec{\pi}_7=(\varphi,\varphi,1+1+1+1+1)$	+1	−5
$\vec{\pi}_8=(5,\varphi,\varphi)$	−1	0
$\vec{\pi}_9=(\varphi,5,\varphi)$	+1	1
$\vec{\pi}_{10}=(\varphi,4+1,\varphi)$	+1	2
$\vec{\pi}_{11}=(4+1,\varphi,\varphi)$	+1	0
$\vec{\pi}_{12}=(4,1,\varphi)$	−1	1
$\vec{\pi}_{13}=(1,4,\varphi)$	−1	1
$\vec{\pi}_{14}=(\varphi,4,1)$	+1	0
$\vec{\pi}_{15}=(\varphi,1,4)$	+1	0
$\vec{\pi}_{16}=(1,\varphi,4)$	−1	−1
$\vec{\pi}_{17}=(4,\varphi,1)$	−1	−1
$\vec{\pi}_{18}=(3+2,\varphi,\varphi)$	+1	0
$\vec{\pi}_{19}=(\varphi,3+2,\varphi)$	+1	2
$\vec{\pi}_{20}=(3,2,\varphi)$	−1	1
$\vec{\pi}_{21}=(2,3,\varphi)$	−1	1
$\vec{\pi}_{22}=(\varphi,3,2)$	+1	0
$\vec{\pi}_{23}=(\varphi,2,3)$	+1	0
$\vec{\pi}_{24}=(3,\varphi,2)$	−1	−1
$\vec{\pi}_{25}=(2,\varphi,3)$	−1	−1
$\vec{\pi}_{26}=(\varphi,3+1+1,\varphi)$	+1	3
$\vec{\pi}_{27}=(3+1,1,\varphi)$	+1	1
$\vec{\pi}_{28}=(1,3+1,\varphi)$	−1	2

$\vec{\pi}_{29}=(\varphi,3+1,1)$	+1	1
$\vec{\pi}_{30}=(\varphi,1,3+1)$	+1	–1
$\vec{\pi}_{31}=(3+1,\varphi,1)$	+1	–1
$\vec{\pi}_{32}=(1,\varphi,3+1)$	–1	–2
$\vec{\pi}_{33}=(3,1+1,\varphi)$	–1	2
$\vec{\pi}_{34}=(\varphi,1+1,3)$	+1	1
$\vec{\pi}_{35}=(\varphi,3,1+1)$	+1	–1
$\vec{\pi}_{36}=(3,\varphi,1+1)$	–1	–2
$\vec{\pi}_{37}=(\varphi,2+2+1,\varphi)$	+1	3
$\vec{\pi}_{38}=(1,2+2,\varphi)$	–1	2
$\vec{\pi}_{39}=(\varphi,2+2,1)$	+1	1
$\vec{\pi}_{40}=(\varphi,1,2+2)$	+1	–1
$\vec{\pi}_{41}=(1,\varphi,2+2)$	–1	–2
$\vec{\pi}_{42}=(2+1,2,\varphi)$	+1	1
$\vec{\pi}_{43}=(2,2+1,\varphi)$	–1	2
$\vec{\pi}_{44}=(\varphi,2,2+1)$	+1	1
$\vec{\pi}_{45}=(\varphi,2+1,2)$	+1	1
$\vec{\pi}_{46}=(2+1,\varphi,2)$	+1	–1
$\vec{\pi}_{47}=(2,\varphi,2+1)$	–1	–2
$\vec{\pi}_{48}=(\varphi,2+2+1,\varphi)$	+1	4
$\vec{\pi}_{49}=(\varphi,2+1+1,1)$	+1	2
$\vec{\pi}_{50}=(\varphi,1,2+1+1)$	+1	–2
$\vec{\pi}_{51}=(1,2+1+1,\varphi)$	–1	3
$\vec{\pi}_{52}=(1,\varphi,2+1+1)$	–1	–3
$\vec{\pi}_{53}=(2+1,1+1,\varphi)$	+1	2
$\vec{\pi}_{54}=(\varphi,2+1,1+1)$	+1	0
$\vec{\pi}_{55}=(\varphi,1+1,2+1)$	+1	0
$\vec{\pi}_{56}=(2+1,\varphi,1+1)$	+1	–2
$\vec{\pi}_{57}=(\varphi,1+1+1,2)$	+1	2
$\vec{\pi}_{58}=(\varphi,2,1+1+1)$	+1	–2
$\vec{\pi}_{59}=(2,1+1+1,\varphi)$	–1	3
$\vec{\pi}_{60}=(2,\varphi,1+1+1)$	–1	–3
$\vec{\pi}_{61}=(\varphi,1+1+1+1+1,\varphi)$	+1	5
$\vec{\pi}_{62}=(\varphi,1+1+1+1,1)$	+1	3
$\vec{\pi}_{63}=(\varphi,1,1+1+1+1)$	+1	–3
$\vec{\pi}_{64}=(1,\varphi,1+1+1+1)$	–1	–4
$\vec{\pi}_{65}=(1,1+1+1+1,\varphi)$	–1	4
$\vec{\pi}_{66}=(\varphi,1+1,1+1+1)$	+1	–1
$\vec{\pi}_{67}=(\varphi,1+1+1,1+1)$	+1	1
$\vec{\pi}_{68}=(1,1,1+1+1)$	–1	–2
$\vec{\pi}_{69}=(1,1+1+1,1)$	–1	2
$\vec{\pi}_{70}=(1,1+1,1+1)$	–1	0
$\vec{\pi}_{71}=(1,1+1,2)$	–1	1
$\vec{\pi}_{72}=(1,2,1+1)$	–1	–1
$\vec{\pi}_{73}=(2,1+1,1)$	–1	1
$\vec{\pi}_{74}=(2,1,1+1)$	–1	–1
$\vec{\pi}_{75}=(2,2,1)$	–1	0
$\vec{\pi}_{76}=(2,1,2)$	–1	0
$\vec{\pi}_{77}=(1,2,2)$	–1	0
$\vec{\pi}_{78}=(3,1,1)$	–1	0
$\vec{\pi}_{79}=(1,3,1)$	–1	0
$\vec{\pi}_{80}=(1,1,3)$	–1	0
$\vec{\pi}_{81}=(1+2,1,1)$	+1	0
$\vec{\pi}_{82}=(1,1+2,1)$	–1	1
$\vec{\pi}_{83}=(1,1,1+2)$	–1	–1

From this table we have;

$$M(0,5)=\omega(\vec{\pi}_8)+\omega(\vec{\pi}_{11})+\omega(\vec{\pi}_{14})+\omega(\vec{\pi}_{15})$$
$$+\omega(\vec{\pi}_{18})+\omega(\vec{\pi}_{22})+\omega(\vec{\pi}_{23})+\omega(\vec{\pi}_{54})+\omega(\vec{\pi}_{55})$$
$$+\omega(\vec{\pi}_{70})+\omega(\vec{\pi}_{75})+\omega(\vec{\pi}_{76})+\omega(\vec{\pi}_{78})+\omega(\vec{\pi}_{79})$$
$$+\omega(\vec{\pi}_{79})+\omega(\vec{\pi}_{80})+\omega(\vec{\pi}_{81})$$
$$=-1+1+1+1+1+1+1+1+1$$
$$-1-1-1-1-1-1-1-1+1=1.$$

$M(1,5)=$ 1–1–1–1–1+1+1+1+1+1+1+1–1–1–1 =1

$M(-1,5)=$ 1–1–1–1–1+1+1+1+1+1+1+1–1–1–1 =1

$M(2,5)=$ 1+1–1–1–1–1+1+1+1–1= 0

$M(-2,5)=$ 1+1–1–1–1–1+1+1+1–1= 0

$M(3,5)=$ 1+1–1–1+1= 1

$M(-3,5)=$ 1+1–1–1+1= 1

$M(4,5)=$ 1–1= 0

$M(-4,5)=$ 1–1= 0

$M(5,5)=1$

$M(-5,5)=1$

$\sum M(m,5)=\sum\omega(\vec{\pi})$;

i.e., $$\sum_{m=-\infty}^{\infty} M(m,5)=\sum_{\substack{\vec{\pi}\in V\\ |\pi|=5\\ \text{crank}(\vec{\pi})=m}}\omega(\vec{\pi})=7$$

i.e., $$\sum_{m=-\infty}^{\infty} M(m,5)=\sum_{\substack{\vec{\pi}\in V\\ |\pi|=5\\ \text{crank}(\vec{\pi})=m}}\omega(\vec{\pi})=P(5).$$

From above discussion we get;

$$\sum_{m=-\infty}^{\infty} M(m,n)=\sum_{\substack{\vec{\pi}\in V\\ |\pi|=n\\ \text{crank}(\vec{\pi})=m}}\omega(\vec{\pi})=P(n).$$

Theorem: The number of partitions π of n with crank $c(\pi)=m$ is $M(m,n)$ for all $n>1$.

Proof: The generating function for $M(m,n)$ is given by;

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)\, z^m x^n = \prod_{n=1}^{\infty}\frac{(1-x^n)}{(1-zx^n)\,(1-z^{-1}x^n)}$$
$$= \frac{(1-x)}{(1-zx)\,(1-zx^2)\,\ldots} + \sum_{j=1}^{\infty}\frac{x^j z^{-j}}{(x^2;x)_{j\text{-}1}\,(zx^{j+1})_{\infty}}. \quad (2)$$

Now we distribute the function into two parts where first one represents the crank with $c(\pi)=l(\pi)$ and second one represents the crank with $c(\pi)=\mu(\pi)-\omega(\pi)$.

The first function is;

$$\frac{(1-x)}{(1-zx)\,(1-zx^2)\,(1-zx^3)\,\ldots}$$
$$=1+(z-1)\,x+z^2x^2+z^3x^3+(z^2+z^4)\,x^4$$
$$+(z^3+z^5)\,x^5+\ldots$$

Counts (for $n>1$) the number of partitions with no 1's and the exponent on z being the largest part of the partition where $c(\pi)=l(\pi)$, like;

Partitions of 4 (π)	Largest part $l(\pi)$	Number of 1's $\omega(\pi)$	Number of parts larger than $\omega(\pi)$ $\mu(\pi)$	Crank $c(\pi)$
4	4	0	1	4
2+2	2	0	2	2

Here $n = 4$, the 5th term is $(z^2+z^4)\,x^4$.

Again second partition is,

$$\sum_{j=1}^{\infty}\frac{x^j z^{-j}}{(x^2,x)_{j\text{-}1}\,(zx^{j+1})_{\infty}}$$
$$=\frac{xz^{-1}}{(1-zx^2)\,(1-zx^3)\,\ldots}+\frac{x^2z^{-2}}{(1-x^2)\,(1-zx^3)\,(1-zx^4)\,\ldots}$$
$$+\frac{x^3z^{-3}}{(1-x^2)\,(1-x^3)\,(1-zx^4)\,(1-zx^5)\,\ldots}+\ldots$$
$$=z^{-1}x+z^{-2}x^2+(1+z^{-3})\,x^3+(1+z^{-2}+z^{-4})\,x^4+\ldots$$

which counts the number of partitions with $\omega(\pi)=j$ and the exponent on z is clearly $c(\pi)=\mu(\pi)-\omega(\pi)$, since $i>0$, like;

Partitions of 4 (π)	Largest part $l(\pi)$	Number of 1's $\omega(\pi)$	Number of parts larger than $\omega(\pi)$ $\mu(\pi)$	Crank $c(\pi)$
3+1	3	1	1	0
2+1+1	2	2	0	–2
1+1+1+1	1	4	0	–4

Here n = 4, the 5th term is $(1+z^{-2}+z^{-4})x^4$ i.e., $(z^0+z^{-2}+z^{-4})x^4$.

Thus in the double series expansion of

$$=\frac{(1-x)}{(1-zx)\,(1-zx^2)\,\ldots}+\sum_{j=1}^{\infty}\frac{x^j z^{-j}}{(x^2,x)_{j\text{-}1}\,(zx^{j+1})_{\infty}},$$

we see that the coefficient of $z^m x^n$ $(n>1)$ is the number of partitions of n in which $c(\pi)=m$. Equating the coefficient of $z^m x^n$ from both sides in (2) we get the number of partitions of n with $c(\pi)=m$ is $M(m,n)$ for all $n>1$. Hence the Theorem.

5. Conclusion

We have verified that the coefficient of x in the right hand side of the generating function for $M(m,n)$ is an explanation of z, the exponents of z represent the crank of partitions, it is already shown with examples for n = 5 and 6. We have satisfied the result $\sum_{m=-\infty}^{\infty} M(m,n)=\sum_{\substack{\vec{\pi}\in V\\ |\pi|=n\\ \text{crank}(\vec{\pi})=m}}\omega(\vec{\pi}) = P(n)$, it is already shown when n = 4 and 5 respectively. For any positive integer of n we can verify the corresponding Theorem. We have already satisfied the Theorem for n = 4 and 5.

Acknowledgment

It is a great pleasure to express our sincerest gratitude to our respected Professor Md. Fazlee Hossain, Department of Mathematics, University of Chittagong, Bangladesh. We will remain ever grateful to our respected Late Professor Dr. Jamal Nazrul Islam, RCMPS, University of Chittagong, Bangladesh.

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