

On Some Inequalities for Functions Whose Second Derivatives Absolute Values Are S-Geometrically Convex

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Abstract In this paper, the authors achieve some new Hadamard type inequalities using elementary well known inequalities for functions whose second derivatives absolute values are s-geometrically and geometrically convex. And also they get some applications for special means for positive numbers.

Keywords: s-geometrically convex, geometrically convex, Hadamard's inequality, Hölder's inequality, power mean inequality, means

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1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [5-9].

In this section we will present definitions and some results used in this paper.

Definition 1. Let I be an interval in \mathbb{R} : Then $f : I \rightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. [5] Let $s \in [0, 1]$. A function $f : I \subset \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}_0$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leq t^3 f(x) + (1-t)^3 f(y) \quad (1.2)$$

for all $x, y \in I$ and $t \in [0, 1]$.

It can be easily checked for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$.

Recently, in [12], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Definition 3. [12] A function $f : I \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$ is said to be a geometrically convex function if

$$f(x^t y^{1-t}) \leq |f(x)|^t |f(y)|^{1-t} \quad (1.3)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 4. [12] A function $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a s-geometrically convex function if

$$f(x^t y^{1-t}) \leq |f(x)|^{t^s} |f(y)|^{(1-t)^s} \quad (1.4)$$

for some $s \in (0, 1]$, where $x, y \in I$ and $t \in [0, 1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on \mathbb{R}_+ .

Example 1. [12] Let $f(x) = x^s / s$; $x \in (0, 1]$, $0 < s < 1$, $q \geq 1$, and then the function

$$|f'(x)|^q = x^{(s-1)q} \quad (1.5)$$

is monotonically decreasing on $(0, 1]$. For $t \in [0, 1]$, we have

$$(s-1)q(t^s - t) \leq 0, (s-1)q((1-t)^s - (1-t)) \leq 0 \quad (1.6)$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0 < s < 1$.

2. Hadamardfis Type Inequalities

In order to prove our main theorems, we need the following lemma [1,3].

Lemma 1. [1,3] Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable mapping on I° , $a, b \in I$ with $a < b$ and f'' is integrable on $[a,b]$, then the following equality holds:

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta+(1-t)b) dt \end{aligned} \tag{2.1}$$

A simple proof of this equality can be also done integrating by parts twice in the right hand side. The details are left to the interested reader.

The next theorems gives a new result of the upper Hermite-Hadamard inequality for s-geometrically and geometrically convex functions.

Theorem 1. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be twice differentiable mapping on I° , $a, b \in I$ with $a < b$ and f'' is integrable on $[a,b]$ and $|f''(a)| \leq 1$. If $|f''|$ is s-geometrically convex and monotonically decreasing on $[a,b]$; and $s \in (0,1]$ then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{12} |f''(b)|^s \left[\frac{\alpha(s,s)+1}{[\ln(\alpha(s,s))]^2} + \frac{2-2\alpha(s,s)}{[\ln(\alpha(s,s))]^3} \right] \end{aligned} \tag{2.2}$$

where

$$\alpha(u,v) = |f''(a)|^u |f''(b)|^{-v}, u, v \geq 0 \tag{2.3}$$

Proof. Since $|f''|$ is a s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1, we get

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \int_0^1 t(1-t) |f''(ta+(1-t)b)| dt \\ & \leq \int_0^1 t(1-t) |f''(a^t b^{(1-t)})| dt \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right) dt \end{aligned}$$

If $0 < \mu \leq 1, 0 < \alpha, s \leq 1$,

$$\mu^{\alpha^s} \leq \mu^{\alpha s} \tag{2.4}$$

When $|f''(a)| \leq 1$, by(2.4), we get

$$\begin{aligned} & \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right) dt \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left(|f''(a)|^{st} |f''(b)|^{s(1-t)} \right) dt \\ & \leq \frac{(b-a)^2}{12} |f''(b)|^s \left[\frac{\left(\frac{f''(a)}{f''(b)} \right)^s + 1}{\left(\ln \left| \frac{f''(a)}{f''(b)} \right|^s \right)^2} + \frac{2 \left(1 - \left(\frac{f''(a)}{f''(b)} \right)^s \right)}{\left(\ln \left| \frac{f''(a)}{f''(b)} \right|^s \right)^3} \right] \\ & = \frac{(b-a)^2}{12} |f''(b)|^s \left[\frac{\alpha(s,s)+1}{[\ln(\alpha(s,s))]^2} + \frac{2-2\alpha(s,s)}{[\ln(\alpha(s,s))]^3} \right] \end{aligned}$$

which completes the proof.

Theorem 2. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$ with $a < b$ and $f'' \in L[a,b]$ and $|f''(a)| \leq 1$. If $|f''|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $p, q > 1$ and $s \in (0,1]$; then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} |f''(b)|^s \left(\Psi(\alpha(sq, sq)) \right)^{\frac{1}{q}} \end{aligned} \tag{2.5}$$

where

$$\Psi(\alpha) = \begin{cases} 1 & \alpha = 1 \\ \frac{\alpha-1}{\ln \alpha} & \alpha \neq 1 \end{cases}$$

$$\alpha(u,v) = |f''(a)|^u |f''(b)|^{-v}, u, v \geq 0$$

Proof. Since $|f''|^q$ is a s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 and the well known Hölder inequality, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) |f''(ta+(1-t)b)| dt \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) |f''(a^t b^{(1-t)})| dt \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right) dt \\ & \leq \frac{(b-a)^2}{2} \left(\int_0^1 [t(1-t)]^p dt \right) \left(\int_0^1 |f''(a)|^{t^s} |f''(b)|^{(1-t)^s} dt \right)^{\frac{1}{q}} \end{aligned} \tag{2.6}$$

If $|f''(a)| \leq 1$, by (2.4), we obtain

$$\begin{aligned} & \int_0^1 |f''(a)|^{qt^s} |f''(b)|^{q(1-t)^s} dt \\ & \leq \int_0^1 |f''(a)|^{sq^t} |f''(b)|^{sq(1-t)} dt \\ & = |f''(a)|^{sq} \int_0^1 (|f''(a)| |f''(b)|^{-1})^{sq^t} dt \\ & = |f''(b)|^{sq} \Psi(\alpha(sq, sq)), \end{aligned} \tag{2.7}$$

and then from (2.6)-(2.7), (2.8) holds.

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left(\int_0^1 [t(1-t)]^p \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a)|^{t^s} |f''(b)|^{(1-t)^s} dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^2}{2} \left(\frac{2^{-1-2p} \sqrt{\pi} \Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{p}} (|f''(b)|^{sq} \Psi(\alpha(sq, sq)))^{\frac{1}{q}} \\ & = \frac{(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{p}} (|f''(b)|^{sq} \Psi(\alpha(sq, sq)))^{\frac{1}{q}} \end{aligned} \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We have to note that, the Beta and Gamma Functions (see [1]), are described respectively, as follows.

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, x, y > 0$$

and

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$$

are used to evaluate the integral

$$\int_0^1 [t(1-t)]^p dt = \int_0^1 t^p (1-t)^p dt = \beta(p+1, p+1)$$

Using the proprieties of Beta function, that is, $\beta(x, x) = 2^{1-2x} \beta\left(\frac{1}{2}, x\right)$ and $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we can achieve that

$$\begin{aligned} \beta(p+1, p+1) &= 2^{-1-2(p+1)} \beta\left(\frac{1}{2}, p+1\right) \\ &= 2^{-1-2p} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \end{aligned}$$

where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which completes the proof.

Corollary 1. Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be differentiable on I° , $a, b \in I$ with $a < b$ and $f'' \in L([a, b])$. If $|f''|^q$ is s-geometrically convex and

monotonically decreasing on $[a, b]$ for $p, q > 1$ and $s \in (0, 1]$, then

i) When $p = q = 2$, one has

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2\sqrt{30}} |f''(b)|^s (\Psi(2s, 2s))^{\frac{1}{2}} \end{aligned}$$

where $\Gamma(3) = 2, \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$.

ii) If we take $s = 1$ in (2.5), we have for geometrically convex, one has

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{p}} |f''(b)| (\Psi(q, q))^{\frac{1}{q}} \end{aligned}$$

Theorem 3. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be twice differentiable on I° , $a, b \in I$ with $a < b$ and $f'' \in L([a, b])$ and $|f''(a)| \leq 1$. If $|f''(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a, b]$, for $q \geq 1$ and $s \in (0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} |f''(b)|^s \end{aligned} \tag{2.9}$$

$$\left[\frac{\alpha(sq, sq) + 1}{[\ln(\alpha(sq, sq))]^2} + \frac{2 - 2\alpha(sq, sq)}{[\ln(\alpha(sq, sq))]^3} \right]$$

where $\alpha(u, v)$ is same with above (2.3).

Proof. Since $|f''|^q$ is a s-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 1 and the well known power mean integral inequality, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left(\int_0^1 t(1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t) |f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^2}{2} \left(\int_0^1 t(1-t) dt \right)^{1-\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} & \left(\int_0^1 t(1-t) \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right)^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t) |f''(a)|^{qt^s} |f''(b)|^{q(1-t)^s} dt \right)^{\frac{1}{q}} \end{aligned}$$

When $|f''(a)| \leq 1$, by (2.4), we get

$$\begin{aligned} & \int_0^1 t(1-t) |f''(a)|^{qt^s} |f''(b)|^{q(1-t)^s} dt \\ & \leq \int_0^1 t(1-t) |f''(a)|^{sqt} |f''(b)|^{sq(1-t)} dt \\ & \leq |f''(a)|^{sq} \int_0^1 t(1-t) |f''(a)|^{sq t} |f''(b)|^{-sq t} dt \\ & = |f''(b)|^{sq} \int_0^1 t(1-t) \left| \frac{f''(a)}{f''(b)} \right|^{sq t} dt \\ & = |f''(b)|^{sq} \left[\frac{\left| \frac{f''(a)}{f''(b)} \right|^{sq} + 1}{\left(\ln \left| \frac{f''(a)}{f''(b)} \right| \right)^2} + \frac{2 \left(1 - \left| \frac{f''(a)}{f''(b)} \right|^{sq} \right)}{\left(\ln \left| \frac{f''(a)}{f''(b)} \right| \right)^3} \right] \end{aligned}$$

which completes the proof.

Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be twice differentiable on I° , $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$ and $|f''(a)| \leq 1$. If $|f''|$ is s -geometrically convex and monotonically decreasing on $[a, b]$ for $\mu, \eta > 0$ with $\mu + \eta = 1$ and $s \in (0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{4} \left[\mu \frac{\sqrt{\pi} \Gamma\left(1 + \frac{1}{\mu}\right)}{4^\mu \Gamma\left(\frac{3}{2} + \frac{1}{\mu}\right)} + \eta |f''(b)|^{\frac{s}{\eta}} \Psi\left(\alpha\left(\frac{s}{\eta}, \frac{s}{\eta}\right)\right) \right] \end{aligned}$$

where $\alpha(u, v)$ is same with above (2.3) and

$$\Psi(\alpha) = \begin{cases} 1 & \alpha = 1 \\ \frac{\eta(\alpha - 1)}{s \ln \alpha} & \alpha \neq 1 \end{cases}$$

Proof. Since $|f''|$ is a s -geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 1, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) |f''(ta + (1-t)b)| dt \tag{2.10} \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) |f''(a^t b^{1-t})| dt \\ & \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) |f''(a)|^{t^s} |f''(b)|^{(1-t)^s} dt \end{aligned}$$

for all $t \in [0, 1]$. Using the well known inequality

$$mn \leq \mu m^\mu + \eta n^\eta, \text{ on the right side of (2.10), we get}$$

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left[\mu \int_0^1 (t(1-t))^{\frac{1}{\mu}} dt + \eta \int_0^1 \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right)^{\frac{1}{\eta}} dt \right] \end{aligned}$$

When $|f''(a)| \leq 1$, by (2.4), we get

$$\begin{aligned} & \int_0^1 |f''(a)|^{\frac{t^s}{\eta}} |f''(b)|^{\frac{(1-t)^s}{\eta}} dt \\ & \leq \int_0^1 |f''(a)|^{\frac{st}{\eta}} |f''(b)|^{\frac{s(1-t)}{\eta}} dt \\ & \leq |f''(b)|^{\frac{s}{\eta}} \int_0^1 \left(|f''(a)| |f''(b)|^{-1} \right)^{\frac{st}{\eta}} dt \\ & = |f''(b)|^{\frac{s}{\eta}} \Psi\left(\alpha\left(\frac{s}{\eta}, \frac{s}{\eta}\right)\right), \end{aligned}$$

and then, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left[\mu \int_0^1 (t(1-t))^{\frac{1}{\mu}} dt + \eta \int_0^1 \left(|f''(a)|^{t^s} |f''(b)|^{(1-t)^s} \right)^{\frac{1}{\eta}} dt \right] \\ & \leq \frac{(b-a)^2}{2} \left[\mu \frac{2^{-1-\frac{2}{\mu}} \sqrt{\pi} \Gamma\left(1 + \frac{1}{\mu}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{\mu}\right)} + \eta |f''(b)|^{\frac{s}{\eta}} \Psi\left(\alpha\left(\frac{s}{\eta}, \frac{s}{\eta}\right)\right) \right] \\ & = \frac{(b-a)^2}{4} \left[\mu \frac{\sqrt{\pi} \Gamma\left(1 + \frac{1}{\mu}\right)}{4^\mu \Gamma\left(\frac{3}{2} + \frac{1}{\mu}\right)} + \eta |f''(b)|^{\frac{s}{\eta}} \Psi\left(\alpha\left(\frac{s}{\eta}, \frac{s}{\eta}\right)\right) \right] \end{aligned}$$

We have to note that, using the Beta and Gamma Functions and evaluating the integral, we get

$$\int_0^1 (t(1-t))^{\frac{1}{\mu}} dt = \int_0^1 t^{\frac{1}{\mu}} (1-t)^{\frac{1}{\mu}} dt = \beta\left(\frac{1}{\mu} + 1, \frac{1}{\mu} + 1\right)$$

And, using the proprieties of Beta function, that is, $\beta(x, x) = 2^{1-2x} \beta\left(\frac{1}{2}, x\right)$ and $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we achieve

$$\begin{aligned} \beta\left(\frac{1}{\mu}+1, \frac{1}{\mu}+1\right) &= 2^{1-2\left(\frac{1}{\mu}+1\right)} \beta\left(\frac{1}{2}, \frac{1}{\mu}+1\right) \\ &= 2^{-1-\frac{2}{\mu}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{\mu}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)} \end{aligned}$$

Where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which completes the proof.

3. Applications to Special Means for Positive Numbers

Let

$$A(a, b) = \frac{a+b}{2}, L(a, b) = \frac{b-a}{\ln b - \ln a} (a \neq b),$$

$$L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, a \neq b, p \in \mathbb{R}, p \neq -1, 0$$

be the arithmetic, logarithmic, generalized logarithmic means for $a, b > 0$ respectively.

Proposition 1. Let $0 < a < b \leq 1, 0 < s \leq 1$. Then, we have

$$\begin{aligned} & \left| A(a^{s+1}, b^{s+1}) - L_{s+1}^{s+1}(a, b) \right| \\ & \leq \frac{(b-a)^2}{12} s(s+1) b^{(s-1)s} \left[\frac{\left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^s + 1}{\left[\ln \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^s \right]^2} + \frac{2 \left(1 - \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^s \right)}{\left[\ln \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^s \right]^3} \right] \end{aligned}$$

Proof. The assertion follows from Theorem 1 applied to s -geometrically convex mapping $f(x) = \frac{x^{s+1}}{s(s+1)}$,

$x \in (0, 1]$.

Proposition 2. Let $0 < a < b \leq 1, 0 < s \leq 1$, and $q \geq 1$. Then, we have

$$\begin{aligned} & \left| A(a^{s+1}, b^{s+1}) - L_{s+1}^{s+1}(a, b) \right| \\ & \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} s(s+1) b^{(s-1)s} \\ & \left[\frac{\left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^{sq} + 1}{\left[\ln \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^{qs} \right]^2} + \frac{2 \left(1 - \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^{sq} \right)}{\left[\ln \left| \frac{a^{(s-1)} }{b^{(s-1)}} \right|^{sq} \right]^3} \right] \end{aligned}$$

Proof. The assertion follows from Theorem 2 applied to s -geometrically convex mapping $f(x) = \frac{x^{s+1}}{s(s+1)}$, $x \in (0, 1]$.

Proposition 3. Let $0 < a < b \leq 1, 0 < s \leq 1$, and $q \geq 1$. Then, we have

$$\begin{aligned} & \left| A(a^{s+1}, b^{s+1}) - L_{s+1}^{s+1}(a, b) \right| \\ & \leq \frac{(b-a)^2}{8} \left(\frac{\pi}{2} \right)^{\frac{1}{q}} \left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)} \right) \\ & s(s+1) \left(b^{(s-1)s} \right)^s \left(\Psi(\alpha(sq, sq)) \right)^{\frac{1}{q}} \end{aligned}$$

In here, $\alpha = \frac{|f''(a)|}{|f''(b)|} = \left| \frac{a^{s-1}}{b^{s-1}} \right| = \left(\frac{a}{b} \right)^{s-1}$ and we can write this; if $a = b, \alpha = 1$ and $\Psi(\alpha) = 1$, and if

$$a \neq b, \alpha \neq 1 \text{ and } \Psi(\alpha) = \frac{\left(\frac{a}{b} \right)^{\frac{(s-1)s}{n}}}{(s-1)s \ln \left(\frac{a}{b} \right)}.$$

Proof. The assertion follows from Theorem 3 applied to s -geometrically convex mapping $f(x) = \frac{x^{s+1}}{s(s+1)}$, $x \in (0, 1]$.

Proposition 4. Let $0 < a < b \leq 1, 0 < s \leq 1$; and $q \geq 1$. Then, we have

$$\begin{aligned} & \left| A(a^{s+1}, b^{s+1}) - L_{s+1}^{s+1}(a, b) \right| \\ & \leq \frac{s(s+1)(b-a)^2}{4} \left(\frac{\sqrt{\pi} \Gamma\left(1+\frac{1}{\mu}\right)}{4^\mu \Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)} \mu \frac{1}{\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)} \right. \\ & \left. + \eta \left(b^{s-1} \right)^{\frac{s}{\eta}} \Psi\left(\alpha \left(\frac{s}{\eta}, \frac{s}{\eta} \right) \right) \right) \end{aligned}$$

In here, $\alpha = \frac{|f''(a)|}{|f''(b)|} = \left| \frac{a^{s-1}}{b^{s-1}} \right| = \left(\frac{a}{b} \right)^{s-1}$ and we can write this; if $a = b, \alpha = 1$ and $\Psi(\alpha) = 1$, and if

$$a \neq b, \alpha \neq 1 \text{ and } \Psi(\alpha) = \frac{\eta \left(\left(\frac{a}{b} \right)^{s-1} - 1 \right)}{(s-1)s \ln \left(\frac{a}{b} \right)}.$$

Proof. The assertion follows from Theorem 4 applied to s -geometrically convex mapping $f(x) = \frac{x^{s+1}}{s(s+1)}$, $x \in (0, 1]$.

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