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Some Inequalities for the Generalized Trigonometric and Hyperbolic Functions

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Abstract In the paper, the authors establish some inequalities of the generalized trigono-metric and hyperbolic functions, partially solve a conjecture posed by R. Klén, M. Vuorinen, and X.-H. Zhang, and finally pose an open problem.

Keywords: *inequality, generalized trigonometric function, generalized hyperbolic functions, Bernoulli inequality, Huygens' inequality, Wilker's inequality, conjecture, open problem*

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1. Introduction

It is well known from calculus that

$$\arcsin x = \int_0^x \frac{1}{\left(1-t^2\right)^{1/2}} dt \tag{1.1}$$

and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{\left(1-t^2\right)^{1/2}} dt \tag{1.2}$$

for $0 \le x \le 1$. For $1 < p < \infty$ and $0 \le x \le 1$, the arc sine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{\left(1-t^p\right)^{1/p}} dt \tag{1.3}$$

and

$$\begin{aligned}\frac{\pi_p}{2} &= \arcsin_p 1 = \int_0^1 \frac{1}{\left(1-t^p\right)^{1/p}} dt = \frac{\pi}{p}\csc\frac{\pi}{p} \\ &= \frac{1}{p}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right) = \frac{1}{p}\int_0^\infty \frac{t^{1/p-1}}{1+t} dt,\end{aligned} \tag{1.4}$$

where Γ denotes the classical gamma function. Hence, we have $\pi_p = \dfrac{2\pi}{p\sin(\pi/p)}$.

The inverse of $\arcsin_p x$ on $\left[0, \frac{\pi_p}{2}\right]$ is called the generalized sine function, denoted by $\sin_p x$, and may be extended to $(-\infty, \infty)$. See [7] and closely related references therein.

For $x \in \left[0, \frac{\pi_p}{2}\right]$, the generalized cosine function $\cos_p x$ is defined by

$$\cos_p x = \frac{d \sin_p x}{dx}. \tag{1.5}$$

It is easy to see that

$$\cos_p x = \left(1-\sin_p^p x\right)^{1/p} \tag{1.6}$$

and

$$\frac{d\cos_p x}{dx} = -\cos_p^{2-p} x \sin_p^{p-1} x. \tag{1.7}$$

The generalized tangent function $\tan_p x$ is defined as

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad x \in \mathbb{R} \setminus \left\{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\right\}. \tag{1.8}$$

From (1.8), it follows that

$$\frac{d\tan_p x}{dx} = 1 + \left|\tan_p x\right|^p, x \in \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right). \tag{1.9}$$

The generalized secant function $\sec_p x$ is defined as

$$\sec_p x = \frac{1}{\cos_p x}, x \in \left[0, \frac{\pi_p}{2}\right). \tag{1.10}$$

It follows from (1.8) and (1.9) that

$$\sec_p^p x = 1 + \tan_p^p x, x \in \left(0, \frac{\pi_p}{2}\right) \tag{1.11}$$

and

$$\frac{d \sec_p x}{dx} = \sec_p x \tan_p^{p-1} x, x \in \left[0, \frac{\pi_p}{2}\right). \tag{1.12}$$

The generalized cosecant function $\csc_p x$ may be defined as

$$\csc_p x = \frac{1}{\sin_p x}, x \in \left(0, \frac{\pi_p}{2}\right]. \tag{1.13}$$

It is clear that

$$\csc_p x = 1 + \frac{1}{\tan_p^p x} \quad and \quad \frac{d \csc_p x}{dx} = -\frac{\csc_p x}{\tan_p x} \tag{1.14}$$

for $x \in \left(0, \frac{\pi_p}{2}\right)$.

The generalized inverse hyperbolic sine function $\arcsin h_p x$ is defined by

$$\arcsin h_p x = \begin{cases} \int_0^x \frac{1}{\left(1+t^p\right)^{1/p}} dt, & x \in [0, \infty), \\ -\arcsin h_p(-x), & x \in (-\infty, 0). \end{cases} \tag{1.15}$$

The inverse of $\arcsin h_p x$ is called the generalized hyperbolic sine function and denoted by $\sin h_p x$.

The generalized hyperbolic cosine function $\cos h_p x$ is defined as

$$\cos h_p x = \frac{d \sin h_p x}{dx}. \tag{1.16}$$

It is easy to show that

$$\cos h_p x - \left|\sin h_p x\right|^p = 1, \quad x \in \mathbb{R} \tag{1.17}$$

and

$$\frac{d \cos h_p x}{dx} = \cos h_p^{2-p} x \sin h_p^{p-1} x, x \geq 0. \tag{1.18}$$

The generalized hyperbolic tangent function and the generalized hyperbolic secant function are defined as

$$\tanh_p x = \frac{\sin h_p x}{\cos h_p x} \quad and \quad \operatorname{sech}_p x = \frac{1}{\cos h_p x}. \tag{1.19}$$

Their derivatives are

$$\frac{d \tanh_p x}{dx} = 1 - \tanh_p^p x = \operatorname{sech}_p^p x \tag{1.20}$$

and

$$\frac{d \operatorname{sech}_p x}{dx} = -\operatorname{sech}_p x \tanh_p^{p-1} x. \tag{1.21}$$

The above formulas (1.5) to (1.9) and (1.15) to (1.19) can be found in [8]. For $r > 0$, the set $S_2 = \left\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = r^2\right\}$ is the equilateral hyperbola in the plane $\mathbb{R}^2$ with the ℓ_2 metric. The connection between the hyperbolic coordinates (r, ϕ) and the rectangular coordinates (x, y) is given by $x = r\cosh\phi$ and $y = r\sinh\phi$. We may easily obtain that, when $x, y > 0$, $x^2 - y^2 = r^2$ and ϕ are related by $\phi = \operatorname{arctanh}\left(\frac{y}{x}\right)$. When $p \neq 2$, the analogue of the equilateral hyperbola is the p-equilateral hyperbola

$$S_p = \left\{(x, y) \in \mathbb{R}^p : x^p - y^p = r^p\right\} \tag{1.22}$$

and the identities $x = r\cosh_p\phi$ and $y = r\sinh_p\phi$ hold. From this, it follows that

$$\left|\cosh_p \phi\right|^p - \left|\sinh_p \phi\right| = 1$$

and

$$\cosh_p \phi = \left[1 + \left(\sinh_p \phi\right)^p\right]^{1/p}, x, y > 0.$$

This gives a geometrical interpretation to $\sinh_p x$ and $\cosh_p x$. Furthermore, we may also define the generalized trigonometric functions by means of Gauss hypergeometric function. Interested readers may refer to [1].

As well as we known, the \hyperbolic" function were introduced in 1760 independent by Vincenzo Riccati and John Heinrid Lambert, the notations sh and ch are still used in some other languages such as European, French, and Russian. The hyperbolic function occurs in the solutions of some linear differential equations, such as defining catenary, Laplaces equations in Cartesian coordinates, and occurs in many important areas in physics, such as special relativity. In complex analysis, the hyperbolic function arises the imaginary parts of sine and cosine. For complex variables, the hyperbolic functions are rational functions of exponential functions and are meromorphic. Therefore, many advantages properties relating to the hyperbolic function have already been applied extensively. For more information on this topic, please read the classical book [5].

During the last decades, many authors have studied the generalized trigonometric functions introduced in [9]. See, for example, [1,2,3,4,7] and plenty of references therein. In [8], some classical inequalities for generalized trigonometric and hyperbolic functions, such as Mitrinović-Adamović inequality, Huygens' inequality, and Wilker's inequality were generalized. In [6], some basic properties of the generalized (p, q) - trigonometric functions were given. Recently, the functions $\arcsin_{p,q} x$ and $\operatorname{arcsinh}_{p,q} x$ were expressed in terms of Gaussian hypergeometric functions and many properties and inequalities for generalized trigonometric and hyperbolic functions were established in [3]. In [1], some Turán type inequalities for generalized trigonometric and hyperbolic functions were presented. Very recently, a conjecture posed in [3] was verified in [7].

In this paper, we will establish some inequalities of the generalized trigonometric and hyperbolic functions, partially solve a conjecture in [8], and finally pose an open problem.

2. Lemmas

For proving our main results, we need the following lemmas.

Lemma 2.1 ([[12], Lemma 2.9]). Let f and g be continuous on $[a,b]$ and differentiable on (a,b) such that $g'(x)\neq 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing, respectively) on (a,b), so are the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$.

Lemma 2.2 ([[10], Bernoulli inequality]). For $t>-1$ and $\alpha>1$, we have

$$(1+t)^{\alpha}>1+\alpha t. \tag{2.1}$$

Lemma 2.3 ([8, Theorem 3.4]). For $p\in[2,\infty)$ and $x\in\left(0,\frac{\pi_p}{2}\right)$, we have

$$\frac{\sin_p x}{x}<\frac{x}{\sinh_p x}. \tag{2.2}$$

Lemma 2.4 ([[8], Theorem 3.16]). If $p>1$, then

$$\frac{p\sin_p x}{x}+\frac{\tan_p x}{x}>1+p, 0<x<\frac{\pi_p}{2} \tag{2.3}$$

and

$$\frac{p\sinh_p x}{x}+\frac{\tanh_p x}{x}>1+p,\ \ x>0. \tag{2.4}$$

Lemma 2.5 ([[8], Theorem 3.22]). For $p\in(1,2]$, the double inequality

$$\frac{\sin_p x}{x}<\frac{\cos_p x+p}{1+p}\le\frac{\cos_p x+2}{3} \tag{2.5}$$

holds for all $x\in\left(0,\frac{\pi_p}{2}\right]$.

Lemma 2.6 ([[8], Theorem 3.24]). For all $x>0$,

1. if $p\in(1,2]$, then

$$\frac{\sinh_p x}{x}<\frac{\cosh_p x+p}{1+p}; \tag{2.6}$$

2. if $p\in(2,\infty)$, then

$$\frac{\sinh_p x}{x}<\frac{\cosh_p x+2}{3}. \tag{2.7}$$

Lemma 2.7 ([8, Theorems 3.6 and 3.7]). For all $p\in(1,\infty)$, we have

$$\cos_p^{\alpha} x<\frac{\sin_p x}{x}<1,\ \ x\in\left(0,\frac{\pi_p}{2}\right) \tag{2.8}$$

and

$$\cosh_p^{\alpha} x<\frac{\sinh_p x}{x}<\cosh_p^{\beta} x,\ \ x>0, \tag{2.9}$$

where the constants $\alpha=\frac{1}{p+1}$ and $\beta=1$ are the best possible.

Lemma 2.8. For $p>1$ and $x\in\left(0,\frac{\pi_p}{2}\right)$, the function $f(x)=\frac{\sinh_p x}{\sin_p x}$ is positive and strictly increasing.

Proof. It is apparent that the function $f(x)$ is positive. An easy computation yields $f'(x)=\frac{g(x)}{\sin_p^2 x}$ and

$$g'(x)=\left(\tanh_p^{p-2} x+\tan_p^{p-2} x\right)\sin_p x\sinh_p x>0,$$

where

$$g(x)=\cosh_p x\sin_p x-\sinh_p x\cos_p x.$$

This means that $g(x)$ is strictly increasing and $g(x)>g(0)=0$. Hence, it follows that $f'(x)>0$ and that $f(x)$ is strictly increasing.

Lemma 2.9. For $p>1$ and $x\in\left(0,\frac{\pi_p}{2}\right)$, the functions $g_1(x)=\sin_p x-x\cos_p x$ and $g_2(x)=x\cosh_p x-\sinh_p x$ are positive.

Proof. An easy computation yields

$$g_1'(x)=x\cos_p x\tan_p^{p-1} x>0$$

and

$$g_2'(x)=x\cosh_p x\tanh_p^{p-1} x>0.$$

These imply that $g_1(x)>g_1(0)=0$ and that $g_2(x)>g_2(0)=0$.

3. Main Results

Now we are in a position to present our main results.

Theorem 3.1. For $p>1$ and $x\in\left(0,\frac{\pi_p}{2}\right)$ we have

$$\left(\frac{\sin_p x}{x}\right)^p+\frac{\tan_p x}{x}>2. \tag{3.1}$$

Proof. Setting $t=\frac{\sin_p x}{x}-1\in(-1,0)$ and $\alpha=p$ in (2.1) leads to

$$\left(\frac{\sin_p x}{x}\right)^p > 1 + p\left(\frac{\sin_p x}{x} - 1\right)$$
$$> 1 - p + 1 + p - \frac{\tan_p x}{x} = 2 - \frac{\tan_p x}{x}.$$

Further using (2.3) results in the inequality (3.1).

Remark 3.1. The inequality (3.1) is an analogy of Wilker's inequality involving the sine and tangent functions. See [[12], Section 8.1].

Theorem 3.2. For $p \in [2, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, we have

$$\left(\frac{x}{\sinh_p x}\right)^p + \frac{\tan_p x}{x} > 2. \qquad (3.2)$$

Proof. Letting $t = \frac{x}{\sinh_p x} - 1$ and $\alpha = p$ in (2.1) gives

$$\left(\frac{x}{\sinh_p x}\right)^p \geq 1 - p + p\frac{x}{\sinh_p x}$$
$$> 1 - p + p\frac{\sin_p x}{x} > 2 - \frac{\tan_p x}{x}.$$

Combining this with (2.2) and (2.3) yields (3.2).

Theorem 3.3. For $p \in (1, 2]$ and $x \in \left(0, \frac{\pi_p}{2}\right)$ we have

$$\frac{px}{\sin_p x} + \frac{x}{\tan_p x} > 1 + p. \qquad (3.3)$$

Proof. This follows from using (2.5) and

$$p + \cos_p x - (1+p)\frac{\sin_p x}{x}$$
$$> p + \cos_p x - (1+p)\frac{\cos_p x + p}{1+p} = 0.$$

Theorem 3.4. For $p \in (1, 2]$ and $x > 0$, we have

$$\frac{px}{\sinh_p x} + \frac{x}{\tanh_p x} > 1 + p. \qquad (3.4)$$

Proof. This follows from using (2.6) and

$$p + \cosh_p x - (1+p)\frac{\sinh_p x}{x}$$
$$> p + \cosh_p x - (1+p)\frac{\cosh_p x + p}{1+p} = 0.$$

Remark 3.2. Inequalities presented in Theorems 3.3 and 3.4 are analogies of Huygens' inequality for the sine and tangent functions. See [11].

Theorem 3.5. For $p \in (1, 2]$, we have

$$\left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x} > 2, \quad x \in \left(0, \frac{\pi_p}{2}\right) \qquad (3.5)$$

and

$$\left(\frac{x}{\sinh_p x}\right)^p + \frac{x}{\tanh_p x} > 2, \quad x > 0. \qquad (3.6)$$

Proof. Taking $t = \frac{x}{\sin_p x} - 1$ and $\alpha = p$ in (2.1) and using the inequality (3.3) result in

$$\left(\frac{x}{\sin_p x}\right)^p > 1 + p\left(\frac{x}{\sin_p x} - 1\right)$$
$$> 1 - p + 1 + p - \frac{x}{\tan_p x} = 2 - \frac{x}{\tan_p x}.$$

The inequality (3.6) may be deduced similarly.

Theorem 3.6. For $p \in (1, \infty)$ and $x \in \left(0, \frac{\pi_p}{2}\right)$ we have

$$\left(\frac{\sin_p x}{x}\right)^p + \frac{\tan_p x}{x} > \left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x}. \qquad (3.7)$$

Proof. When letting $a = \left(\frac{\sin_p x}{x}\right)^p$ and $b = \frac{\tan_p x}{x}$ the inequality (3.7) becomes $a + b > \frac{1}{a} + \frac{1}{b}$ which is equivalent to $ab > 1$, that is,

$$\left(\frac{\sin_p x}{x}\right)^{p+1} \frac{1}{\cos_p x} > 1.$$

This can be derived from utilizing (2.8) as follows

$$\left(\frac{\sin_p x}{x}\right)^{p+1} \frac{1}{\cos_p x} > \left[\cos_p^{1/(p+1)} x\right]^{p+1} \frac{1}{\cos_p x} = 1.$$

Theorem 3.7. For $p \in (1, \infty)$ and $x > 0$, we have

$$\left(\frac{\sinh_p x}{x}\right)^p + \frac{\tanh_p x}{x} > \left(\frac{x}{\sinh_p x}\right)^p + \frac{x}{\tanh_p x}. \qquad (3.8)$$

Proof. This follows from using the inequality (2.9).

Remark 3.3. Inequalities (3.7) and (3.8) imply inequalities (3.1) and (3.20) in [8, Corollary 3.19].

Theorem 3.8. For $p > 1$ and $x \in \left(0, \frac{\pi_p}{2}\right)$, we have

$$\frac{\tan_p x}{x} > \frac{x}{\sin_p x} \qquad (3.9)$$

Proof. Let $f(x) = \tan_p x \sin_p x - x^2$. An easy computation yields

$$f'(x) = \sin_p x \sec_p^p x + \sin_p x - 2x,$$
$$f''(x) = \cos_p x \sec_p^p x + p \sin_p x \sec_p^p x \tan_p^{p-1} x + \cos_p x - 2,$$
$$f'''(x) = \left[(2p-1)\sec_p^p x - 1\right]\sin_p^{p-1} x \sec_p^{p-2} x + p^2 \sin_p^{2p-1} x \sec_p^{3p-2} x + p(p-1)\sin_p^{p-1} x_p^{3p-2} x \geq 0.$$

Hence, $f''(x) > f''(0) = 0$, $f'(x)$ is strictly increasing, $f'(x) > f'(0) = 0$, and $f(x) > f(0) = 0$. The inequality (3.9) follows.

Theorem 3.9. For $p > 1$ and $x \in (0, \infty)$, we have

$$\frac{\tanh_p x}{x} > \frac{p+1}{p\cosh_p x + 1} \tag{3.10}$$

or, equivalently,

$$\frac{\sinh_p x}{x} > \frac{(p+1)\cosh_p x}{p\cosh_p x + 1}. \tag{3.11}$$

Proof. Set $f(x) = (p+1)x - \tanh_p x\left(p\cosh_p x + 1\right)$. Then

$$f'(x) = p - p\cosh_p x + \tanh_p^p x$$

and

$$f''(x) = p\tanh_p^{p-1} x\left(\sec_p^p x - \cosh_p x\right) < 0.$$

Accordingly, it follows that $f'(x) < f'(0) = 0, f'(x)$ is strictly decreasing, and $f(x) < f(0) = 0$. The inequalities (3.10) and (3.11) are proved.

Remark 3.4. Considering Theorem 3.9 and $\cosh_p x > 1$ reveals

$$\frac{\sinh_p x}{x} > \frac{p+1}{p + \cosh_p x}. \tag{3.12}$$

Theorem 3.10. For $p \ge 3$ and $x \in \left(x_p^*, \frac{\pi_p}{2}\right)$, the function $f(x) = \dfrac{\ln \frac{x}{\sin_p x}}{\ln \frac{\sinh_p x}{x}}$ is strictly increasing, where x_p^* satisfies the equation $\cosh_p^p x = p - 2$.

Proof. Let $f_1(x) = \ln \dfrac{x}{\sin_p x}$ and $f_2(x) = \ln \dfrac{\sinh_p x}{x}$. Then $f_1(0) = \lim_{x\to 0} f_1(x) = 0$, $f_2(0) = \lim_{x\to 0} f_2(x) = 0$, and

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sinh_p x}{\sin_p x}\frac{g_1(x)}{g_2(x)}, \tag{3.13}$$

where $g_1(x) = \sin_p x - x\cos_p x$ and $g_2(x) = x\cosh_p x - \sinh_p x$ satisfy $g_1(0) = g_2(0) = 0$ and

$$\frac{g_1'(x)}{g_2'(x)} = \frac{\sin_p x\tan_p^{p-2} x}{\sinh_p x\tanh_p^{p-2} x},$$

$$\frac{d}{dx}\left[\frac{g_1'(x)}{g_2'(x)}\right] = \frac{\tan_p^{p-3} x\tanh_p^{p-3} xh(x)}{\left(\sinh_p x\tanh_p^{p-2} x\right)^2},$$

$$h(x) = \left[(p-2)\sec_p^p x - 1\right]\tan_p x + \left[1 - (p-2)\operatorname{sech}_p^p x\right]\tan_p x.$$

Owing to $p \ge 3$ and $x \in \left(x_p^*, \frac{\pi_p}{2}\right)$, by the monotonicity of $\sec_p x$ and $\operatorname{sech}_p x$, we obtain $(p-2)\sec_p^p x - 1 > 0$ and $1 - (p-2)\operatorname{sech}_p^p x > 0$. So the ratio $\dfrac{g_1'(x)}{g_2'(x)}$ is strictly increasing for $x \in \left(x_p^*, \frac{\pi_p}{2}\right)$. By Lemma 2.1, we see that $\dfrac{g_1(x)}{g_2(x)}$ is strictly increasing for $x \in \left(x_p^*, \frac{\pi_p}{2}\right)$. On account of Lemmas 2.8 and 2.9, we obtain that the quotient $\dfrac{f_1'(x)}{f_2'(x)}$ is strictly increasing. Thus, by Lemma 2.1, the ratio $\dfrac{f_1(x)}{f_2(x)}$ is also strictly increasing.

Remark 3.5. If $p = 3$, we obtain $x_3^* = 0$. This means that the function $f(x) = \dfrac{\ln \frac{x}{\sin_3 x}}{\ln \frac{\sinh_3 x}{x}}$ is strictly increasing on $\left(0, \frac{\pi_3}{2}\right)$.

Notice that Theorem 3.10 partially solves Conjecture 3.12 in [8].

Theorem 3.11. For $p > 1$ and $x \in (0, 1)$, we have

$$\int_0^1 \frac{\cos_p x}{\sqrt[p]{1 - x^p}}dx > \int_0^1 \frac{x^{p-2}\sin_p x}{\sqrt[p]{1 - x^p}}dx. \tag{3.14}$$

Proof. If putting $t = \arcsin_p x$, the left hand side of (3.14) becomes

$$\int_0^1 \frac{\cos_p x}{\sqrt[p]{1 - x^p}}dx = \int_0^{\pi_p/2} \cos_p \sin_p t\, dt. \tag{3.15}$$

Similarly, taking $t = \arccos_p x$, the right hand side of (3.14) reduces to

$$\int_0^1 \frac{x^{p-2}\sin_p x}{\sqrt[p]{1 - x^p}}dx = \int_0^{\pi_p/2} \sin_p \cos_p t dt. \tag{3.16}$$

Making use of the monotonicity of $\sin_p x$ and $\cos_p x$ acquires

$$\cos_p \sin_p t > \cos_p t > \sin_p \cos_p t.$$

The inequality (3.14) is thus proved.

4. An Open Problem

Finally, we pose an open problem.

$$\frac{p\sin_p x}{x} + \frac{\tan_p x}{x} > \frac{px}{\sin_p x} + \frac{x}{\tan_p x} \tag{4.1}$$

is valid on $\left(0, \frac{\pi_p}{2}\right)$.

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