

# Using Area Mean Value Theorem to Solve Some Double Integrals

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**Abstract** The present paper studies six types of double integrals and uses Maple for verification. These double integrals can be solved using area mean value theorem. On the other hand, some examples are used to demonstrate the calculations.

**Keywords:** double integrals, area mean value theorem, Maple

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## 1. Introduction

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and so on. In this paper, we study the following six types of double integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \int_0^R A(r, \theta, s, \phi, n) dr d\theta, \quad (1)$$

$$\int_0^{2\pi} \int_0^R B(r, \theta, s, \phi, n) dr d\theta, \quad (2)$$

$$\int_0^{2\pi} \int_0^R C(r, \theta, s, \phi, n) dr d\theta, \quad (3)$$

$$\int_0^{2\pi} \int_0^R D(r, \theta, s, \phi, n) dr d\theta, \quad (4)$$

$$\int_0^{2\pi} \int_0^R E(r, \theta, s, \phi, n) dr d\theta, \quad (5)$$

$$\int_0^{2\pi} \int_0^R F(r, \theta, s, \phi, n) dr d\theta, \quad (6)$$

where  $s, \phi, R$  are real numbers,  $R > 0$ , and  $n$  is a positive integer,

$$\begin{aligned} &A(r, \theta, s, \phi, n) \\ &= r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned} \quad (7)$$

$$\begin{aligned} &B(r, \theta, s, \phi, n) \\ &= r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned} \quad (8)$$

$$\begin{aligned} &C(r, \theta, s, \phi, n) \\ &= r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \cosh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned} \quad (9)$$

$$\begin{aligned} &D(r, \theta, s, \phi, n) \\ &= r \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \sinh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned} \quad (10)$$

$$\begin{aligned} &E(r, \theta, s, \phi, n) \\ &= r \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \cosh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned} \quad (11)$$

and

$$\begin{aligned} &F(r, \theta, s, \phi, n) \\ &= r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \sinh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right]. \end{aligned} \quad (12)$$

We can obtain the solutions of these double integrals using area mean value theorem; these are the major results of this paper (i.e., Theorems 1-3). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-29], Yu and B. -H. Chen [30], and T. -J. Chen and Yu [31,32,33] used complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, and generalized Cauchy integral formula to solve some types of integrals. In this paper, three examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

## 2. Main Results

Some formulas and theorems used in this paper are introduced below.

### 2.1. Euler's Formula

$e^{ix} = \cos x + i \sin x$ , where  $i = \sqrt{-1}$ , and  $x$  is any real number.

### 2.2. DeMoivre's Formula

$(\cos x + i \sin x)^m = \cos mx + i \sin mx$ , where  $m$  is any integer, and  $x$  is any real number.

The following Formula 2.3 and Formula 2.4 can be found in [[34], p62]

### 2.3.

$\sin(a + ib) = \sin a \cosh b + i \cos a \sinh b$ , where  $a, b$  are real numbers.

### 2.4.

$\cos(a + ib) = \cos a \cosh b - i \sin a \sinh b$ , where  $a, b$  are real numbers.

### 2.5. Binomial Theorem:

$(u + v)^n = \sum_{k=0}^n \frac{(n)_k}{k!} u^{n-k} v^k$ , where  $u, v$  are complex numbers,  $n$  is a positive integer,  $(n)_k = n(n-1) \cdots (n-k+1)$  for positive integers  $k$ , and  $(n)_0 = 1$ .

An important theorem used in this study is introduced below, which can be found in [[35], p147].

### 2.6. Area Mean Value Theorem

Suppose that  $z, \lambda$  are complex numbers, and  $R > 0$ . If  $f(z)$  is analytic in a domain which contains the closed disc  $\{z \in \mathbb{C} \mid |z - \lambda| \leq R\}$ , then:

$$\frac{1}{\pi R^2} \iint_{|z-\lambda| \leq R} f(z) r dr d\theta = f(\lambda).$$

Firstly, we determine the solutions of the double integrals (1) and (2).

**Theorem 1** Suppose that  $s, \phi, R$  are real numbers,  $R > 0$ , and  $n$  is a positive integer. Then the double integrals

$$\int_0^{2\pi} \int_0^R A(r, \theta, s, \phi, n) r dr d\theta = \pi R^2 \exp(s^n \cos n\phi) \cdot \cos(s^n \sin n\phi), \tag{13}$$

and

$$\int_0^{2\pi} \int_0^R B(r, \theta, s, \phi, n) r dr d\theta = \pi R^2 \exp(s^n \cos n\phi) \cdot \cos(s^n \sin n\phi), \tag{14}$$

where

$$\begin{aligned} &A(r, \theta, s, \phi, n) \\ &= r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right], \end{aligned}$$

and

$$\begin{aligned} &B(r, \theta, s, \phi, n) \\ &= r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \\ &\times \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right]. \end{aligned}$$

**Proof** Using area mean value theorem for analytic function  $f(z) = \exp(z^n)$  yields:

$$\int_0^{2\pi} \int_0^R \exp[(z + re^{i\theta})^n] r dr d\theta = \pi R^2 \exp(z^n). \tag{15}$$

Let  $z = se^{i\phi}$ , then by Euler's formula, DeMoivre's formula and binomial theorem, we obtain:

$$\begin{aligned} &\int_0^{2\pi} \int_0^R r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} (se^{i\phi})^{n-k} (re^{i\theta})^k \right] r dr d\theta \\ &= \pi R^2 \exp(s^n e^{in\phi}). \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned} &\int_0^{2\pi} \int_0^R r \exp \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k e^{i[(n-k)\phi + k\theta]} \right] r dr d\theta \\ &= \pi R^2 \exp(s^n \cos n\phi + is^n \sin n\phi). \end{aligned} \tag{17}$$

Using the equality of real parts of both sides of Eq. (17) yields Eq. (13) holds. Also by the equality of imaginary parts of both sides of Eq. (17), we obtain Eq. (14).

Next, the solutions of the double integrals (3) and (4) can be obtained below.

**Theorem 2** If the assumptions are the same as Theorem 1, then the double integrals

$$\begin{aligned} &\int_0^{2\pi} \int_0^R C(r, \theta, s, \phi, n) r dr d\theta \\ &= \pi R^2 \sin(s^n \cos n\phi) \cdot \cosh(s^n \sin n\phi), \end{aligned} \tag{18}$$

and

$$\int_0^{2\pi} \int_0^R D(r, \theta, s, \phi, n) dr d\theta = \pi R^2 \cos(s^n \cos n\phi) \cdot \sinh(s^n \sin n\phi), \tag{19}$$

where

$$C(r, \theta, s, \phi, n) = r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \times \cosh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right],$$

and

$$D(r, \theta, s, \phi, n) = r \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \times \sinh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right].$$

**Proof** By area mean value theorem for analytic function  $g(z) = \sin(z^n)$ , we have:

$$\int_0^{2\pi} \int_0^R \sin[(z + re^{i\theta})^n] r dr d\theta = \pi R^2 \sin(z^n). \tag{20}$$

Let  $z = se^{i\phi}$ , then:

$$\int_0^{2\pi} \int_0^R r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} (se^{i\phi})^{n-k} (re^{i\theta})^k \right] dr d\theta = \pi R^2 \sin(s^n e^{in\phi}). \tag{21}$$

It follows that:

$$\int_0^{2\pi} \int_0^R r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k e^{i[(n-k)\phi + k\theta]} \right] dr d\theta = \pi R^2 \sin(s^n \cos n\phi + is^n \sin n\phi). \tag{22}$$

By Formula 2.3 and the equality of real parts of both sides of Eq. (22), we obtain Eq. (18). Also using Formula 2.3 and the equality of imaginary parts of both sides of Eq. (22) yields Eq. (19) holds.

Finally, we solve the double integrals (5) and (6).

**Theorem 3** If the assumptions are the same as Theorem 1, then the double integrals

$$\int_0^{2\pi} \int_0^R E(r, \theta, s, \phi, n) dr d\theta = \pi R^2 \cos(s^n \cos n\phi) \cdot \cosh(s^n \sin n\phi), \tag{23}$$

and

$$\int_0^{2\pi} \int_0^R F(r, \theta, s, \phi, n) dr d\theta = \pi R^2 \sin(s^n \cos n\phi) \cdot \sinh(s^n \sin n\phi), \tag{24}$$

where

$$E(r, \theta, s, \phi, n) = r \cos \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \times \cosh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right],$$

and

$$F(r, \theta, s, \phi, n) = r \sin \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \cos[(n-k)\phi + k\theta] \right] \times \sinh \left[ \sum_{k=0}^n \frac{(n)_k}{k!} s^{n-k} r^k \sin[(n-k)\phi + k\theta] \right].$$

**Proof** Using area mean value theorem for analytic function  $h(z) = \cos(z^n)$  and Formula 2.4, we can easily obtain the desired results.

### 3. Examples

In the following, for the six types of double integrals in this study, we provide some examples and use Theorems 1-3 to determine their solutions. On the other hand, Maple is used to calculate the approximations of some double integrals and their solutions for verifying our answers.

**Example 1** In Eq. (13), let  $s = 2, \phi = \frac{\pi}{3}, n = 1$ , and  $R = 2$ , we obtain:

$$\int_0^{2\pi} \int_0^2 A\left(r, \theta, 2, \frac{\pi}{3}, 1\right) dr d\theta = 4\pi e \cdot \cos(\sqrt{3}), \tag{25}$$

where

$$A\left(r, \theta, 2, \frac{\pi}{3}, 1\right) = r \exp(1 + r \cos \theta) \cdot \cos(\sqrt{3} + r \sin \theta). \tag{26}$$

Next, we use Maple to verify the correctness of Eq. (25).  
`>evalf(Doubleint(r*exp(1+r*cos(theta))*cos(sqrt(3)+r*sin(theta)),r=0..2,theta=0..2*Pi),18);`

-5.48444066856117700

`>evalf(4*Pi*exp(1)*cos(sqrt(3)),18);`

-5.48444066856117688

Also using Eq. (14) yields:

$$\int_0^{2\pi} \int_0^2 B\left(r, \theta, 2, \frac{\pi}{3}, 1\right) dr d\theta = 4\pi e \cdot \sin(\sqrt{3}), \tag{27}$$

where

$$B\left(r, \theta, 2, \frac{\pi}{3}, 1\right) = r \exp(1 + r \cos \theta) \cdot \sin(\sqrt{3} + r \sin \theta). \tag{28}$$

`>evalf(Doubleint(r*exp(1+r*cos(theta))*sin(sqrt(3)+r*sin(theta)),r=0..2,theta=0..2*Pi),18);`

33.7157808756591908

`>evalf(4*Pi*exp(1)*sin(sqrt(3)),18);`

33.7157808756591909

**Example 2** In Eq. (18), if  $s = 1, \phi = \frac{\pi}{2}, n = 2$ , and  $R = 3$ , then:

$$\int_0^{2\pi} \int_0^3 C\left(r, \theta, 1, \frac{\pi}{2}, 2\right) dr d\theta = -9\pi \sin(1), \tag{29}$$

where

$$C\left(r, \theta, 1, \frac{\pi}{2}, 2\right) = r \sin(-1 - 2r \sin \theta + r^2 \cos 2\theta) \quad (30)$$

$$\times \cosh(2r \cos \theta + r^2 \sin 2\theta).$$

We also use Maple to verify the correctness of Eq. (29).  
`>evalf(Doubleint(r*sin(-1-2r*sin(theta)+r^2*cos(2*theta))  
*cosh(2*r*cos(theta)+r^2*sin(2*theta)),r=0..3,theta=0..2*  
Pi));`

$$-23.79203158$$

`>evalf(-9*Pi*sin(1));`

$$-23.79203158$$

Also by Eq. (19), we have:

$$\int_0^{2\pi} \int_0^3 D\left(r, \theta, 1, \frac{\pi}{2}, 2\right) dr d\theta = 0, \quad (31)$$

where

$$D\left(r, \theta, 1, \frac{\pi}{2}, 2\right) = r \cos(-1 - 2r \sin \theta + r^2 \cos 2\theta) \quad (32)$$

$$\times \sinh(2r \cos \theta + r^2 \sin 2\theta).$$

`>evalf(Doubleint(r*cos(-1-2r*sin(theta)+r^2*cos(2*theta))  
*sinh(2*r*cos(theta)+r^2*sin(2*theta)),r=0..3,theta=0..2*  
Pi),14);`

$$-1.987124 \cdot 10^{-10}$$

**Example 3** In Eq. (23), let  $s = 2, \phi = \pi, n = 1$ , and  $R = 1$ , then:

$$\int_0^{2\pi} \int_0^1 E(r, \theta, 2, \pi, 1) dr d\theta = \pi \cos(2), \quad (33)$$

where

$$E(r, \theta, 2, \pi, 1) = r \cos(-2 + r \cos \theta) \cdot \cosh(r \sin \theta). \quad (34)$$

`>evalf(Doubleint(r*cos(-2+r*cos(theta))*cosh(r*sin(theta))  
),r=0..1,theta=0..2*Pi));`

$$-1.307363845$$

`>evalf(Pi*cos(2));`

$$-1.307363845$$

On the other hand, using Eq. (24) yields:

$$\int_0^{2\pi} \int_0^1 F(r, \theta, 2, \pi, 1) dr d\theta = 0, \quad (35)$$

where

$$F(r, \theta, 2, \pi, 1) = r \sin(-2 + r \cos \theta) \cdot \sinh(r \sin \theta). \quad (36)$$

`>evalf(Doubleint(r*sin(-2+r*cos(theta))*sinh(r*sin(theta))  
),r=0..1,theta=0..2*Pi));`

$$4.9526896950035 \cdot 10^{-16}.$$

## 4. Conclusion

In this paper, we use area mean value theorem to solve some types of double integrals. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also

plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

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