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Integral Inequalities of Hermite–Hadamard Type for m-AH Convex Functions

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Abstract In the paper, the authors introduce the concept "m-AH convex functions" and establish some inequalities of Hermite-Hadamard type for m-AH convex functions.

Keywords: *Hermite-Hadamard's inequality, m-AH convex function, Hölder's inequality*

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1. Introduction

Throughout this paper, we use the following notations: $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}_0=[0,+\infty)$, $\mathbb{R}_+=(0,+\infty)$.

We first recall several definitions.

Definition 1.1. A function $f: I\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

Definition 1.2. A function $f: I\subset\mathbb{R}_0\to\mathbb{R}_+$ is said to be geometrically convex if

$$f(x^t y^{(1-t)})\le[f(x)]^t[f(y)]^{(1-t)} \tag{1.2}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

Definition 1.3 ([3]). A function $f:[0,b]\to\mathbb{R}$ is said to be m-convex if

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y) \tag{1.3}$$

holds for all $x,y\in[0,b]$, $t\in[0,1]$, and some $m\in(0,1]$.

Definition 1.4 ([11]). Let $f:[0,b]\to\mathbb{R}_+$ be a positive function on $[0,b]$ and $m\in(0,1]$. If

$$f(x^t y^{m(1-t)})\le[f(x)]^t[f(y)]^{m(1-t)} \tag{1.4}$$

holds for all $x,y\in[0,b]$, $t\in[0,1]$, then we say that the function $f(x)$ is m-geometrically convex on $[0,b]$.

Definition 1.5 ([12]). A function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex where $(\alpha,m)\in[0,1]^2$, if we have

$$f(tx+m(1-t)y)\le t^\alpha f(x)+m(1-t^\alpha)f(y) \tag{1.5}$$

for all $x,y\in[0,b]$, $t\in[0,1]$.

We now recall some inequalities of Hermite- Hadamard type.

Theorem1.1 ([[1]. Theorem 2.2]). Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$.

(i) If $|f'(x)|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8} \tag{1.6}$$

(ii) If $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$ for $p>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le\frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.7}$$

Theorem 1.2 ([[2], Theorems 2.3 and 2.4]). Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|^p$ is a convex function on $[a,b]$ for $p>1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}(|f'(a)|+|f'(b)|). \tag{1.8}$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \times\left[\left(|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right)^{(p-1)/p} +\left(3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right)^{(p-1)/p}\right]. \tag{1.9}$$

Theorem 1.3 ([[4], Theorem 2]). Let $f:\mathbb{R}_0\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $f\in L([a,b])$ for $0\le a<b<\infty$, then

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le\min\left\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\right\}. \tag{1.10}$$

Theorem 1.4. ([6], Theorem 2.2]). Let $I \subset \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a,b]$ for some given numbers $m \in (0,1]$ and $q \ge 1$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \le \frac{b-a}{4}\min\left\{ \left(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2} \right)^{1/q}, \left(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2} \right)^{1/q} \right\}. \quad (1.11)$$

In this paper, we will introduce the concept "m-AH convex functions" and establish some inequalities of Hermite-Hadamard type for m-AH convex functions.

2. Definition and lemmas

The concept of m-AH convex function may be introduced as follows.

Definition 2.1. A function $f : I \subset \mathbb{R} \to \mathbb{R}_0$ is said to be AH convex if for all $x, y \in I$ and $t \in [0,1]$ the inequality

$$f(tx + (1-t)y) \le \left[t(f(x))^{-1} + (1-t)(f(y))^{-1} \right]^{-1} \quad (2.1)$$

holds. If the inequality (2.1) is reversed then $f(x)$ is said to be AH concave function.

Definition 2.2. A function $f : [0,b] \to \mathbb{R}_0$ is said to be m-AH convex for some given number $m \in (0,1]$, if the inequality

$$f(tx + m(1-t)y) \le \left[t(f(x))^{-1} + m(1-t)(f(y))^{-1} \right]^{-1} \quad (2.2)$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$. if the inequality (2.2) reverses, then $f(x)$ is said to be m-AH concave.

When $m = 1$, the m-AH convex function is AH convex function on $[0,b]$.

In order to establish some inequalities of Hermite–Hadamard type for m-AH convex functions, we find the following lemmas.

Lemma 2.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L([a,b])$. Then

$$\begin{aligned} f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u &= \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t f'(t\tfrac{a+x}{2} + (1-t)a)\mathrm{d}t + \int_0^1 (1+t) f'(tx + (1-t)\tfrac{a+x}{2})\mathrm{d}t \right) \\ &- \frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (2-t) f'(t\tfrac{b+x}{2} + (1-t)x)\mathrm{d}t + \int_0^1 (1-t) f'(tb + (1-t)\tfrac{b+x}{2})\mathrm{d}t \right). \end{aligned} \quad (2.3)$$

Proof. Integrating by part gives

$$\begin{aligned} &\int_0^1 t f'(t\tfrac{a+x}{2} + (1-t)a)\mathrm{d}t \\ &= \frac{2}{x-a}\int_0^1 t\mathrm{d}f(t\tfrac{a+x}{2} + (1-t)a) \\ &= \frac{2}{x-a}\left(f(\tfrac{a+x}{2}) - \frac{2}{x-a}\int_a^{\frac{a+x}{2}} f(u)\mathrm{d}u \right). \end{aligned} \quad (2.4)$$

Similarly, we have

$$\begin{aligned} &\int_0^1 (1+t) f'(tx + (1-t)\tfrac{a+x}{2})\mathrm{d}t \\ &= \frac{2}{x-a}\left(2f(x) - f(\tfrac{a+x}{2}) - \frac{2}{x-a}\int_{\frac{a+x}{2}}^x f(u)\mathrm{d}u \right), \end{aligned} \quad (2.5)$$

$$\begin{aligned} &\int_0^1 (2-t) f'(t\tfrac{b+x}{2} + (1-t)x)\mathrm{d}t \\ &= \frac{2}{b-x}\left(f(\tfrac{b+x}{2}) - 2f(x) + \frac{2}{b-x}\int_x^{\frac{b+x}{2}} f(u)\mathrm{d}u \right), \end{aligned} \quad (2.6)$$

$$\begin{aligned} &\int_0^1 (1-t) f'(tb + (1-t)\tfrac{b+x}{2})\mathrm{d}t \\ &= \frac{2}{b-x}\left(-f(\tfrac{b+x}{2}) + \frac{2}{b-x}\int_{\frac{b+x}{2}}^b f(u)\mathrm{d}u \right). \end{aligned} \quad (2.7)$$

From (2.4)-(2.7), the identity (2.3) follows. The proof is complete.

Lemma 2.2. For $m \in (0,1]$ and $r, s > 0$, we have

$$H_1(r,s) = \int_0^1 \frac{t\mathrm{d}t}{ts^{-1} + m(1-t)r^{-1}} = \begin{cases} \dfrac{rs(r - ms + ms\ln(ms/r))}{(r-ms)^2}, & r - ms \ne 0, \\ \dfrac{r}{2m}, & r - ms = 0, \end{cases} \quad (2.8)$$

and

$$H_2(r,s) = \int_0^1 \frac{\mathrm{d}t}{ts^{-q} + m(1-t)r^{-q}} = \begin{cases} \dfrac{\ln r^q - \ln(ms^q)}{r^q - ms^q}(rs)^q, & r^q - ms^q \ne 0, \\ \dfrac{r^q}{m}, & r^q - ms^q = 0. \end{cases} \quad (2.9)$$

3. Main Results

In this section, we will present several Hermite-Hadamard type inequalities for the m-AH convex functions.

Theorem 3.1. Let $f : \mathbb{R}_0 \to \mathbb{R}_+$ be differentiable, $f' \in L([a,b])$ for $0 \le a < b$. If $|f'|$ is an m-AH convex function on $[a, b/m]$ for $m \in (0,1]$, then

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(x) \right| \\ &\le \frac{(x-a)^2}{4(b-a)}\left[\begin{array}{l} 2H_2(\gamma^{1/q}(x), \alpha(x)^{1/q}) \\ -H_1(\gamma(x), \alpha(x)) + H_1(u, \alpha(x)) \end{array} \right] \\ &+ \frac{(b-x)^2}{4(b-a)}\left[\begin{array}{l} 2H_2(\gamma^{1/q}(x), \beta^{1/q}(x)) \\ -H_1(\gamma(x), \beta(x)) + H_1(v, \beta(x)) \end{array} \right], \end{aligned} \quad (3.1)$$

where

$$u = |f'(a/m)|,\ v = |f'(b/m)|,$$
$$\alpha(x) = \left|f'(\tfrac{a+x}{2})\right|,\ \beta(x) = \left|f'(\tfrac{b+x}{2})\right|, \tag{3.2}$$
$$\gamma(x) = |f'(x/m)|.$$

Proof. By Lemmas 2.1 and 2.2 and the m-AH convexity of $|f'(x)|$ on $[a, b/m]$, we have

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(x)\right|$$
$$\le \frac{(x-a)^2}{4(b-a)}\Big(\int_0^1 t\left|f'(t\tfrac{a+x}{2}+(1-t)a)\right|\mathrm{d}t$$
$$+\int_0^1 (1+t)\left|f'(tx+(1-t)\tfrac{a+x}{2})\right|\mathrm{d}t\Big)$$
$$+\frac{(b-x)^2}{4(b-a)}\Big(\int_0^1 (2-t)\left|f'(t\tfrac{b+x}{2}+(1-t)x)\right|\mathrm{d}t$$
$$+\int_0^1 (1-t)\left|f'(tb+(1-t)\tfrac{b+x}{2})\right|\mathrm{d}t\Big)$$
$$\le \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 \frac{t\mathrm{d}t}{t\left|f'(\frac{a+x}{2})\right|^{-1}+m(1-t)\left|f'(a/m)\right|^{-1}}\right.$$
$$\left.+\int_0^1 \frac{(1+t)\mathrm{d}t}{mt\left|f'(x/m)\right|^{-1}+(1-t)\left|f'(\frac{a+x}{2})\right|^{-1}}\right)$$
$$+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 \frac{(2-t)\mathrm{d}t}{t\left|f'(\frac{b+x}{2})\right|^{-1}+m(1-t)\left|f'(x/m)\right|^{-1}}\right.$$
$$\left.+\int_0^1 \frac{(1-t)\mathrm{d}t}{mt\left|f'(b/m)\right|^{-1}+(1-t)\left|f'(\frac{b+x}{2})\right|^{-1}}\right)$$
$$= \frac{(x-a)^2}{4(b-a)}\Big[2H_2(\gamma^{1/q}(x), \alpha(x)^{1/q})$$
$$-H_1(\gamma(x),\alpha(x)) + H_1(u,\alpha(x))\Big]$$
$$+\frac{(b-x)^2}{4(b-a)}\Big[2H_2(\gamma^{1/q}(x),\beta^{1/q}(x))$$
$$-H_1(\gamma(x),\beta(x)) + H_1(v,\beta(x))\Big].$$

So the inequality (3.1) holds, which complete the proof.

Corollary 3.1.1. Under the conditions of Theorem 3.1,

(1) if $x = a$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(a)\right|$$
$$\le \frac{b-a}{4}\Big[2H_2(\gamma^{1/q}(a),\beta^{1/q}(a)) \tag{3.3}$$
$$-H_1(\gamma(a),\beta(a)) + H_1(v,\beta(a))\Big].$$

(2) if $x = b$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(b)\right|$$
$$\le \frac{b-a}{4}\Big[2H_2(\gamma^{1/q}(b),\alpha^{1/q}(b)) \tag{3.4}$$
$$-H_1(\gamma(b),\alpha(b)) + H_1(u,\alpha(b))\Big].$$

(3) if $x = (a+b)/2$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(\tfrac{a+b}{2})\right|$$
$$\le \frac{b-a}{16}\Big\{\Big[2H_2(\gamma^{1/q}(\tfrac{a+b}{2}),\alpha^{1/q}(\tfrac{a+b}{2}))$$
$$-H_1(\gamma(\tfrac{a+b}{2}),\alpha(\tfrac{a+b}{2})) + H_1(u,\alpha(\tfrac{a+b}{2}))\Big] \tag{3.5}$$
$$+\Big[2H_2(\gamma^{1/q}(\tfrac{a+b}{2}),\beta^{1/q}(\tfrac{a+b}{2}))$$
$$-H_1(\gamma(\tfrac{a+b}{2}),\beta(\tfrac{a+b}{2})) + H_1(v,\beta(\tfrac{a+b}{2}))\Big]\Big\}.$$

Theorem 3.2. Let $f:\mathbb{R}_0 \to \mathbb{R}_+$ be differentiable, $f' \in L([a,b])$ for $0 \le a < b$. If $|f'|^q$ is an m-AH convex function on $[a, b/m]$ for $m \in (0,1]$, $p, q > 1$, and $(1/p)+(1/q) = 1$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(x)\right|$$
$$\le \frac{(x-a)^2}{2^{2+1/p}(b-a)}\Big[3^{1/p}\big(2H_2(\gamma(x),\alpha(x))$$
$$-H_1(\gamma^q(x),\alpha^q(x))\big)^{1/q} + H_1^{1/q}(u^q,\alpha^q(x))\Big] \tag{3.6}$$
$$+\frac{(b-x)^2}{2^{2+1/p}(b-a)}\Big[3^{1/p}\big(2H_2(\gamma(x),\beta(x))$$
$$-H_1(\gamma^q(x),\beta^q(x))\big)^{1/q} + H_1^{1/q}(v^q,\beta^q(x))\Big].$$

where u, v, $\alpha(x), \beta(x), \gamma(x)$ are defined in (3.2).

Proof. By Lemma 2.1 and 2.2, the m-AH convexity of $|f'(x)|^q$ on $[a, b/m]$, and Hölder's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(x)\right|$$
$$\le \frac{(x-a)^2}{4(b-a)}\Big(\int_0^1 t\left|f'(t\tfrac{a+x}{2}+(1-t)a)\right|\mathrm{d}t$$
$$+\int_0^1 (1+t)\left|f'(tx+(1-t)\tfrac{a+x}{2})\right|\mathrm{d}t\Big)$$
$$+\frac{(b-x)^2}{4(b-a)}\Big(\int_0^1 (2-t)\left|f'(t\tfrac{b+x}{2}+(1-t)x)\right|\mathrm{d}t$$
$$+\int_0^1 (1-t)\left|f'(tb+(1-t)\tfrac{b+x}{2})\right|\mathrm{d}t\Big)$$
$$\le \frac{(x-a)^2}{4(b-a)}\Big[\Big(\Big(\int_0^1 t\mathrm{d}t\Big)^{1/p}$$
$$\times\Big(\int_0^1 t\left|f'(t\tfrac{a+x}{2}+m(1-t)a/m)\right|^q \mathrm{d}t\Big)^{1/q} + \Big(\int_0^1 (1+t)\mathrm{d}t\Big)^{1/p}$$
$$\times\Big(\int_0^1 (1+t)\left|f'(mtx/m+(1-t)\tfrac{a+x}{2})\right|^q \mathrm{d}t\Big)^{1/q}\Big]$$
$$+\frac{(b-x)^2}{4(b-a)}\Big[\Big(\Big(\int_0^1 (2-t)\mathrm{d}t\Big)^{1/p}$$
$$\times\Big(\int_0^1 (2-t)\left|f'(t\tfrac{b+x}{2}+m(1-t)x/m)\right|^q \mathrm{d}t\Big)^{1/q}$$
$$+\Big(\int_0^1 (1-t)\mathrm{d}t\Big)^{1/p}$$
$$\times\Big(\int_0^1 (1-t)\left|f'(mtb/m+(1-t)\tfrac{b+x}{2})\right|^q \mathrm{d}t\Big)^{1/q}\Big]$$

$$\le \frac{(x-a)^2}{2^{2+1/p}(b-a)}$$

$$\times\left[\left(\int_0^1 \frac{t\mathrm{d}t}{t\left|f'(\frac{a+x}{2})\right|^{-q}+m(1-t)\left|f'(a/m)\right|^{-q}}\right)^{1/q}\right.$$

$$\left.+3^{1/p}\left(\int_0^1 \frac{(1+t)\mathrm{d}t}{mt\left|f'(x/m)\right|^{-q}+(1-t)\left|f'(\frac{a+x}{2})\right|^{-q}}\right)^{1/q}\right]$$

$$+\frac{(b-x)^2}{2^{2+1/p}(b-a)}$$

$$\times\left[3^{1/p}\left(\int_0^1 \frac{(2-t)\mathrm{d}t}{t\left|f'(\frac{b+x}{2})\right|^{-q}+m(1-t)\left|f'(x/m)\right|^{-q}}\right)^{1/q}\right.$$

$$\left.+\left(\int_0^1 \frac{(1-t)\mathrm{d}t}{mt\left|f'(b/m)\right|^{-q}+(1-t)\left|f'(\frac{b+x}{2})\right|^{-q}}\right)^{1/q}\right]$$

$$=\frac{(x-a)^2}{2^{2+1/p}(b-a)}\Big[3^{1/p}\big(2H_2(\gamma(x),\alpha(x))$$

$$-H_1(\gamma^q(x),\alpha^q(x))\big)^{1/q}+H_1^{1/q}(u^q,\alpha^q(x))\Big]$$

$$+\frac{(b-x)^2}{2^{2+1/p}(b-a)}\Big[3^{1/p}\big(2H_2(\gamma(x),\beta(x))$$

$$-H_1(\gamma^q(x),\beta^q(x))\big)^{1/q}+H_1^{1/q}(v^q,\beta^q(x))\Big].$$

So the inequality (3.6) holds, which complete the proof.

Corollary 3.2.1. Under the conditions of Theorem 3.2,

(1) if $x=a$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y-f(a)\right|\le\frac{b-a}{2^{2+1/p}}\Big[3^{1/p}$$
$$\times\big(2H_2(\gamma(a),\beta(a))-H_1(\gamma^q(a),\beta^q(a))\big)^{1/q}\qquad(3.7)$$
$$+H_1^{1/q}(v^q,\beta^q(a))\Big].$$

(2) if $x=b$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y-f(b)\right|\le\frac{b-a}{2^{2+1/p}}\Big[3^{1/p}$$
$$\times\big(2H_2(\gamma(b),\alpha(b))-H_1(\gamma^q(b),\alpha^q(b))\big)^{1/q}+H_1^{1/q}(u^q,\alpha^q(b))\Big].\qquad(3.8)$$

(3) if $x=(a+b)/2$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y-f(\tfrac{a+b}{2})\right|$$
$$\le\frac{b-a}{2^{4+1/p}}\Big[3^{1/p}\big(2H_2(\gamma(\tfrac{a+b}{2}),\alpha(\tfrac{a+b}{2}))$$
$$-H_1(\gamma^q(\tfrac{a+b}{2}),\alpha^q(\tfrac{a+b}{2}))\big)^{1/q}+H_1^{1/q}(u^q,\alpha^q(\tfrac{a+b}{2}))\Big]\qquad(3.9)$$
$$+\frac{b-a}{2^{4+1/p}}\Big[3^{1/p}\big(2H_2(\gamma(\tfrac{a+b}{2}),\beta(\tfrac{a+b}{2}))$$
$$-H_1(\gamma^q(\tfrac{a+b}{2}),\beta^q(\tfrac{a+b}{2}))\big)^{1/q}+H_1^{1/q}(v^q,\beta^q(\tfrac{a+b}{2}))\Big].$$

Theorem 3.3. Let $f:\mathbb{R}_0\to\mathbb{R}_+$ be differentiable, $f'\in L([a,b])$ for $0\le a<b$. If $|f'|^q$ is an m-AH convex function on $[a,b/m]$ for $m\in(0,1]$, $p,q>1$, and $(1/p)+(1/q)=1$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y-f(x)\right|\le\frac{(a-x)^2}{4(p+1)^{1/p}(b-a)}$$
$$\times\Big[(2^{p+1}-1)^{1/p}H_2^{1/q}(\gamma(x),\alpha(x))+H_2^{1/q}(u,\alpha(x))\Big]\qquad(3.10)$$
$$+\frac{(b-x)^2}{4(p+1)^{1/p}(b-a)}$$
$$\times\Big[(2^{p+1}-1)^{1/p}H_2^{1/q}(\gamma(x),\beta(x))+H_2^{1/q}(v,\beta(x))\Big].$$

Proof. By Lemma 2.1 and 2.2, the m-AH convexity of $|f'(x)|^q$ on $[0,b/m]$, and Hölder's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(\mu)\mathrm{d}\mu-f(x)\right|$$

$$\le\frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t\left|f'(t\tfrac{a+x}{2}+(1-t)a)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1(1+t)\left|f'(tx+(1-t)\tfrac{a+x}{2})\right|\mathrm{d}t\right)$$
$$+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1(2-t)\left|f'(t\tfrac{b+x}{2}+(1-t)x)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1(1-t)\left|f'(tb+(1-t)\tfrac{b+x}{2})\right|\mathrm{d}t\right)$$

$$\le\frac{(x-a)^2}{4(b-a)}\left(\left(\int_0^1 t^p\mathrm{d}t\right)^{1/p}\right.$$
$$\times\left(\int_0^1\left|f'\left[t\tfrac{a+x}{2}+m(1-t)a/m\right]\right|^q\mathrm{d}t\right)^{1/q}+\left(\int_0^1(1+t)^p\mathrm{d}t\right)^{1/p}$$
$$\left.\times\left(\int_0^1\left|f'\left[mtx/m+(1-t)\tfrac{a+x}{2}\right]\right|^q\mathrm{d}t\right)^{1/q}\right)+\frac{(b-x)^2}{4(b-a)}\left(\left(\int_0^1(2-t)^p\mathrm{d}t\right)^{1/p}\right.$$
$$\times\left(\int_0^1\left|f'\left[t\tfrac{b+x}{2}+m(1-t)x/m\right]\right|^q\mathrm{d}t\right)^{1/q}+\left(\int_0^1(1-t)^p\mathrm{d}t\right)^{1/p}$$
$$\left.\times\left(\int_0^1\left|f'\left[mtb/m+(1-t)\tfrac{b+x}{2}\right]\right|^q\mathrm{d}t\right)^{1/q}\right)$$

$$\le\frac{(a-x)^2}{4(p+1)^{1/p}(b-a)}$$

$$\times\left[\left(\int_0^1\frac{\mathrm{d}t}{t\left|f'(\frac{a+x}{2})\right|^{-q}+m(1-t)\left|f'(a/m)\right|^{-q}}\right)^{1/q}\right.$$

$$+(2^{p+1}-1)^{1/p}$$

$$\left.\times\left(\int_0^1\frac{\mathrm{d}t}{mt\left|f'(x/m)\right|^{-q}+(1-t)\left|f'(\frac{a+x}{2})\right|^{-q}}\right)^{1/q}\right]$$

$$+\frac{(b-x)^2}{4(p+1)^{1/p}(b-a)}\Big[(2^{p+1}-1)^{1/p}$$

$$\left.+\left(\int_0^1\frac{\mathrm{d}t}{mt\left|f'(b/m)\right|^{-q}+(1-t)\left|f'(\frac{b+x}{2})\right|^{-q}}\right)^{1/q}\right]$$

$$
\begin{aligned}
&= \frac{(a-x)^2}{4(p+1)^{1/p}(b-a)} \\
&\times\left[(2^{p+1}-1)^{1/p} H_2^{1/q}(\gamma(x),\alpha(x)) + H_2^{1/q}(u,\alpha(x))\right] \\
&+ \frac{(b-x)^2}{4(p+1)^{1/p}(b-a)} \\
&\times\left[(2^{p+1}-1)^{1/p} H_2^{1/q}(\gamma(x),\beta(x)) + H_2^{1/q}(v,\beta(x))\right].
\end{aligned}
$$

So the inequality (3.10) holds, which completes the proof.

Corollary 3.3.1. Under the conditions of Theorem 3.3, if $x = a$, then

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(a)\right| \le \frac{b-a}{4(p+1)^{1/p}} \\
&\times\left[(2^{p+1}-1)^{1/p} H_2^{1/q}(\gamma(a),\beta(a)) + H_2^{1/q}(v,\beta(a))\right].
\end{aligned} \tag{3.11}
$$

(2) if $x = b$, then

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(b)\right| \le \frac{b-a}{4(p+1)^{1/p}} \\
&\times\left[\begin{aligned}&(2^{p+1}-1)^{1/p} H_2^{1/q}(\gamma(b),\alpha(b)) \\ &+H_2^{1/q}(u,\alpha(b))\end{aligned}\right].
\end{aligned} \tag{3.12}
$$

(3) if $x = (a+b)/2$, then

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y - f(\tfrac{a+b}{2})\right| \\
&\le \frac{b-a}{16(p+1)^{1/p}}\Big[(2^{p+1}-1)^{1/p} \\
&\times H_2^{1/q}(\beta(\tfrac{a+b}{2}),\alpha(\tfrac{a+b}{2})) + H_2^{1/q}(u,\alpha(\tfrac{a+b}{2}))\Big] \\
&+ \frac{b-a}{16(p+1)^{1/p}}\Big[(2^{p+1}-1)^{1/p} \\
&\times H_2^{1/q}(\gamma(\tfrac{a+b}{2}),\beta(\tfrac{a+b}{2})) + H_2^{1/q}(v,\beta(\tfrac{a+b}{2}))\Big].
\end{aligned} \tag{3.13}
$$

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