Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 1, 23-28
Available online at http://pubs.sciepub.com/tjant/2/1/6

DOI:10.12691/tjant-2-1-6

Hermite-Hadamard Type Inequalities for (m, h_1, h_2)-Convex Functions Via Riemann-Liouville Fractional Integrals

De-Ping Shi[1], Bo-Yan Xi[1,*], Feng Qi[1,2,3]

[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, China
[2]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, China
[3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, China
*Corresponding author: baoyintu78@qq.com

Received February 04, 2014; Revised March 06, 2014; Accepted March 16, 2014

Abstract In the paper, via Riemann-Liouville fractional integration, the authors present some new inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are (m, h_1, h_2)-convex.

Keywords: *Riemann-Liouville fractional integral, (m, h1, h2)-convex function, integral inequality of Hermite-Hadamard type*

Cite This Article: De-Ping Shi, Bo-Yan Xi, and Feng Qi, "Hermite-Hadamard Type Inequalities for (m, h_1, h_2)-Convex Functions Via Riemann-Liouville Fractional Integrals." *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 1 (2014): 23-28. doi: 10.12691/tjant-2-1-6.

1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0.1]$. If the inequality (1) reverses, then f is said to be concave on I.

The well-known Hermite-Hadamard inequality reads that for every convex function $f: I \subseteq \mathbb{R} \to \mathbb{R}$, we have $f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}$, where $a, b \in I$ with $a<b$. If f is concave, the above inequalities reverse.

Definition 1.2. ([2]) For $s \in (0,1]$, a function $f: I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be s-convex if

$$f(\lambda x+(1-\lambda)y) \le \lambda^s f(x)+(1-\lambda)^s f(y) \tag{2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality (2) reverses, then f is said to be s-concave on I.

Definition 1.3. ([6]) Let $(0,1) \subseteq J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$ be an interval, and $h: J \to \mathbb{R}_0$. A function $f: I \to \mathbb{R}_0$ is said to be h-convex if the inequality

$$f(\lambda x+(1-\lambda)y) \le h(\lambda)f(x)+h(1-\lambda)f(y) \tag{3}$$

If the above inequality (3) reverses, then f is said to be h-concave on I.

Definition 1.4. ([10]) For $f:[0,b] \to \mathbb{R}, \alpha, m \in (0,1]$, if

$$f(\lambda x+m(1-\lambda)y) \le \lambda^\alpha f(x)+m(1-\lambda^\alpha)f(y)$$

is valid for all $x, y \in [0,b]$ and $\lambda \in [0.1]$, then we say that $f(x)$ is a (α, m)-convex function on $[0,b]$.

There have been many inequalities of Hermite-Hadamard type for the above convex functions. Some of them may be recited as follows.

Theorem 1.1. ([2]) Let $f: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I^\circ$ and $a<b$. If $|f'(x)|$ is a convex function on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.2. ([3]) Let f is a differentiable function on $[a,b]$, $|f'|^q$ is convex function on $[a,b]$, where $q>1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.3. ([4]) Let $f: I \to \mathbb{R}$ is a differentiable function on I°, where $a,b \in I^\circ, a<b, p>1$, if $|f'(x)|^{p/(p-1)}$ is convex function on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}$$
$$\times\left\{\left[\left|f'(a)\right|^{p/(p-1)}+3\left|f'(b)\right|^{p/(p-1)}\right]^{1-1/p}\right.$$
$$\left.+\left[3\left|f'(a)\right|^{p/(p-1)}+\left|f'(b)\right|^{p/(p-1)}\right]^{1-1/p}\right\}.$$

Theorem 1.4. ([5]) Let $f: \mathbb{R}_0 \to \mathbb{R}_0$ is s-convex function, where $s\in(0,1), a,b\in\mathbb{R}_0, a<b$, if $f'\in L_1([a,b])$, then

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{s+1}.$$

Definition 1.6. ([1]) Let $f\in L_1([a,b])$, The Riemann-Liouville integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\mathrm{d}t, x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\mathrm{d}t, x<b$$

Respectively, where $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$. and $\Gamma(\alpha)$ is the classical Euler gamma function be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}t^{\alpha-1}e^{-t}\mathrm{d}t.$$

Theorem 1.5. ([12]) Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b, f\in L_1([a,b])$, If f is a convex function on $[a,b]$ then

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}(b)+J_{b-}^{\alpha}(a)]$$
$$\leq\frac{f(a)+f(b)}{2}.$$

with $\alpha>0$.

Theorem 1.6. ([12]) Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$, If $|f'|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}(b)+J_{b-}^{\alpha}(a)]\right|$$
$$\leq\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[f'(a)+f'(b)].$$

In this paper, motivated by the above results, we will establish a Riemann-Liouville fractional integral identity involving a differentiable mapping and present some new inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals for (m, h_1, h_2)-convex functions.

2. A Definition and A Lemma

In the most recent paper [11], Maksa and Palés introduced even more general notion of convexity. More precisely, (α,β,a,b)-convex functions are defined as solutions f of the functional inequality

$$f(\alpha(t)x+\beta(t)y)\leq a(t)f(x)+b(t)f(y),$$

where $0\neq T\subseteq[0,1]$ and $\alpha,\beta,a,b:T\to\mathbb{R}$ are given functions.

We first introduce a definition of (m,h_1,h_2)-convex functions.

Definition 2.1. Assume $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$, $h_1,h_2:[0,1]\to\mathbb{R}_0$, and $m\in(0,1]$. Then f is said to be (m, h_1, h_2)-convex if the inequality

$$f(\lambda x+m(1-\lambda)y)\leq h_1(\lambda)f(x)+mh_2(\lambda)f(y)$$

holds for all $x,y\in I$ and $\lambda\in[0.1]$. If the above inequality reverses, then f is said to be (m, h_1, h_2)-concave on I.

Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $f'\in L_1([a,b])$. Denote $M_\alpha(a,b)$ by

$$M_\alpha(a,b)=\frac{1}{3}\left[f(a)+\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)+f(b)\right]$$
$$-\frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^{\alpha}}\times\left[J_{a+}^{\alpha}f\left(\frac{2a+b}{3}\right)+J_{\frac{2a+b}{3}+}^{\alpha}f\left(\frac{a+2b}{3}\right)+J_{\frac{a+2b}{3}+}^{\alpha}f(b)\right].$$

Specially, when $\alpha=1$, we have

$$M_1(a,b)=\frac{1}{3}\left[f(a)+\frac{1}{2}\left(f\left(\frac{2a+b}{3}\right)\right.\right.$$
$$\left.\left.+f\left(\frac{a+2b}{3}\right)\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

Lemma 2.1. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L_1([a,b])$. Then

$$M_\alpha(a,b)=\frac{b-a}{9}$$
$$\times\left\{\int_0^1(0-t^{\alpha})f'\left(ta+(1-t)\frac{2a+b}{3}\right)\mathrm{d}t\right.$$
$$+\int_0^1\left(\frac{1}{2}-t^{\alpha}\right)f'\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\mathrm{d}t$$
$$\left.+\int_0^1(1-t^{\alpha})f'\left(t\frac{a+2b}{3}+(1-t)b\right)\mathrm{d}t\right\}.$$

Proof. Letting $u=at+(1-t)\frac{2a+b}{3}$. By integration by parts, we have

$$\frac{b-a}{9}\int_0^1(-t^\alpha)f'\left(ta+(1-t)\frac{2a+b}{3}\right)\mathrm{d}t$$
$$=\frac{1}{3}\left[f(a)-\alpha\int_0^1 f\left(ta+(1-t)\frac{2a+b}{3}\right)t^{\alpha-1}\mathrm{d}t\right]$$
$$=\frac{1}{3}f(a)-\frac{\alpha 3^{\alpha-1}}{(b-a)^\alpha}\int_a^{\frac{2a+b}{3}}\left(\frac{2a+b}{3}-u\right)^{\alpha-1}f(u)\mathrm{d}u$$
$$=\frac{1}{3}f(a)-\frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha}J_{a+}^\alpha f\left(\frac{2a+b}{3}\right).$$

Similarly, we obtain

$$\frac{b-a}{9}\int_0^1\left(\frac{1}{2}-t^\alpha\right)f'\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\mathrm{d}t$$
$$=\frac{1}{6}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right),$$
$$\frac{b-a}{9}\int_0^1\left(\frac{1}{2}-t^\alpha\right)f'\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\mathrm{d}t$$
$$=\frac{1}{6}\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)$$
$$-\frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha}J_{\frac{2a+b}{3}+}^\alpha f\left(\frac{a+2b}{3}\right)$$

and

$$\frac{b-a}{9}\int_0^1(1-t^\alpha)f'\left(t\frac{a+2b}{3}+(1-t)b\right)\mathrm{d}t$$
$$=\frac{1}{3}f(b)-\frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha}J_{\frac{a+2b}{3}+}^\alpha f(b).$$

The proof of Lemma 2.1 is complete.

3. New Inequalities for (*m*, *h*₁, *h*₂)-Convex Functions

Theorem 3.1. Let $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable function such that $f'\in L_1([a,b])$ for $0<a<b$. If $|f'|^q$ is (m,h_1,h_2) -convex on $[a,b]$ for $q>1$ and $h_1,h_2\in L_1([a,b])$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\right.$$
$$\times\left(|f'(a)|^q\|h_1\|_1+m\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q}$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}$$
$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\|h_1\|_1+m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\left.\times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q\|h_1\|_1+m\left|f'\left(\frac{b}{m}\right)\right|^q\|h_2\|_1\right)^{1/q}\right],$$

where $\|h_1\|_p=\left(\int_0^1 h_1^p(t)\mathrm{d}t\right)^{1/p}$ for $p\ge1$ and $B(x,y)$ is the classical Beta function which may be defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad x>0,y>0.$$

Proof. From Lemma 2.1, Hölder inequality, and the (m, h_1, h_2)-convexity of $|f'|^q$, we obtain

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left[\left(\int_0^1 t^{\alpha q/(q-1)}\mathrm{d}t\right)^{1-1/q}\right.$$
$$\times\left(|f'(a)|^q\|h_1\|_1+m\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q}$$
$$+\left(\frac{1}{2}\right)^{1/q}\left(\int_0^1\left|\frac{1}{2}-t^\alpha\right|\mathrm{d}t\right)^{1-1/q}$$
$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\|h_1\|_1+m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q}$$
$$+\left(\int_0^1(1-t^\alpha)^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}$$
$$\left.\times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q\|h_1\|_1+m\left|f'\left(\frac{b}{m}\right)\right|^q\|h_2\|_1\right)^{1/q}\right],$$

where

$$\int_0^1 t^{\alpha q/(q-1)}\mathrm{d}t=\frac{q-1}{\alpha q+q-1},$$
$$\int_0^1(1-t^\alpha)^{q/(q-1)}\mathrm{d}t=\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right),$$

and

$$\int_0^1\left|\frac{1}{2}-t^\alpha\right|\mathrm{d}t=\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{2(\alpha+1)}.$$

The proof of Theorem 3.1 is complete.

Corollary 3.1.1. Under the conditions of Theorem 3.1, if $h_1(t)=h(t),\ h_2(t)=h(1-t)$, then

$$|M_\alpha(a,b)|\le\frac{(b-a)\|h_1\|_1^{1/q}}{9}\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\right.$$
$$\times\left(|f'(a)|^q+m\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\right)^{1/q}$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}$$

$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\times\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right].$$

Furthermore, if $m=1$, then

$$|M_\alpha(a,b)|\le\frac{(b-a)\|h\|_1^{1/q}}{9}\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\right.$$
$$\times\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{1/q}$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}$$
$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\times\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right].$$

Corollary 3.1.2. Under the conditions of Corollary 3.1.1, if $h_1(t)=h(t)=t^s$, $m=1$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{1/q}\right.$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}\times\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right];$$

Specially, if $\alpha=s=m=1$, then

$$|M_1(a,b)|\le\frac{b-a}{9}\left(\frac{1}{2}\right)^{1/q}$$
$$\times\left[\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{1/q}\right.$$
$$+\left(\frac{1}{2}\right)^{2-1/q}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right].$$

Corollary 3.1.3. Under the conditions of Theorem 3.1,if $h_1(t)=t^{\alpha_1}$ and $h_2(t)=1-t^{\alpha_1}$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{\alpha_1+1}\right)^{1/q}$$
$$\times\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\times\left(|f'(a)|^q+m\alpha_1\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\right)^{1/q}\right.$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}$$
$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+m\alpha_1\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\times\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+m\alpha_1\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right].$$

Specially, if $m=1$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{\alpha_1+1}\right)^{1/q}$$
$$\times\left[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-1/q}\left(|f'(a)|^q+\alpha_1\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{1/q}\right.$$
$$+\frac{1}{2}\left(\frac{\alpha 2^{(\alpha-1)/\alpha}-\alpha+1}{\alpha+1}\right)^{1-1/q}$$
$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\alpha_1\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{1/q}$$
$$+\left(\frac{1}{\alpha}B\left(\frac{2q-1}{q-1},\frac{1}{\alpha}\right)\right)^{1-1/q}$$
$$\times\left.\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+\alpha_1|f'(b)|^q\right)^{1/q}\right].$$

Theorem 3.2. Let $f: I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function such that $f' \in L_1([a,b])$ and $0 < a < b$. If $|f'|^q$ is (m,h_1,h_2)-convex on $[a,b]$ for $h_1,h_2 \in L_1([a,b])$ and $q \geq 1$, then

$$|M_\alpha(a,b)| \leq \frac{b-a}{9}\left(\frac{1}{\alpha+1}\right)^{1-1/q} \times\left\{\left[\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\|h_2\|_2^2\right)\right]^{1/q} + \frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q} \times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\|h_1\|_1 + m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q} + \alpha^{1-1/q}\left[\frac{1}{2}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\|h_2\|_2^2\right)\right]^{1/q}\right\},$$

where where $\|h_1\|_p$ is as given in Theorem 3.1.

Proof. From Lemma 2.1, Hölder inequality, and the (m,h_1,h_2)-convexity of $|f'|^q$, we obtain

$$|M_\alpha(a,b)| \leq \frac{b-a}{9}\left[\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-1/q} \times\left(|f'(a)|^q\int_0^1 t^\alpha h_1(t)\mathrm{d}t + m\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\int_0^1 t^\alpha h_2(t)\mathrm{d}t\right)^{1/q} + \left(\int_0^1\left|\frac{1}{2}-t^\alpha\right|\mathrm{d}t\right)^{1-1/q}\left(\frac{1}{2}\right)^{1/q}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\int_0^1 h_1(t)\mathrm{d}t + m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\int_0^1 h_2(t)\mathrm{d}t\right)^{1/q} + \left(\int_0^1(1-t^\alpha)\mathrm{d}t\right)^{1-1/q} \times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q\int_0^1(1-t^\alpha)h_1(t)\mathrm{d}t + m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1(1-t^\alpha)h_2(t)\mathrm{d}t\right)^{1/q}\right]$$

$$\leq \frac{b-a}{9}\left[\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-1/q}\left(|f'(a)|^q\int_0^1\frac{t^{2\alpha}+h_1^2(t)}{2}\mathrm{d}t\right.\right.$$

$$+\left(\frac{1}{2}\right)^{1/q}\left(\int_0^1\left|\frac{1}{2}-t^\alpha\right|\mathrm{d}t\right)^{1-1/q}\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\int_0^1 h_1(t)\mathrm{d}t + m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\int_0^1 h_2(t)\mathrm{d}t\right)^{1/q}$$

$$+\left(\int_0^1(1-t^\alpha)\mathrm{d}t\right)^{1-1/q}\times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q\int_0^1\frac{(1-t^\alpha)^2+h_1^2(t)}{2}\mathrm{d}t + m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1\frac{(1-t^\alpha)^2+h_2^2(t)}{2}\mathrm{d}t\right)^{1/q}\Bigg]$$

$$=\frac{b-a}{9}\left(\frac{1}{\alpha+1}\right)^{1-1/q}\left[\left(\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\|h_2\|_2^2\right)\right)^{1/q} + \frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q} \times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\|h_1\|_1 + m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\|h_2\|_1\right)^{1/q} + \alpha^{1-1/q}\left(\frac{1}{2}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\|h_1\|_2^2\right) + \frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\|h_2\|_2^2\right)\right)^{1/q}\right].$$

The proof of Theorem 3.2 is complete.

Corollary 3.2.1. Under the conditions of Theorem 3.2,

(1) if $h_1(t) = t^{\alpha_1}$ and $h_2(t) = 1 - t^{\alpha_1}$, then

$$|M_\alpha(a,b)| \leq \frac{b-a}{9}\left(\frac{1}{\alpha+1}\right)^{1-1/q} \times\left[\left(\frac{1}{2}|f'(a)|^q\left(\frac{1}{2\alpha+1}+\frac{1}{2\alpha_1+1}\right) + \frac{m}{2}\left|f'\left(\frac{2a+b}{3m}\right)\right|^q\left(\frac{1}{2\alpha+1}+\frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}\right)\right)^{1/q} + \frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q} \times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q\frac{1}{\alpha_1+1} + m\left|f'\left(\frac{a+2b}{3m}\right)\right|^q\frac{\alpha_1}{\alpha_1+1}\right)^{1/q}\right.$$

$$+\alpha^{1-1/q}\left(\frac{1}{2}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$

$$\times\left(\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\frac{1}{2\alpha_1+1}\right)+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\right.$$

$$\left.\left.\times\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}\right)\right)^{1/q}\right];$$

(2) if $h_1(t)=h(t)$, $h_2(t)=h(1-t)$, and $m=1$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{\alpha+1}\right)^{1-1/q}$$

$$\times\left[\left(\frac{1}{2}\left(\frac{1}{2\alpha+1}+\|h\|_2^2\right)\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)\right)^{1/q}\right.$$

$$+\frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q}$$

$$\times\left(\|h\|_1\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)\right)^{1/q}$$

$$+\alpha^{1-1/q}\left(\frac{1}{2}\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\|h\|_2^2\right)\right.$$

$$\left.\left.\times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)\right)^{1/q}\right];$$

(3) if $h_1(t)=h(t)=t^s$, $h_2(t)=h(1-t)$, and $m=1$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{\alpha+1}\right)^{1-1/q}$$

$$\times\left[\left(\frac{1}{2}\left(\frac{1}{2\alpha+1}+\frac{1}{2s+1}\right)\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)\right)^{1/q}\right.$$

$$+\frac{1}{2}\left(\alpha 2^{(\alpha-1)/\alpha}-\alpha+1\right)^{1-1/q}$$

$$\times\left(\frac{1}{s+1}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)\right)^{1/q}$$

$$+\alpha^{1-1/q}\left(\frac{1}{2}\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)}+\frac{1}{2s+1}\right)\right.$$

$$\left.\left.\times\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)\right)^{1/q}\right];$$

(4) if $h_1(t)=h(t)=t^s$, $h_2(t)=h(1-t)$ and $\alpha=s=m=1$, then

$$|M_\alpha(a,b)|\le\frac{b-a}{9}\left(\frac{1}{2}\right)^{1-1/q}$$

$$\times\left[\left(\frac{1}{3}\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)\right)^{1/q}\right.$$

$$+\frac{1}{2}\left(\frac{1}{2}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)\right)^{1/q}$$

$$\left.+\left(\frac{1}{3}\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)\right)^{1/q}\right];$$

Acknowledgement

This work was partially supported by the NNSF under Grant No. 11361038 of China and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY13159, China.

References

[1] R. Gorenflo and F. Mainardi, Fractional Calculus, Integral and Differential Equations of Fractional order, Springer, Wien, 1997.

[2] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math., 48 (1994), no. 1, 100-111.

[3] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11 (1998), no. 5, 91-95.

[4] C. E. M. Pearce and J. Pečarič, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13 (2000), no. 2, 51-55.

[5] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl. Math. Comput., 147 (2004), no. 1, 137-146.

[6] S. S. Dragomir and S. Fitzpatrik, The Hadamard's inequality for s-convex functions in the second sense, Demonstratio Math., 32 (1999), no. 4, 687-696.

[7] S. Varosanec, on h-convexity, J. Math. Anal. Appl., 326 (2007), 303-311.

[8] R.-F. Bai, F Qi, and B.-Y. Xi, Hermite-Hadamard type inequalities for the m- and (a, m) -logarithmically convex functions, Filomat, 27 (1) (2013), 1-7.

[9] Z. Dahmani, New inequalities in fractional integrals, Int, J. Nonlinear Sci., 9 (2010), no. 4, 155-160.

[10] E. K. Godunova and V. I. Levin, Neravenstva dlja funkcil sirokogo klassa, soderzascego vypuklye, monotonnye I nekotorye drugie vidy funkii, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, 138-142.

[11] V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. Convex, Clujnapoca, 1993. (Romania).

[12] G. Maksa, Z. S. Páles, The equality case in some recent convexity inequalities, Opuscula Math., 31 (2011), no. 2, 269-277.

[13] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model. 57 (2013), 2403-2407.