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q-Analogue of p-Adic log Γ Type Functions Associated with Modified q-Extension of Genocchi Numbers with Weight α and β

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Abstract The p-adic log gamma functions associated with q-extensions of Genocchi and Euler polynomials with weight α were recently studied [6]. By the same motivation, we aim in this paper to describe q-analogue of p-adic log gamma functions with weight alpha and beta. Moreover, we give relationship between p-adic q-log gamma functions with weight (α,β) and q-extension of Genocchi numbers with weight alpha and beta and modified q-Euler numbers with weight α.

Keywords: *modified q-Genocchi numbers with weight alpha and beta, modified q-Euler numbers with weight alpha and beta, p-adic log gamma functions*

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1. Introduction

Assume that p is a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote by the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Also we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|.|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one talks of q-extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the following notation.

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

where $\lim_{q \to 1} [x]_q = x$; cf. [1-23].

For a fixed positive integer d with $(d,f) = 1$, we set

$$X = X_d = \varprojlim_N \mathbb{Z} / dp^N \mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

and

$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod\, dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

It is known that

$$\mu_q\left(x + p^N \mathbb{Z}_p\right) = \frac{q^x}{\left[p^N\right]_q}$$

is a distribution on X for $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

has a limit $f'(a)$ as $(x,y)\to(a,a)$ and denote this by $f\in UD(\mathbb{Z}_p)$. The p-adic q-integral of the function $f\in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x. \tag{1.2}$$

The bosonic integral is considered by Kim as the bosonic limit $g_{n,q}^{(\alpha,\beta)}(0) := g_{n,q}^{(\alpha,\beta)}$ $I_1(f)=\lim_{q\to 1} I_q(f)$. Similarly, the p-adic fermionic integration on $\mathbb{Z}_p$ defined by Kim as follows:

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x).$$

Let $q\to 1$, then we have p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x.$$

Stirling asymptotic series are known as

$$\log\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right) = \left(x-\frac{1}{2}\right)\log x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\frac{B_{n+1}}{x^n} - x \tag{1.3}$$

where B_n are familiar n-th Bernoulli numbers cf. [6,8,9,23].

Recently, Araci et al. defined modified q-Genocchi numbers and polynomials with weight α and β in [4,5] by the means of generating function:

$$\sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!} = t\int_{\mathbb{Z}_p} q^{-\beta\xi} e^{[x+\xi]_{q^\alpha} t}\,d\mu_{-q^\beta}(\xi). \tag{1.4}$$

So from the above, we easily get Witt's formula of modified q-Genocchi numbers and polynomials with weight α and β as follows:

$$\frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{-\beta\xi}[x+\xi]_{q^\alpha}^n\,d\mu_{-q^\beta}(\xi) \tag{1.5}$$

where $g_{n,q}^{(\alpha,\beta)}(0) := g_{n,q}^{(\alpha,\beta)}$ are modified q-extension of Genocchi numbers with weight α and β cf. [4,5].

In [21], Rim and Jeong are defined modified q-Euler numbers with weight α as follows:

$$\tilde{\xi}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} q^{-t}[t]_{q^\alpha}\,d\mu_{-q}(t) \tag{1.6}$$

From expressions of (1.5) and (1.6), we get the following Proposition 1.

Proposition 1. *The following*

$$\tilde{\xi}_{n,q}^{(\alpha)} = \frac{g_{n+1,q}^{(\alpha,1)}}{n+1}$$

is true.

In previous paper [6], Araci, Acikgoz and Park introduced weighted q-analogue of p-adic log gamma type functions and derived some interesting identities. They were motivated from paper of T. Kim by "On a q-analogue of the p-adic log gamma functions and related integrals, J. Number Theory, 76 (1999), no. 2, 320-329." By the same motivation, we introduce q-analogue of p-adic log gamma type function with weight α and β. We derive in this paper some interesting identities including this type of functions.

2. On P-Adic log Γ Function with Weight α and β

In this part, from (1.2), we start at the following nice identity:

$$I_{-q}^{(\beta)}\left(q^{-\beta x} f_n\right) + (-1)^{n-1} I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) = [2]_{q^\beta}\sum_{l=0}^{n-1}(-1)^{n-1-l} f(l) \tag{2.1}$$

where $f_n(x) = f(x+n)$ and $n\in\mathbb{N}$ (see [4]).

In particular for $n=1$ into (2.1), we easily see that

$$I_{-q}^{(\beta)}\left(q^{-\beta x} f_1\right) + I_{-q}^{(\beta)}\left(q^{-\beta x} f\right) = [2]_{q^\beta} f(0). \tag{2.2}$$

By simple an application, it is easy to indicate as follows:

$$\left((1+x)\log(1+x)\right)' = 1+\log(1+x) = 1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}x^n \tag{2.3}$$

where $\left((1+x)\log(1+x)\right)' = \frac{d}{dx}\left((1+x)\log(1+x)\right)$.

By expression of (2.3), we can derive

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}x^{n+1} + x + c \tag{2.4}$$

where c is constant.

If we take $x=0$, so we get $c=0$. By expression of (2.3) and (2.4), we easily see that,

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}x^{n+1} + x. \tag{2.5}$$

It is considered by T. Kim for q-analogue of p-adic locally analytic function on $\mathbb{C}_p\setminus\mathbb{Z}_p$ as follows:

$$G_{p,q}(x) = \int_{\mathbb{Z}_p}[x+\xi]_q\left(\log[x+\xi]_q - 1\right)d\mu_{-q}(\xi) \tag{2.6}$$

(for detail, see [5,6]).

By the same motivation of (2.6), in previous paper [6], q-analogue of p-adic locally analytic function on $\mathbb{C}_p \setminus \mathbb{Z}_p$ with weight α is considered

$$G_{p,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+\xi]_{q^\alpha} \left(\log[x+\xi]_{q^\alpha} - 1\right) d\mu_{-q}(\xi) \tag{2.7}$$

In particular $\alpha = 1$ in (2.7), we easily see that, $G_{p,q}^{(1)}(x) = G_{p,q}(x)$.

With the same manner, we introduce q-analogue of p-adic locally analytic function on p with weight α and β as follows:

$$G_{p,q}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p} q^{-\beta\xi} [x+\xi]_{q^\alpha} \left(\log[x+\xi]_{q^\alpha} - 1\right) d\mu_{-q^\beta}(\xi). \tag{2.8}$$

From expressions of (2.2) and (2.8), we state the following Theorem:

Theorem 1. *The following identity holds:*

$$G_{p,q}^{(\alpha,\beta)}(x+1) + G_{p,q}^{(\alpha,\beta)}(x) = [2]_{q^\beta} [x]_{q^\alpha} \left(\log[x]_{q^\alpha} - 1\right).$$

It is easy to show that,

$$\begin{aligned}[x+\xi]_{q^\alpha} &= \frac{1-q^{\alpha(x+\xi)}}{1-q^\alpha} \\ &= \frac{1-q^{\alpha x}+q^{\alpha x}-q^{\alpha(x+\xi)}}{1-q^\alpha} \\ &= \left(\frac{1-q^{\alpha x}}{1-q^\alpha}\right) + q^{\alpha x}\left(\frac{1-q^{\alpha\xi}}{1-q^\alpha}\right) \\ &= [x]_{q^\alpha} + q^{\alpha x}[\xi]_{q^\alpha}\end{aligned} \tag{2.9}$$

Substituting $x \to \frac{q^{\alpha x}[\xi]_{q^\alpha}}{[x]_{q^\alpha}}$ into (2.5) and by using (2.9), we get interesting formula:

$$\begin{aligned}&[x+\xi]_{q^\alpha}\left(\log[x+\xi]_{q^\alpha} - 1\right) \\ &= \left([x]_{q^\alpha} + q^{\alpha x}[\xi]_{q^\alpha}\right)\log[x]_{q^\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{\left(-q^{\alpha x}\right)^{n+1}}{n(n+1)} \frac{[\xi]_{q^\alpha}^{n+1}}{[x]_{q^\alpha}^{n}} - [x]_{q^\alpha}\end{aligned} \tag{2.10}$$

If we substitute $\alpha = 1$ into (2.10), we get Kim's q-analogue of p-adic log gamma function (for detail, see [8]). From expression of (1.2) and (2.10), we obtain the following worthwhile and interesting theorems.

Theorem 2. *For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ the following*

$$\begin{aligned}&G_{p,q}^{(\alpha,\beta)}(x) \\ &= \left(\frac{[2]_{q^\beta}}{2}[x]_{q^\alpha} + q^{\alpha x}\frac{g_{2,q}^{(\alpha,\beta)}}{2}\right)\log[x]_{q^\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{\left(-q^{\alpha x}\right)^{n+1}}{n(n+1)(n+2)} \frac{g_{n+1,q}^{(\alpha,\beta)}}{[x]_{q^\alpha}^{n}} - [x]_{q^\alpha}\frac{[2]_{q^\beta}}{2}\end{aligned}$$

is true.

Corollary 1. *Taking $q \to 1$ in Theorem 2, we get nice identity:*

$$\begin{aligned}&G_{p,1}^{(\alpha,\beta)}(x) \\ &= \left(x + \frac{G_2}{2}\right)\log x \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)(n+2)} \frac{G_{n+1}}{x} - x\end{aligned}$$

where G_n are called famous Genocchi numbers.

Theorem 3. *The following nice identity*

$$\begin{aligned}&G_{p,q}^{(\alpha,1)}(x) \\ &= \left(\frac{[2]_q}{2}[x]_{q^\alpha} + q^{\alpha x}\widehat{\xi}_{1,q}^{(\alpha)}\right)\log[x]_{q^\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{\left(-q^{\alpha x}\right)^{n+1}}{n(n+1)} \frac{\widehat{\xi}_{n,q}^{(\alpha)}}{[x]_{q^\alpha}^{n}} - \frac{[2]_q}{2}[x]_{q^\alpha}\end{aligned}$$

is true.

Corollary 2. *Putting $q \to 1$ into Theorem 3, we have the following identity:*

$$G_{p,1}^{(\alpha,\beta)}(x) = (x+E_1)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{E_n}{x^n} - x$$

where E_n are familiar Euler numbers.

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