q-Analogue of p-Adic log Γ Type Functions Associated with Modified q-Extension of Genocchi Numbers with Weight α and β

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Received September 14, 2013; Revised September 21, 2013; Accepted September 23, 2013

Abstract The p-adic log gamma functions associated with q-extensions of Genocchi and Euler polynomials with weight α were recently studied [6]. By the same motivation, we aim in this paper to describe q-analogue of p-adic log gamma functions with weight alpha and beta. Moreover, we give relationship between p-adic q-log gamma functions with weight (α,β) and q-extension of Genocchi numbers with weight alpha and beta and modified q-Euler numbers with weight α .

Keywords: modified q-Genocchi numbers with weight alpha and beta, modified q-Euler numbers with weight alpha and beta, p-adic log gamma functions

Cite This Article: Erdoğan Şen, Mehmet Acikgoz, and Serkan Araci, "q-Analogue of p-Adic log Γ Type Functions Associated with Modified q-Extension of Genocchi Numbers with Weight α and β ." *Turkish Journal of Analysis and Number Theory* 1, no. 1 (2013): 9-12. doi: 10.12691/tjant-1-1-3.

1. Introduction

notation.

Assume that p is a fixed odd prime number. Throughout this paper Z, Z_p , Q_p and C_p will denote by the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of Q_p , respectively.

Also we denote $N^* = N \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : C_p \to Q \cup \{\infty\}$ (Q is the field of rational numbers) denote the p-adic valuation of C_p normalized so that $v_p(p) = 1$. The absolute value on C_p will be denoted as $|\cdot|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in C_p$. When one talks of q-extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in C$, or a p-adic number $q \in C_p$. If $q \in C$, we assume that |q| < 1. If $q \in C_p$, we assume $|1-q|_p < p^{-\frac{1}{p-1}}$, so that

 $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the following

$$[x]_q = \frac{1-q^x}{1-a}, \ [x]_{-q} = \frac{1-(-q)^x}{1+a}$$
 (1.1)

where
$$\lim_{q \to 1} [x]_q = x$$
; cf. [1-23].

For a fixed positive integer d with (d, f) = 1, we set

$$X = X_d = \lim_{\overline{N}} \mathbb{Z}/dp^N \mathbb{Z},$$

$$X^* = \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp \mathbb{Z}_p$$

and

$$a + dp^N Z_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. It is known that

$$\mu_q \left(x + p^N Z_p \right) = \frac{q^x}{\left[p^N \right]_q}$$

is a distribution on X for $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$.

Let $UD(Z_p)$ be the set of uniformly differentiable function on Z_p . We say that f is a uniformly differentiable function at a point $a \in Z_p$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

has a limit f'(a) as $(x,y) \rightarrow (a,a)$ and denote this by $f \in UD(\mathbb{Z}_p)$. The p-adic q-integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_{q}(f)$$

$$= \int_{Z_{p}} f(x) d\mu_{q}(x)$$

$$= \lim_{N \to \infty} \frac{1}{\left[p^{N}\right]_{q}} \sum_{x=0}^{p^{N}-1} f(x) q^{x}.$$

$$(1.2)$$

The bosonic integral is considered by Kim as the bosonic limit $g_{n,q}^{(\alpha,\beta)}(0) \coloneqq g_{n,q}^{(\alpha,\beta)} \quad I_1(f) = \lim_{q \to 1} I_q(f)$. Similarly, the p-adic fermionic integration on \mathbf{Z}_p defined by Kim as follows:

$$I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{Z_p} f(x) d\mu_{-q}(x).$$

Let $q \rightarrow 1$, then we have p-adic fermionic integral on Z_p as follows:

$$I_{-1}(f) = \lim_{q \to -1} I_q(f) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x)(-1)^x.$$

Stirling asymptotic series are known as

$$\log\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right)$$

$$=\left(x-\frac{1}{2}\right)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{B_{n+1}}{x^n} - x$$
(1.3)

where B_n are familiar n-th Bernoulli numbers cf. [6,8,9,23].

Recently, Araci et al. defined modified q-Genocchi numbers and polynomials with weight α and β in [4,5] by the means of generating function:

$$\sum_{n=0}^{\infty} g_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!} = t \int_{Z_p} q^{-\beta\xi} e^{\left[x+\xi\right]_q \alpha t} d\mu_{-q\beta}(\xi). (1.4)$$

So from the above, we easily get Witt's formula of modified q-Genocchi numbers and polynomials with weight α and β as follows:

$$\frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} = \int_{Z_p} q^{-\beta\xi} \left[x + \xi \right]_{q^{\alpha}}^n d\mu_{-q^{\beta}}(\xi) \quad (1.5)$$

where $g_{n,q}^{(\alpha,\beta)}(0) := g_{n,q}^{(\alpha,\beta)}$ are modified q-extension of Genocchi numbers with weight α and β cf. [4,5].

In [21], Rim and Jeong are defined modified q-Euler numbers with weight α as follows:

$$\hat{\xi}_{n,q}^{(\alpha)} = \int_{Z_p} q^{-t} \left[t \right]_{q^{\alpha}} d\mu_{-q} \left(t \right)$$
 (1.6)

From expressions of (1.5) and (1.6), we get the following Proposition 1.

Proposition 1. The following

$$\widehat{\xi}_{n,q}^{(\alpha)} = \frac{g_{n+1,q}^{(\alpha,1)}}{n+1}$$

is true.

In previous paper [6], Araci, Acikgoz and Park introduced weighted q-analogue of p-adic log gamma type functions and derived some interesting identities. They were motivated from paper of T. Kim by "On a q-analogue of the p-adic log gamma functions and related integrals, J. Number Theory, 76 (1999), no. 2, 320-329." By the same motivation, we introduce q-analogue of p-adic log gamma type function with weight α and β . We derive in this paper some interesting identities including this type of functions.

2. On P-Adic log Γ Function with Weight α and β

In this part, from (1.2), we start at the following nice identity:

$$I_{-q}^{(\beta)} \left(q^{-\beta x} f_n \right) + (-1)^{n-1} I_{-q}^{(\beta)} \left(q^{-\beta x} f \right)$$

$$= \left[2 \right]_{q\beta} \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l)$$
(2.1)

where $f_n(x) = f(x+n)$ and $n \in \mathbb{N}$ (see [4]).

In particular for n = 1 into (2.1), we easily see that

$$I_{-q}^{(\beta)} \left(q^{-\beta x} f_1 \right) + I_{-q}^{(\beta)} \left(q^{-\beta x} f \right) = [2]_{a\beta} f(0).$$
 (2.2)

By simple an application, it is easy to indicate as follows:

$$((1+x)\log(1+x))'$$
= 1 + log (1+x) (2.3)
$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n$$

where $((1+x)\log(1+x)) = \frac{d}{dx}((1+x)\log(1+x))$.

By expression of (2.3), we can derive

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x + c \qquad (2.4)$$

where c is constant.

If we take x = 0, so we get c = 0. By expression of (2.3) and (2.4), we easily see that,

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x.$$
 (2.5)

It is considered by T. Kim for q-analogue of p-adic locally analytic function on $C_p \setminus Z_p$ as follows:

$$G_{p,q}(x) = \int_{\mathbb{Z}_p} [x + \xi]_q \left(\log[x + \xi]_q - 1 \right) d\mu_{-q}(\xi)$$
 (2.6)

(for detail, see [5,6]).

By the same motivation of (2.6), in previous paper [6], q-analogue of p-adic locally analytic function on $\mathbf{C}_p \setminus \mathbf{Z}_p$ with weight α is considered

$$G_{p,q}^{(\alpha)}(x) = \int_{Z_p} [x+\xi]_{q^{\alpha}} \left(\log[x+\xi]_{q^{\alpha}} -1 \right) d\mu_{-q}(\xi)$$
(2.7)

In particular $\alpha = 1$ in (2.7), we easily see that, $G_{p,q}^{(1)}(x) = G_{p,q}(x)$.

With the same manner, we introduce q-analogue of p-adic locally analytic function on p with weight α and β as follows:

$$G_{p,q}^{(\alpha,\beta)}(x) = \int_{Z_p} q^{-\beta\xi} \left[x + \xi \right]_{q^{\alpha}} \left(\log \left[x + \xi \right]_{q^{\alpha}} - 1 \right) d\mu_{-q^{\beta}}(\xi).$$
(2.8)

From expressions of (2.2) and (2.8), we state the following Theorem:

Theorem 1. The following identity holds:

$$G_{p,q}^{\left(\alpha,\beta\right)}\left(x+1\right)+G_{p,q}^{\left(\alpha,\beta\right)}\left(x\right)=\left[2\right]_{q}\beta\left[x\right]_{q^{\alpha}}\left(\log\left[x\right]_{q^{\alpha}}-1\right).$$

It is easy to show that,

$$[x+\xi]_{q^{\alpha}} = \frac{1-q^{\alpha(x+\xi)}}{1-q^{\alpha}}$$

$$= \frac{1-q^{\alpha x}+q^{\alpha x}-q^{\alpha(x+\xi)}}{1-q^{\alpha}}$$

$$= \left(\frac{1-q^{\alpha x}}{1-q^{\alpha}}\right)+q^{\alpha x}\left(\frac{1-q^{\alpha \xi}}{1-q^{\alpha}}\right)$$

$$= [x]_{q^{\alpha}} + q^{\alpha x}[\xi]_{q^{\alpha}}$$

$$(2.9)$$

Substituting $x \to \frac{q^{\alpha x} [\xi]_q \alpha}{[x]_q \alpha}$ into (2.5) and by using

(2.9), we get interesting formula:

$$\begin{aligned} & \left[x + \xi \right]_{q} \alpha \left(\log \left[x + \xi \right]_{q} \alpha - 1 \right) \\ &= \left(\left[x \right]_{q} \alpha + q^{\alpha x} \left[\xi \right]_{q} \alpha \right) \log \left[x \right]_{q} \alpha \\ &+ \sum_{n=1}^{\infty} \frac{\left(-q^{\alpha x} \right)^{n+1}}{n(n+1)} \frac{\left[\xi \right]_{q}^{n+1}}{\left[x \right]_{\alpha}^{n}} - \left[x \right]_{q} \alpha \end{aligned} \tag{2.10}$$

If we substitute $\alpha = 1$ into (2.10), we get Kim's q-analogue of p-adic log gamma function (for detail, see [8]). From expression of (1.2) and (2.10), we obtain the following worthwhile and interesting theorems.

Theorem 2. For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ the following

$$\begin{split} &G_{p,q}^{\left(\alpha,\beta\right)}\left(x\right)\\ &= \left(\frac{\left[2\right]_{q}\beta}{2}\left[x\right]_{q}\alpha + q^{\alpha x}\,\frac{g_{2,q}^{\left(\alpha,\beta\right)}}{2}\right)\!\log\!\left[x\right]_{q}\alpha\\ &+ \sum_{n=1}^{\infty}\!\frac{\left(-q^{\alpha x}\right)^{n+1}}{n(n+1)(n+2)}\frac{g_{n+1,q}^{\left(\alpha,\beta\right)}}{\left[x\right]_{\alpha}^{n}}\!-\!\left[x\right]_{q}\alpha\,\frac{\left[2\right]_{q}\beta}{2} \end{split}$$

is true.

Corollary 1. Taking $q \rightarrow 1$ in Theorem 2, we get nice identity:

$$G_{p,1}^{(\alpha,\beta)}(x) = \left(x + \frac{G_2}{2}\right) \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)(n+2)} \frac{G_{n+1}}{x} - x$$

where G_n are called famous Genocchi numbers.

Theorem 3. The following nice identity

$$G_{p,q}^{(\alpha,1)}(x)$$

$$= \left(\frac{[2]_q}{2} [x]_{q^{\alpha}} + q^{\alpha x} \hat{\xi}_{1,q}^{(\alpha)}\right) \log[x]_{q^{\alpha}}$$

$$+ \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)} \frac{\hat{\xi}_{n,q}^{(\alpha)}}{[x]_{q^{\alpha}}^n} - \frac{[2]_q}{2} [x]_{q^{\alpha}}$$

is true

Corollary 2. Putting $q \rightarrow 1$ into Theorem 3, we have the following identity:

$$G_{p,1}^{(\alpha,\beta)}(x) = (x+E_1)\log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{E_n}{x^n} - x$$

where E_n are familiar Euler numbers.

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