

The Hyper-Geometric Daehee Numbers and Polynomials

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Abstract We consider the hyper-geometric Daehee numbers and polynomials and investigate some properties of those numbers and polynomials.

Keywords: Daehee numbers, Hyper-geometric Daehee numbers and polynomials

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1. Introduction

As is known, the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad (1.1)$$

(see [5,6,7,9,10,11,12]).

In the special case, $x=0, D_n = D_n(0)$ are called the Daehee numbers.

Let $\mathbb{Z}_p, \mathbb{Q}_p$ and \mathbb{C}_p denote the rings of p-adic integers, the fields of p-adic numbers and the completion of algebraic closure of \mathbb{Q}_p . The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = 1/p$. Let (\mathbb{Z}_p) be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on \mathbb{Z}_p is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x), \quad (1.2)$$

(see [7,8]).

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.2), we get

$$I(f_1) = I(f) + f'(0), \text{ where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}. \quad (1.3)$$

As is known, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (1.4)$$

and the Stirling number of the second kind is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \quad (1.5)$$

(see [2,3,4]).

For $\alpha \in \mathbb{N}$, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (1.6)$$

(see [1,2,9]).

When $x=0, B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α .

A hyper-geometric series $\sum_k c_k$ is a series for which $c_0=1$ and the ratio of consecutive terms is a rational function of the summation index k , i.e., one for which

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)},$$

with $P(k)$ and $Q(k)$ polynomials. In this case, c_k is called a hyper-geometric term. The functions generated by hyper-geometric series are called generalized hyper-geometric functions. If the polynomials are completely factored, the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_p)(k+1)} \quad (1.7)$$

(see [13]),

where the factor of $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hyper-geometric function is written

$${}_pF_q \left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix} \middle| x \right] = \sum_{k=0}^{\infty} c_k x^k \quad (1.8)$$

(see [13]).

If $p = 2$ and $q = 1$, the function becomes a traditional hyper-geometric function ${}_2F_1(a, b; c; x)$. Many sums can be written as generalized hyper-geometric functions by inspections of the ratios of consecutive terms in the generating hyper-geometric series.

We introduce the hyper-geometric Daehee numbers and polynomials. From our definition, we can derive some interesting properties related to the hyper-geometric Daehee numbers and polynomials.

2. The Hyper-Geometric Daehee Numbers and Polynomials

First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p} (x)_n d\mu_0(x), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \quad (2.1)$$

By (2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n d\mu_0(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x}{n} t^n d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} (1+t)^x d\mu_0(x), \end{aligned} \quad (2.2)$$

(see [6]), where $t \in \mathbb{C}_p$ with $|t|_p < p^{-1}$.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-1}$, let us take $f(x) = (1+t)^x$. Then, from (1.3), we have

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_0(x) = \frac{\log(1+t)}{t}. \quad (2.3)$$

By (1.1) and (2.3), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} &= \frac{\log(1+t)}{t} \\ &= \int_{\mathbb{Z}_p} (1+t)^x d\mu_0(x) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n d\mu_0(x) \frac{t^n}{n!}. \end{aligned} \quad (2.4)$$

(see [6]).

Therefore, by (2.4), we obtain the following Lemma.

Lemma 1. For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} (x)_n d\mu_0(x) = D_n.$$

For $n \in \mathbb{Z}$, it is known that

$$\left(\frac{t}{\log(1+t)} \right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!}, \quad (2.5)$$

(see [4,5,6]).

Thus, by (2.5), we get

$$D_k = \int_{\mathbb{Z}_p} (x)_k d\mu_0(x) = B_k^{(k+2)}(1), (k \geq 0), \quad (2.6)$$

where $B_k^{(n)}(x)$ are the Bernoulli polynomials of order n .

In the special case, $x = 0$, $B_k^{(n)} = B_k^{(n)}(0)$ are called the n -th Bernoulli numbers of order n .

From (2.4), we note that

$$\begin{aligned} (1+t)^x \int_{\mathbb{Z}_p} (1+t)^y d\mu_0(y) &= \left(\frac{\log(1+t)}{t} \right) (1+t)^x \\ &= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \end{aligned} \quad (2.7)$$

(see [6]).

Thus, by (2.7), we get

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y) = D_n(x), (n \geq 0), \quad (2.8)$$

and, from (2.5), we have

$$D_n(x) = B_n^{(n+2)}(x+1). \quad (2.9)$$

(see [6]).

Therefore, by (2.8) and (2.9), we obtain the following Lemma.

Lemma 2. For $n \geq 0$, we have

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y),$$

and

$$D_n(x) = B_n^{(n+2)}(x+1).$$

By Lemma 1, we easily see that

$$D_n = \sum_{l=0}^n S_1(n, l) B_l, \quad (2.10)$$

(see [6]), where B_l are the ordinary Bernoulli numbers.

From Lemma 2, we have

$$\begin{aligned} D_n(x) &= \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y) \\ &= \sum_{l=0}^n S_1(n, l) B_l(x), \end{aligned} \quad (2.11)$$

(see [6]), where $B_l(x)$ are the Bernoulli polynomials defined by generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Therefore, by (2.10) and (2.11), we obtain the following corollary.

Corollary 3. For $n \geq 0$, we have

$$D_n(x) = \sum_{l=0}^n S_1(n, l) B_l(x).$$

In (2.4), we have

$$\begin{aligned} \frac{t}{e^t - 1} &= \sum_{n=0}^{\infty} D_n \frac{1}{n!} (e^t - 1)^n \\ &= \sum_{n=0}^{\infty} D_n \frac{1}{n!} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m D_n S_2(m, n) \right) \frac{t^m}{m!} \end{aligned} \tag{2.12}$$

and

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \tag{2.13}$$

(see [6]).

Therefore, by (2.12) and (2.13), we obtain the following Lemma.

Lemma 4. For $n \geq 0$, we have

$$B_m = \sum_{n=0}^m D_n S_2(m, n).$$

In particular,

$$\int_{\mathbb{Z}_p} x^m d\mu_0(x) = \sum_{n=0}^m D_n S_2(m, n).$$

Remark. For $m \geq 0$, by (2.11), we have

$$\int_{\mathbb{Z}_p} (x+y)^m d\mu_0(y) = \sum_{n=0}^m D_n(x) S_2(m, n).$$

(see [6]).

Now, we define the hyper-geometric Daehee polynomials

$$F \left(\begin{matrix} a & b \\ c \end{matrix} \middle| x \right) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n} \binom{-b}{n}}{\binom{-c}{n}} (-x)^n, \tag{2.14}$$

where $\binom{x}{n} = \frac{x(x+1)\cdots(x+n-1)}{n!}$.

For example, we have

$$\begin{aligned} F \left(\begin{matrix} 1 & 1 \\ 2 \end{matrix} \middle| x \right) &= \sum_{n=0}^{\infty} \frac{\binom{-1}{n} \binom{-1}{n}}{\binom{-2}{n}} (-x)^n \\ &= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \\ &= \frac{1}{x} \log(1+x) \\ &= \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}. \end{aligned} \tag{2.15}$$

Thus the hyper-geometric Daehee number are defined by

$$2F \left(\begin{matrix} 1 & N \\ N+1 \end{matrix} \middle| -t \right) = \sum_{n=0}^{\infty} D_{N,n} \frac{t^n}{n!} \tag{2.16}$$

Note that $D_{1,N} = D_n$ is the Daehee number.

$$F \left(\begin{matrix} 1 & N \\ N+1 \end{matrix} \middle| -x \right) = \sum_{n=0}^{\infty} \frac{(1)_{\infty} \binom{N}{n}}{(N+1)_n} \frac{(-x)^n}{n!}, \tag{2.17}$$

where $(a)_n = a(n+1)\cdots(a_n-1)$.

$$\begin{aligned} &\sum_{n=0}^{\infty} D_{1,N} \frac{(-x)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{n! N(N+1)\cdots(N+n-1) (-x)^n}{(N+1)\cdots(N+n)n!} \\ &= \frac{N!}{(N-1)!} \sum_{n=0}^{\infty} \frac{(N+n-1)!}{(N+n)!} (-x)^n \\ &= \frac{N!}{(N-1)!} \sum_{n=0}^{\infty} \frac{1}{(N+n)} (-x)^n. \end{aligned} \tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 5. For $n \geq 0$, we have

$$\frac{D_{1,N}}{n!} = \frac{N}{N+n} (-1)^n.$$

In (2.17), we have

$$\begin{aligned} &N(-1)^{N-1} \sum_{n=N}^{\infty} \frac{(-1)^{n-1}}{n} x^{n-N} \\ &= \frac{(-1)^{N-1} N}{x^N} \sum_{n=N}^{\infty} \frac{(-1)^{n-1}}{n} x^n \\ &= \frac{(-1)^{N-1} N}{x^N} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \sum_{n=1}^{N-1} \frac{(-1)^{n-1}}{n} x^n \right\} \\ &= (-1)^{N-1} \frac{N}{x^N} \left\{ \log(1+x) - \sum_{n=1}^{N-1} \frac{(-1)^{n-1}}{n} x^n \right\} \end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 6. For $n \geq 0$, we have

$$\begin{aligned} &(-1)^{N-1} \frac{x^N}{N} F \left(\begin{matrix} 1 & N \\ N+1 \end{matrix} \middle| -x \right) \\ &= \log(1+x) - \sum_{n=1}^{N-1} \frac{(-1)^{n-1}}{n} x^n. \end{aligned}$$

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