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On the Cohomological Impact of a Quasi-Poisson Structure for Some Poisson Cohomology Spaces

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Abstract In order to construct a Poisson cohomology complex in the quasi-Poisson context, we establish an isomorphism between interesting Poisson cohomology groups for a quasi-Poisson algebra and Poisson cohomology groups for Poisson algebra coming from the Jacobiator of the quasi-Poisson algebra.

Keywords: *(Quasi-) Poisson algebra, Jacobiator, coboundary, Poisson cohomology*

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1. Introduction

The Poisson structures were first introduced and discussed in the not so well know paper by S. Lie in 1875 [1], whose use the name of function groups. The classical Poisson bracket

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right) \tag{1}$$

defined on the algebra of smooth functions on $\mathbb{R}^{2n}$ plays a fundamental role in the analytical mechanics. It was discovered by D. Poisson in 1809. The Poisson bracket (1) is derived from a symplectic structure on $\mathbb{R}^{2n}$ and it appears as one of the main ingredients of symplectic geometry. The basic properties of the bracket (1) are that it yields the structure of a Lie algebra on the space of functions and it has a natural compatibility with the usual associative product of functions. These facts are of algebraic nature and it is natural to define an abstract notion of a *Poisson algebra*. Following A. Vinogradov and I. Krasil'shchik in [2], J. Braconnier (in [3]) has developed the algebraic version of Poisson geometry.

One of the most important notion related to the Poisson geometry is Poisson cohomology which was introduced by A. Lichnerowicz (in [4]) and in algebraic setting by I. Krasil'shchik (in [5]). Unlike the De Rham cohomology, Poisson cohomology spaces are almost irrelevant to the topology of the manifold and moreover they have bad functorial properties. They are very large and their actual computation is both more complicated and less significant than in the case of the De Rham cohomology. However they are very interesting because they allow us to describe various results concerning Poisson structures in particular one important result about the geometric quantization of the manifold.

According to [6], a Poisson algebra is a commutative associative algebra A over $\mathbb{R}$ carrying a Lie algebra bracket $\{\cdot,\cdot\}$ for which each adjoint operator $X_f=\{\cdot,f\}$ is a derivation of the associative algebra structure. Poisson algebras appear naturally in Hamiltonian mechanics, and are also central in the study of quantum groups. Manifolds with a Poisson algebra structure are known as Poisson manifolds, of which the symplectic manifolds and the Lie-Poisson groups are a special case. There are also non commutative Poisson algebras [7,8,9,10], but we will not treat them in this paper. Of course, one can replace $\{\cdot,\cdot\}$ by another bracket $\{\cdot,\cdot\}_0$ whose the Jacobi identity is not always verified. We will call it a quasi-Poisson structure.

The *quasi-Poisson* structures was introduced by Alekseev, Yvette Kosmann-Schwarzbach and Meinrenken in [11,12]. It appeared as a finite-dimensional alternative to infinite-dimensional constructions of Poisson structures on moduli spaces, proposed in particular by Huebschmann [13], Goldman [14,15], Jerrey and Weitsman [16]. According to Vaisman in [17], Poisson cohomology plays an important role in obstruction to quantification. Example of Poisson cohomolgy of Poisson algebras are given in [18] and [19]. A usefull reference for Poisson cohomology of Poisson algebras is [20]. However, the construction of a Poisson structure often requires a choice of r-matrices, even if the bi-derivation (deduced from the Jacobi quasi-identity) obtained in the quotient does not seem to depend on this choice. Quasi-Poisson structures then appear as a more natural technique for constructing Poisson structures.

They have indeed good reduction properties, allowing to obtain a Poisson structure when we proceed to the quotient. For most of this paper, A will be the algebra $C^{\infty}(M)$ of smooth functions on a manifold M, in which case the bracket is called a Poisson structure on M, and $(M,\{\cdot,\cdot\})$ is called a *Poisson manifold. The derivations* X_f are represented by vector fields, which are called hamiltonian vector fields.

Since the bracket $\{f,g\}$ of functions on a Poisson manifold M is a derivation in each argument, it depends only on the first derivatives f of and g, and hence it can be written in the form

$$\{f,g\}=\pi(df,dg) \qquad (2)$$

where $\pi\in\Gamma(\Lambda^2 TM)$ is a field of skew-symmetric bilinear forms on T^*M, i.e., a *bivector field.*

We call π the *Poisson tensor*. The Jacobi identity for the bracket implies that π satisfies an integrability condition which is a quadratic first-order (semilinear) partial differential equation in local coordinates and has the invariant form $[\pi,\pi]=0$, where the bracket here is the Schouten-Nijenhuis bracket on multivector fields (see [21]).

Lichnerowicz (in [4]) observed that the operation $[\pi,\cdot]$ of Schouten bracket with a Poisson tensor is a differential on multivector fields, and he began the study of the resulting cohomology theory for Poisson manifolds. In particular, he showed that the map from differential forms to multivector fields determined by $\tilde{\pi}:T^*M\to TM$ is a morphism from the de Rham complex to the Poisson complex.

According to [6], in the symplectic case this map is an isomorphism, but the Poisson cohomology spaces $H^*_\pi(M)$ are in general quite different from the de Rham cohomology. Then $H^0_\pi(M)$ consists of the functions which Poisson commute with everything, the so-called *Casimir functions* on M, $H^1_\pi(M)$ is the space of infinitesimal Poisson automorphisms modulo hamiltonian vector fields. $H^2_\pi(M)$ can be interpret as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations, while $H^3_\pi(M)$ receives the possible obstructions to extending infinitesimal deformations.

In the following, $A=C^\infty(M)$ where $M=\mathbb{C}[x,y,z]$. It is clair that $(A,\cdot,\{\cdot,\cdot\})$ where $\{\cdot,\cdot\}:(x,y)\mapsto\{x,y\}=y^n$ is a Poisson algebra.

Let $n\geq 1$ be an integer, $A_0=C^\infty(M_0)$ where $M_0=\mathbb{C}[x,y,z]/\langle y=0\rangle$ and let $(A_0,\cdot,\{\cdot,\cdot\}_0)$ with bracket is associated to the bivector field defined by

$$\pi_0=-y^n\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\frac{1}{n}x\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\frac{1}{n}y\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

In section 2, we establish that $(A_0,\cdot,\{\cdot,\cdot\}_0)$ is a quasi-Poisson algebra and there exist a biderivation ϕ_{M_0} such that $[\pi_0,\pi_0]=\phi_{M_0}$.

This derivation is given by the bracket $\{\cdot,\cdot\}:(x,y)\mapsto\{x,y\}=y^n$ of the algebra $(A,\cdot,\{\cdot,\cdot\})$. The question arises is: there exists an isomorphism between the Poisson cohomology spaces of the quasi-Poisson algebra $(A_0,\cdot,\{\cdot,\cdot\}_0)$ and those of the Poisson algebra $(A,\cdot,\{\cdot,\cdot\})$ induced by the Jacobi identity? In other words, under what criteria can we obtain an isomomorphism between $H^*_{\pi_0}(A_0)$ and $H^*_\pi(A)$? Is it related to the cobord? Is it not related to the associated differentials? The same type of problem appears in [22], when Alekseev proposes in 1994, a finite dimensional construction of a Poisson structure on a moduli space by Hamiltonian reduction of a quasi-Poisson biderivation. There are other constructions on moduli space, proposed in particular by Huebschmann [23,24], Goldman [25,26], Jerey and Weitsman [27]. We do not details these works which are totally independent of the techniques we develop here.

In this paper, we fix a ground field $\mathbb{F}$ of characteristic zero, which the reader may think of as being $\mathbb{R}$ or $\mathbb{C}$, especially in the context of varieties. The Poisson complex considered is described in [20]

Definition 1.1 [20] Let $(A,\cdot,\{\cdot,\cdot\})$ be a Poisson algebra. For $k\in A$, the space of $k-$ cochains of the Poisson cohomology complex denoted by $\chi^k(A)$ is the $\mathbb{F}-$vector space of skew-symmetric $k-$ derivations of A. The Poisson coboundary operator is the graded $\mathbb{F}-$linear map (of degree 1)

$$\delta_\pi:\chi^\bullet(A)\to\chi^{\bullet+1}(A)$$

defined, for $Q\in\chi^q(A)$, where $q\in\mathbb{N}$, by a skew-symmetric multi-derivation of A:

$$\begin{aligned}\delta^q_\pi(Q)\left[F_0,\dots,F_q\right]&:=\sum_{i=0}^{q}(-1)^i\left\{F_i,Q\left[F_0,\dots,\widehat{F_i},\dots,F_q\right]\right\}\\&+\sum_{0\leq i<j\leq q}(-1)^{i+j}Q\left[\left\{F_i,F_j\right\},F_0,\dots,\widehat{F_i},\dots,\widehat{F_j},\dots,F_q\right],\end{aligned} \qquad (3)$$

For all $F_0,\dots,F_q\in A$ and $\delta^{q+1}_\pi\circ\delta^q_\pi=0$, for $q\in\mathbb{N}$.

We obtain the Poisson cohomology complex of A:

$$\dots\to\chi^q(A)\xrightarrow{\delta^q_\pi}\chi^{q+1}(A)\xrightarrow{\delta^{q+1}_\pi}\chi^{q+2}(A)\xrightarrow{\delta^{q+2}_\pi}\dots$$

The elements of $Ker\delta^{q+1}_\pi$ are called Poisson $(q-1)$-cocycles, while the elements of $\operatorname{Im}\delta^q_\pi$ are called Poisson $q-$coboundaries.

The elements of q**-th Poisson cohomology space** are Poisson $q-$*cocycles* modulo Poisson $q-$*coboundaries*, $H^q_\pi(A):=Ker\delta^q_\pi/\operatorname{Im}\delta^{q-1}_\pi$ for all $q\in N^*$ and $H^q_\pi(A):=Ker\delta^0_\pi$.

The graded vector space $H^\bullet_\pi(A):=\bigoplus_{q\in N}H^q_\pi(A)$ is called the *Poisson cohomology of* $(A,\cdot,\{\cdot,\cdot\})$.

In the following, $\mathbb{F}=\mathbb{C}$. As graded-$\mathbb{C}$ vector spaces, the Poisson cohomology of the quasi-Poisson algebra $(A_0,\cdot,\{\cdot,\cdot\}_0)$ will be denoted by $H^*_{\pi_0}(A_0)$.

Our main result is the construction of a Poisson cohomology complex on which $H^*_{\pi_0}(A_0)$ is isomorphic to $H^*_{\pi}(A)$.

The general notions of a Poisson algebra and of a Poisson cohomology is described in section 2. The Poisson cohomology complex of the cohomology spaces are detailed in section 3 of this article.

2. On Quasi-Poisson Algebras and Poisson Cohomology

In this section, we first specify the notations and recall some classical results that will be useful later. Let M be a smooth variety of dimension d.

We use the following notations : $C^\infty(M)$ denotes the commutative algebra of $C^\infty-$ *functions* on M, $\chi^1(M)$ the space of vector fields over M, i.e. the $C^\infty-$sec*tions* of the tangent bundle $p_M : TM \to M$. More generally, for any integer q $(2 \le q \le d)$, $\chi^q(M)$ the space of multivectors field of degree q.

Consider the coordinates system $(x_1,\dots,x_d)$, a multivectors field Q is written:

$$Q = \sum_{1\le j_1<\dots<j_q\le d} Q_{j_1\cdots j_q}\partial_{j_1}\wedge\dots\wedge\partial_{j_q}.$$

Let $\chi(M) = C^\infty(M)\oplus\chi^1(M)\oplus\dots\oplus\chi^d(M)$.

Let $2\le q\le d$, $\Omega^q(M)$ is the space of q-differential forms on M.

A differential form $\alpha\in\Omega^q(M)$ is defined by $\alpha = \sum_{1\le j_1<\dots<j_q\le d} \alpha_{j_1\cdots j_q} dx_{j_1}\wedge\dots\wedge dx_{j_q}$.

Let $\Omega(M) = C^\infty(M)\oplus\Omega^1(M)\oplus\dots\oplus\Omega^d(M)$.

Definition 2.1. [21] Let q be an integer $(2\le q\le d)$. The Leibniz bracket of order q or multi-derivation on M, is the application

$$\{\cdot,\dots,\cdot\} : C^\infty(M)\times\dots\times C^\infty(M)$$
$$\to C^\infty(M), (f_1,\dots,f_q)\mapsto\{f_1,\dots,f_q\}$$

Such that

a) $\{\cdot,\dots,\cdot\}$ is the alternating $\mathbb{R}$-multilinear.

b) $\{\cdot,\dots,\cdot\}$ verifies Leibniz's rule.

Let $D^q(M)$ be the space of Leibniz brackets of order q on M.

Proposition 2.1. [21] Let q be an integer $(1\le q\le d)$. The application which to a field of multivectors Q associates the Leibniz bracket of order q defined by $\{f_1,\dots,f_q\} = Q(df_1,\dots,df_q)$, for all $f_1,\dots,f_q\in C^\infty(M)$, induces a bijection between $\chi^q(M)$ and $D^q(M)$.

Proof. (see [21]).

Definition 2.2. (Jacobiator) For any Leibniz bracket of order 2, we call the Jacobiator of $\{\cdot,\cdot\}$ the application

$$J : C^\infty(M)\times C^\infty(M)\times C^\infty(M)$$
$$\to C^\infty(M), (x,y,z)\mapsto J(x,y,z)$$

with $J(x,y,z) = \{x,\{y,z\}\}+\{y,\{z,x\}\}+\{z,\{x,y\}\}$.

Proposition 2.2. [21] The Jacobiator is a Leibniz bracket of order 3.

Proof. It is clear that J is $\mathbb{R}$-multilinear and that if $x=y, x=z$ or $y=z$ then $J(x,y,z)=0$.

Now,

$$J(xw,y,z) = \{xw,\{y,z\}\}+\{y,\{z,xw\}\}+\{z,\{xw,y\}\}.$$

We have $J(xw,y,z) = xJ(w,y,z)+wJ(x,y,z)$. Therefore J is a Leibniz bracket of order 3.

Considering the bivector field

$$\pi_0 = -y^n\partial_1\wedge\partial_2 + \frac{1}{n}x\partial_1\wedge\partial_3 - \frac{1}{n}y\partial_2\wedge\partial_3 \tag{4}$$

for any integer $n\ge 2$ where ∂_1, ∂_2 and ∂_3 are respectively the partial derivatives in x, y and z.

Proposition 2.3. π_0 is a Poisson quasi-structure on M_0 for which the Jacobiator is

$$J : C^\infty(M_0)\times C^\infty(M_0)\times C^\infty(M_0)$$
$$\to C^\infty(M_0), (x,y)\mapsto y^n \tag{5}$$

with $J(x,y,z) := y^n$.

Moreover, the Schouten-Nijenhuis bracket that we will now introduce is an extension of the Lie bracket $\chi^1(M)$ to the whole algebra of multivectors fields $\chi(M)$ as the following theorem shows.

Theorem. 2.1. (Schouten-Nijenhuis) Let M be a smooth variety. Then there exists on $\chi(M)$ a bracket $[\cdot,\cdot]$ called the Schouten-Nijenhuis bracket and verifying the following properties :

a) If $A\in\chi^a(M)$ and $B\in\chi^b(M)$ then $[A,B]\in\chi^{a+b-1}(M)$

b) (Graded anti-commutativity) If $A\in\chi^a(M)$ and $B\in\chi^b(M)$ then $[A,B] = -(-1)^{(a-1)(b-1)}[B,A]$.

c) (The graded Leibniz rule) If $A\in\chi^a(M)$, $B\in\chi^b(M)$ and $C\in\chi^c(M)$ then

$$[A,B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b}B\wedge[A,C],$$

$$[A\wedge B,C] = A\wedge[B,C] + (-1)^{(c-1)b}[A,C]\wedge B.$$

d) (The graded Jacobi identity) If $A\in\chi^a(M)$, $B\in\chi^b(M)$ and $C\in\chi^c(M)$ then

$$u[A,[B,C]] + v[B,[C,A]] + w[C,[A,B]] = 0.$$

with $u=(-1)^{(a-1)(c-1)}$, $v=(-1)^{(b-1)(a-1)}$,
$w=(-1)^{(c-1)(b-1)}$.

Note that if $X\in\chi^1(M)$, $A\in\chi^a(M)$ and $f\in C^\infty(M)$, the Schouten-Nijenhuis bracket of X and A coincides with the Lie bracket and the Schouten-Nijenhuis bracket of X and f is $X(f)$.

Consider the Jacobiator J of proposition 2.3. According to proposition 2.1, there exists a field of $3-vectors$ such that $J(x,y,z)=\pi_J(dx,dy,dz)$.

Proposition 2.4. $2\pi_J=[\pi_0,\pi_0]:=y^n\partial_1\wedge\partial_2\wedge\partial_3$.

Proof. By a simple computation, we have and

$$\{x,\{y,z\}\}=4J(x,y,z).$$

Since $\pi_J(dx,dy,dz)=J(x,y,z)$, we deduce that

$$[\pi_0,\pi_0](dx,dy,dz)=2\pi_J(dx,dy,dz).$$

This completes the proof of the proposition.

Note that $[\pi_0,\pi_0]=\phi_{M_0}$ implies that

$$\phi_{M_0}=2\pi_J$$

where π_J is the Jacobiator associated to π_0.

Then ϕ_{M_0} is a Leibniz bracket of order 3. On a Poisson manifold (M,π), $C^\infty(M;\wedge^\bullet TM)$, with the differential $d_\pi=[\pi,\bullet]$, is a complex.

In fact, since $d_\pi^2=\frac{1}{2}[[\pi,\pi],\cdot]$, the property $[\pi,\pi]=0$ ensures that d_π squares to zero.

The cohomology of d_π is called the Poisson cohomology of (M,π).

Let (M_0,π_0) be a quasi-Poisson manifold. Then, $d_{\pi_0}:=[\pi_0,\bullet]$ defines an operator on the space of multivectors. Its square is in general non-vanishing, $d_{\pi_0}^2=\frac{1}{2}[\phi_{M_0},\cdot]$. In the following, we study conditions for which d_{π_0} becomes a differential.

Precisely, we are using the Poisson complex described in [20]. According to Definition 1.1, we shall use the following isomorphisms :

$$\chi^1(A)\xrightarrow{\cong}A\times A\times A,\ f_1\partial_x+f_2\partial_y+f_3\partial_z\mapsto(f_1,f_2,f_3),$$

$$\chi^1(A)\times\chi^1(A)\times\chi^1(A)\xrightarrow{\cong}A,\ g\partial_x\wedge\partial_y\wedge\partial_z\mapsto g.$$

With these isomorphisms, we deduce the following proposition.

Proposition 2.1. The following sequence is a Poisson cohomology complex of A.

$$d_\pi:0\xrightarrow{0}\chi^0(A)\xrightarrow{\delta_\pi^0}\chi^1(A)\xrightarrow{\delta_\pi^1}\chi^2(A)\xrightarrow{\delta_\pi^k}0\quad(10)$$

where $\delta_{\pi_0}^0(f)=(y^n\partial_2 f,-y^n\partial_1 f,0)$ for f in A,

$$\delta_{\pi_0}^1(g_1,g_2,g_3)=y^n\partial_2 g_2+y^n\partial_1 g_1-ny^{n-1}g_2+g_3$$

for all g_1,g_2,g_3 belongs to $\chi^1(A)$ and $\delta_{\pi_0}^k=0$ where $k\geq 2$ be an integer.

Proof. It follows from definition 1.1 that it suffices to show that $\delta_\pi^1\circ\delta_\pi^0=0$. *Let* f in A, by a simple computation we obtain $\delta_\pi^1\circ\delta_\pi^0(f)=0$. Thus, d_π is a Poisson cohomology complex of A.

Let's consider now the quasi-Poisson structure on A_0. In order to describe the probably Poisson cohomology complex of A_0, we will need the following lemma.

Lemma 2.1. For a f in A_0, the map $D_f:A_0\to A_0$, defined by

$$D_f:=-y^n\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right)+\frac{1}{n}x\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial}{\partial x}\right)-\frac{1}{n}y\left(\frac{\partial f}{\partial y}\frac{\partial}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial}{\partial y}\right)$$

is a derivation and the bracket $\{\cdot,\cdot\}_0$ induces a Hamiltonian H_0 defined as

$$H_0(df):=\left(y^n f_2-\frac{1}{n}xf_3\right)\frac{\partial}{\partial x}+\left(-y^n f_1+\frac{1}{n}yf_3\right)\frac{\partial}{\partial y}+\left(\frac{1}{n}xf_1-\frac{1}{n}yf_2\right)\frac{\partial}{\partial z}$$

with $f_1,f_2,f_3\in A_0$.

Proof.

Let $F,G\in A_0$. It's clair that D_f is a derivation.

Like any (quasi)-Poisson structure, $\{\cdot,\cdot\}_0$ *induces a Hamiltonian* H_0 *defined for all* F,G *in* A_0 *by* $H_0(dF)(G)=\{F,G\}_0$ *which means that* $H_0(dF)(\bullet)=\{F,\bullet\}$.

Let f_1,f_2,f_3 be three elements of A_0 such that $df=f_1dx+f_2dy+f_3dz$.

For that, H_0 is defined by

$$H_0(df)=(y^n f_2-\frac{1}{n}xf_3)\frac{\partial}{\partial x}+(-y^n f_1+\frac{1}{n}yf_3)\frac{\partial}{\partial y}+(\frac{1}{n}xf_1-\frac{1}{n}yf_2)\frac{\partial}{\partial z}.$$

For the sake of simplicity we shall use the following isomorphisms:

$$\chi^1(A_0)\xrightarrow{\cong} A_0\times A_0\times A_0,\ f_1\partial_x+f_2\partial_y+f_3\partial_z\mapsto(f_1,f_2,f_3),$$

$$\chi^1(A_0)\times\chi^1(A_0)\times\chi^1(A_0)\xrightarrow{\cong} A_0,\ g\partial_x\wedge\partial_y\wedge\partial_z\mapsto g.$$

With these isomorphisms and considering the following sequence

$$d_{\pi_0}:0\xrightarrow{0}\chi^0(A_0)\xrightarrow{\delta^0_{\pi_0}}\chi^1(A_0)\xrightarrow{\delta^1_{\pi_0}}\chi^2(A_0)\xrightarrow{\delta^k_{\pi_0}}0. \quad (11)$$

where $\delta^k_{\pi_0}=0$ $(k=0,1)$ are associated differentials defined by:

$$\delta^0_{\pi_0}(f)=\begin{pmatrix}-y^n\partial_2 f+\frac{1}{n}x\partial_3 f\\ y^n\partial_1 f-\frac{1}{n}y\partial_3 f\\ -\frac{1}{n}x\partial_1 f+\frac{1}{n}y\partial_2 f\end{pmatrix}$$

for all f in A_0 , $\delta^1_{\pi_0}(f_1,f_2,f_3)=af_1+bf_2+cf_3$ for all f_1,f_2,f_3 in $\chi^1(A_0)$ where

$$a=(-y^n+\frac{1}{n}x)\partial_x-\frac{1}{n}y\partial_y+\frac{1}{n}y\partial_z-\frac{1}{n},$$

$$b=\frac{1}{n}x\partial_x+(-y^n-\frac{1}{n}y)\partial_y+\frac{1}{n}x\partial_z+ny^{n-1}+\frac{1}{n},$$

$$c=y^n\partial_x-y^n\partial_y+(\frac{1}{n}x-\frac{1}{n}y)\partial_z.$$

$\delta^k_{\pi_0}=0$ where $k\geq 2$ be an integer.

By a simple computation, we deduce that:

Proposition 2.5. d_{π_0} is non-vanishing.

We call d_{π_0} is a quasi-Poisson cohomology complex of $(A,\cdot,\{\cdot,\cdot\})$. In the following section, we study conditions for which d_{π_0} becomes a differential and we construct a sub-algebra of $(A_0,\cdot,\{\cdot,\cdot\}_0)$ where the Poisson cohomology of $(A,\cdot,\{\cdot,\cdot\})$ is isomorphic to the Poisson cohomology of $(A_0,\cdot,\{\cdot,\cdot\}_0)$.

3. An Isomorphism between the Quasi-Poisson Cohomology and Poisson Cohomology Associated to the Jacobiator

In this section, we establish an isomorphism between Poisson cohomology groups for the quasi-Poisson algebra $(A_0,\cdot,\{\cdot,\cdot\}_0)$ and Poisson cohomology of $(A,\cdot,\{\cdot,\cdot\})$.

Recall the quasi-differential d_{π_0} defined in (11) as

$$d_{\pi_0}:0\xrightarrow{0}\chi^0(A_0)\xrightarrow{\delta^0_{\pi_0}}\chi^1(A_0)\xrightarrow{\delta^1_{\pi_0}}\chi^2(A_0)\xrightarrow{\delta^k_{\pi_0}}0.$$

Let $f,f_1,f_2,f_3\in\chi^0(A_0):=A.$ We have:

$$\delta^0_{\pi_0}(f)=\begin{pmatrix}-y^n\partial_2 f+\frac{1}{n}x\partial_3 f\\ y^n\partial_1 f-\frac{1}{n}y\partial_3 f\\ -\frac{1}{n}x\partial_1 f+\frac{1}{n}y\partial_2 f\end{pmatrix},$$

$\delta^1_{\pi_0}(f_1,f_2,f_3)=a_1f_1+a_2f_2+a_3f_3$ with

$$a_1=(-y^n+\frac{1}{n}x)\partial_1-\frac{1}{n}y\partial_2+\frac{1}{n}y\partial_3-\frac{1}{n},$$

$$a_2=\frac{1}{n}x\partial_1+(-y^n-\frac{1}{n}y)\partial_2+\frac{1}{n}x\partial_3+ny^{n-1}+\frac{1}{n}$$

and $a_3=y^n\partial_1-y^n\partial_2+(\frac{1}{n}x-\frac{1}{n}y)\partial_3.$

We proved that $\delta^1_{\pi_0}\circ\delta^0_{\pi_0}\neq 0$ i.e. d_{π_0} is a quasi-differential.

Let's consider the following partial differential equations:

$$(E_0):\partial_x f-\partial_y f+\partial_z f=-x\partial_x f+y\partial_y f=x\partial_z f=y\partial_z f=0.$$

Let $\widetilde{A_0}=\left(A_0\big/\mathbb{C}[z]\right)\cap\widetilde{E_0}$ where $\widetilde{E_0}$ denotes the solution set of the system of partial differential equations (E_0).

Therefore,

$$\widetilde{A_0}=\left(x\mathbb{C}[x]\oplus y\mathbb{C}[y]\oplus xy\mathbb{C}[x,y]\right)\cap\widetilde{E_0}.$$

Let's consider the isomorphisms

$$\chi^1(A_0)\xrightarrow{\cong} A_0\times A_0\times A_0,\ f_1\partial_x+f_2\partial_y+f_3\partial_z\mapsto(f_1,f_2,f_3),$$

$$\chi^1(A_0)\times\chi^1(A_0)\times\chi^1(A_0)\xrightarrow{\cong} A_0,\ g\partial_x\wedge\partial_y\wedge\partial_z\mapsto g.$$

Considering the following sequence

$$\widetilde{d_{\pi_0}}:0\xrightarrow{0}\chi^0(\widetilde{A_0})\xrightarrow{\widetilde{\delta^0_{\pi_0}}}\chi^1(\widetilde{A_0})\xrightarrow{\widetilde{\delta^1_{\pi_0}}}\chi^2(\widetilde{A_0})\xrightarrow{\widetilde{\delta^k_{\pi_0}}}0$$

with $\widetilde{\delta^0_{\pi_0}}(f)=(y^n\partial_2 f,-y^n\partial_1 f,0)$ for all f belongs to $\widetilde{A_0}$, $\widetilde{\delta^1_{\pi_0}}(f_1,f_2,f_3)=uf_1+vf_2$ for all f_1,f_2,f_3 in $\chi^1(\widetilde{A_0})$

where $u=(y^n+\frac{1}{n}x)\partial_x+\frac{1}{n}y\partial_y-\frac{1}{n}$,

$$v=\frac{1}{n}x\partial_x+(y^n+\frac{1}{n}y)\partial_y-ny^{n-1}-\frac{1}{n}.$$

$\widetilde{\delta^k_{\pi_0}}=0$ where $k\geq 2$ be an integer.

By a simple computation, we obtain the following result.

Proposition 3.1.

$\widetilde{d_{\pi_0}}$ is a differential and $H^*_{\pi_0}(\widetilde{A_0})\cong H^*_\pi(A)$.

Since $(\widetilde{A_0},\cdot,\{\cdot,\cdot\}_0)$ is a sub-algebra of $(A_0,\cdot,\{\cdot,\cdot\}_0)$, the proposition 3.1 completes the proof of the following main result.

Theorem. Let $n\geq 1$ be an integer, $M=\mathbb{C}[x,y,z]$, $M_0=\mathbb{C}[x,y,z]/\langle y=0\rangle$ and let $(A_0,\cdot,\{\cdot,\cdot\}_0)$ be the quasi-Poisson algebra defined on $A_0=C^\infty(M_0)$ with quasi-Poisson structure $\{\cdot,\cdot\}_0$ associated to the bivector field π_0 defined by

$$\pi_0=-y^n\partial_x\wedge\partial_y+\frac{1}{n}x\partial_x\wedge\partial_z-\frac{1}{n}y\partial_y\wedge\partial_z.$$

- The Jacobiator of $\{\cdot,\cdot\}_0$ is equivalent to the Poisson structure $\{\cdot,\cdot\}:(x,y)\mapsto\{x,y\}=y^n$.

- As graded-$\mathbb{C}$ vector spaces, the Poisson cohomology for the quasi-Poisson algebra $(\widetilde{A_0},\cdot,\{\cdot,\cdot\}_0)$ is isomorphic to Poisson cohomology for the Poisson algebra $(A,\cdot,\{\cdot,\cdot\})$ according to the differential $\widetilde{d_{\pi_0}}$.

This means that there exists an isomorphism between Poisson cohomology groups for a quasi-Poisson algebra and Poisson cohomology groups for Poisson algebra coming from the Jacobiator of the quasi-Poisson algebra. An extension of the work in this context is to explicitly calculate the Poisson cohomology of $(A_0,\cdot,\{\cdot,\cdot\}_0)$.

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