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A Note on Admissible Monomials of Degree $2^{\lambda}-1$

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Abstract Let $\mathbf{P}(n)=\mathbb{F}_2[x_1,\ldots,x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well known rules. A major problem in algebraic topology is that of determining $\mathcal{A}^+\mathbf{P}(n)$, the image of the action of the positively graded part of A. We are interested in the related problem of determining a basis for the quotient vector space $\mathbf{Q}(n)=\mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$. Both $\mathbf{P}(n)=\bigoplus_{d\geq 0}\mathbf{P}^d(n)$ and $\mathbf{Q}(n)$ are graded, where $\mathbf{P}^d(n)$ denotes the set of homogeneous polynomials of degree d. In this note we show that the monomial $a_n=x_1x_2^2x_3^4\ldots x_j^{2^{j-1}}\ldots x_n^{2^{n-1}}\in\mathbf{P}^{2^n-1}(n)$ is the only one among all its permutation representatives that is admissible, (that is, a_n meets a criterion to be in a certain basis for $\mathbf{Q}(n)$). We show further that if $b_m=x_1x_2^2x_3^4\ldots x_j^{2^{j-1}}\ldots x_m^{2^{m-1}}\in\mathbf{P}^{2^m-1}(m)$ with $m\geq n$, then there are exactly $\binom{m+n}{m}\left(1-\frac{n}{m+1}\right)$ permutation representatives of the product monomial a_nb_m that are admissible.

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1. Introduction

Let $\mathbf{P}(n)=\mathbb{F}_2[x_1,x_2,\ldots,x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ by the formula

$$Sq^i(x_j)=\begin{cases}x_j, & i=0\\ x_j^2, & i=1\\ 0, & \text{otherwise}\end{cases}$$

and subject to the Cartan formula

$$Sq^k(uv)=\sum_{i=0}^{k}Sq^i(u)Sq^{k-i}(v)$$

for $u,v\in\mathbf{P}(n)$.

A polynomial $f\in\mathbf{P}(n)$ is in the image of the action of the Steenrod algebra if

$$f=\sum_{i>0}Sq^i(f_i)$$

for some polynomials $f_i\in\mathbf{P}(n)$. That means f belongs to $\mathcal{A}^+\mathbf{P}(n)$, the subspace of all hit polynomials. The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors. Our work is motivated by the related problem of finding a basis for the quotient vector space

$$\mathbf{Q}(n)=\mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n).$$

We define what it means for a monomial $a=x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}\in\mathbf{P}(n)$ to be admissible. Write $a_i=\sum_{j\geq 0}\alpha_j(a_i)2^j$ for the binary expansion of each exponent a_i. We then associate with a, two sequences,

$$\omega(a)=(\omega_0(a),\omega_1(a),\ldots,\omega_j(a),\ldots),$$
$$\sigma(a)=(a_1,a_2,\ldots,a_n),$$

where $\omega_j(a)=\sum_{1\leq i\leq n}\alpha_j(a_i)$ for each $j\geq 0$. $\omega(a)$ is called the **weight vector** of the monomial a and $\sigma(a)$ is called the **exponent vector** of the monomial a.

Given two sequences $p=(u_0,u_1,\ldots,u_l,0,\ldots)$, $q=(v_0,v_1,\ldots,v_l,0,\ldots)$, we say $p<q$ if there is a positive integer k such that $u_i=v_i$ for all $i<k$ and $u_k<v_k$. We are now in a position to define an order relation on monomials.

Definition 1.1 Let a, b be monomials in $\mathbf{P}(n)$. We say that $a<b$ if and only if one of the following holds:

1. $\omega(a)<\omega(b)$,
2. $\omega(a)=\omega(b)$ and $\sigma(a)<\sigma(b)$.

The order on the set of sequences of nonnegative integers is the lexicographical one.

Following Kameko [1], we define:

Definition 1.2 A monomial a is said to be **inadmissible** if there exists monomials $y_1,y_2,\ldots,y_k$ such that

$$a\equiv y_1+y_2+\ldots+y_k \bmod \mathcal{A}^+\mathbf{P}(n) \text{ and } y_j<x,\ j=1,2,\ldots,k$$

a is said to be **admissible** if it is not inadmissible. The set of all the classes represented by the admissible monomials in $\mathbf{P}(n)$ is a basis for $\mathbf{Q}(n)$.

Determination of all admissible monomials therefore provides a general approach in solving the problem of finding a basis for the quotient vector space $\mathbf{Q}(n)$. This approach was used by Kameko [1] in solving the problem in the case $n=3$ and by Sum [4,5] in the case $n=4$. The problem is unknown in general. Our aim is to show that a certain class of monomials is admissible and to obtain a procedure for counting admissible permutation representatives. Both $\mathbf{P}(n)$ and $\mathbf{Q}(n)$ are graded by degree and we shall denoted by $\mathbf{P}^d(n)$, the subspace of $\mathbf{P}(n)$ consisting of all the homogeneous polynomials of degree *d*.

We shall require the following result.

Theorem 1.3 (Kameko [1]; Sum [4]). *Let x, w be monomials in* $\mathbf{P}(n)$ *such that* $\omega_i(x)=0$ *for* $i>r>0$. *If* w *is inadmissible, then* xw^{2^r} *is also inadmissible*.

For any $I=(i_0,i_1,\ldots,i_r)$, $0<i_0<i_1<\ldots<i_r\le n$, $0\le r<n$, we define the homomorphism $p_I:\mathbf{P}(n)\to\mathbf{P}(n-1)$ of algebras by substituting

$$p_I\left(x_j\right)=\begin{cases} x_j, & \text{if } 1\le j<i_0,\\ \sum\limits_{0\le s\le r} x_{i_s-1}, & \text{if } j=i_0,\\ x_{j-1}, & \text{if } i_0<j\le n.\end{cases}$$

Then p_I is a homomorphism of $\mathcal{A}$-modules. In particular, for $I=(i)$, we have $p_i(x_i)=0$.

2. Main Results

The following result identifies a class of admissible monomials in $\mathbf{P}^{2^\lambda-1}(n)$, $1\le n\le 2^\lambda-1$.

Proposition 2.1 *Let* $\lambda\ge1$ *and for each integer n with,* $1\le n\le 2^\lambda-1$, *let* $a_n=x_1^{m_1}\ldots x_n^{m_n}\in\mathbf{P}^{2^\lambda-1}(n)$ *be a monomial for which* $m_1m_2\ldots m_n\ne0$. *If* a_n *is the monomial of least order in* $\mathbf{P}^{2^\lambda-1}(n)$, *then* a_n *admissible.*

Proof. Our proof of the proposition makes use of the fact that the homomorphism $p:\mathbf{P}(n)\to\mathbf{P}(n-1)$ of algebras defined by substituting

$$p\left(x_j\right)=\begin{cases} x_j, & \text{if } 1\le j<n,\\ x_{j-1}, & \text{if } j=n\end{cases}$$

is a homomorphism of $\mathcal{A}$-modules.

If $n=2^\lambda-1$, then $a_n=x_1x_2\ldots x_n$ is a spike, hence is admissible. Starting with $n=2^\lambda-1$ we form a sequence of monomials $a_{n-1},a_{n-2},a_{n-3},\ldots$ in $\mathbf{P}^{2^\lambda-1}(n-1)$, $\mathbf{P}^{2^\lambda-1}(n-2)$, $\mathbf{P}^{2^\lambda-1}(n-3),\ldots$ respectively as follows. Setting $x_n=x_{n-1}$ in a_n, we obtain the monomial $a_{n-1}=x_1x_2\ldots x_{n-1}^2$ of least order in $\mathbf{P}^{2^\lambda-1}(n-1)$. Setting $x_{n-3}=x_{n-2}$ in a_{n-1}, we obtain the monomial $a_{n-2}=x_1x_2\ldots x_{n-3}^2x_{n-2}^2$ of least order in $\mathbf{P}^{2^\lambda-1}(n-2)$. Setting $x_{n-5}=x_{n-4}$ in a_{n-2}, we obtain the monomial $a_{n-3}=x_1x_2\ldots x_{n-5}^2x_{n-4}^2x_{n-3}^2$ of least order in $\mathbf{P}^{2^\lambda-1}(n-3)$. Proceeding in this manner we eventually obtain the monomial of least order

$$a_{n'}=x_1(x_2x_3\ldots x_{n'-1}x_{n'})^2\in\mathbf{P}^{2^\lambda-1}(n')$$

where $n'=2^{\lambda-1}$.

Starting with $n'=2^{\lambda-1}$ we continue our sequence of monomials of least order by forming $a_{n'-i}\in\mathbf{P}^{2^\lambda-1}(n'-i)$ as follows. Setting $x_{n'}=x_{n'-1}$ in $a_{n'}$, we obtain the monomial $a_{n-1}=x_1(x_2\ldots x_{n'-2})^2x_{n'-1}^4$ of least order in $\mathbf{P}^{2^\lambda-1}(n'-1)$. By setting $x_{n'-3}=x_{n'-2}$ in $a_{n'-1}$ we obtain $a_{n'-2}$, then $x_{n'-5}=x_{n'-4}$ in $a_{n'-2}$ and so on we obtain a sequence of monomials of least order $a_{n'-i}\in\mathbf{P}^{2^\lambda-1}(n'-i)$. Eventually we obtain the monomial of least order

$$a_{n''}=x_1x_2^2(x_3x_4\ldots x_{n''-1}x_{n''})^4\in\mathbf{P}^{2^\lambda-1}(n'').$$

where $n''=2^{\lambda-2}+1$.

Proceeding further in a similar manner we eventually obtain the monomial of least order

$$a_n=x_1x_2^2x_3^4\ldots x_j^{2^{j-1}}\ldots x_n^{2^{n-1}}\in\mathbf{P}^{2^n-1}(n).$$

With $\lambda=n$ we continue our sequence of monomials of least order by forming $a_{\lambda-i}\in\mathbf{P}^{2^\lambda-1}(\lambda-i)$ by setting $x_\lambda=x_{\lambda-1}$ in a_λ to obtain $a_{\lambda-1}$, then $x_{\lambda-3}=x_{\lambda-2}$ in

$a_{\lambda-1}$ to obtain $a_{\lambda-2}$ and so on. Proceeding in this manner we eventually obtain the monomial of least order

$$x_1 x_2^{2^{\lambda}-2} \in \mathbf{P}^{2^{\lambda}-1}(2)$$

which is known to be admissible. Thus all the monomials a_n, $1 \le n \le 2^{\lambda}-1$, are not in the image of the action of the Steenrod algebra. Since each a_n is a monomial of least order in $\mathbf{P}^{2^{\lambda}-1}(n)$ it follows than a_n is admissible for each n.

Finally we note if

$$a_n = x_1 x_2^2 x_3^4 \ldots x_j^{2^{j-1}} \ldots x_n^{2^{n-1}} \in \mathbf{P}^{2^n-1}(n),$$

then no other permutation representative of a_n is admissible. This is the case since for any pair of permutations $b, c \in \mathbf{P}^{2^n-1}(n)$ of a_n we have $b + c \in A^{+}\mathbf{P}(n)$.

In [2] we prove that:

Lemma 2.2 (Mothebe [2]). *If* $u = x_1^{m_1} \ldots x_k^{m_k} \in \mathbf{P}^d(k)$ *and* $v = x_1^{e_1} \ldots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ *are admissible monomials, then for each permutation* $\sigma \in S_{k+r}$ *for which* $\sigma(i) < \sigma(j)$, $i < j \le k$ *and* $\sigma(s) < \sigma(t)$, $k < s < t \le k+r$, *the monomial*

$$x_{\sigma(1)}^{m_1} \cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \cdots x_{\sigma(k+r)}^{e_r} \in \mathbf{P}^{d+d'}(k+r)$$

is admissible.

As a consequence of Proposition 2.1 and Lemma 2.2 we have:

Theorem 2.3 *For each pair of integers* $n, m \ge 1$ *the monomials* $a_n = x_1 x_2^2 x_3^4 \ldots x_j^{2^{j-1}} \ldots x_n^{2^{n-1}} \in \mathbf{P}^{2^n-1}(n)$ *and* $b_m = x_1 x_2^2 x_3^4 \ldots x_j^{2^{j-1}} \ldots x_m^{2^{m-1}} \in \mathbf{P}^{2^m-1}(m)$ *are admissible. Further if* $\sigma \in S_{n+m}$ *and* $d = 2^m - 1 + 2^n - 1$,

$$x_{\sigma(1)} x_{\sigma(2)}^2 \cdots x_{\sigma(n)}^{2^{n-1}} x_{\sigma(n+1)} x_{\sigma(n+2)}^2 \cdots x_{\sigma(n+m)}^{2^{m-1}} \in \mathbf{P}^d(n+m)$$

is admissible if and only if $\sigma(i) < \sigma(j)$, $i < j \le n$ *and* $\sigma(s) < \sigma(t)$, $n < s < t \le n+m$.

Proof. By the result of Lemma 2.2 if $\sigma \in S_{n+m}$ and $\sigma(i) < \sigma(j)$, $i < j \le n$ and $\sigma(s) < \sigma(t)$, $n < s < t \le n+m$, then the monomial

$$x_{\sigma(1)} x_{\sigma(2)}^2 \cdots x_{\sigma(n)}^{2^{n-1}} x_{\sigma(n+1)} x_{\sigma(n+2)}^2 \cdots x_{\sigma(n+m)}^{2^{m-1}}$$

is admissible.

Conversely we note that the result of the theorem is true if $n \le m \le 2$. Proceeding by induction on m suppose the result of the theorem is true for some integer $m-1 \ge 2$ and all $n \le m-1$. Let

$$a = x_{\sigma(1)} x_{\sigma(2)}^2 \cdots x_{\sigma(n)}^{2^{n-1}} x_{\sigma(n+1)} x_{\sigma(n+2)}^2 \cdots x_{\sigma(n+m)}^{2^{m-1}}$$

where $n \le m$ and the permutation $\sigma \in S_{n+m}$ does not satisfy the condition $\sigma(i) < \sigma(j)$, $i < j \le n$ and $\sigma(s) < \sigma(t)$, $n < s < t \le n+m$. If a is admissible, then $x_1 x_2 x_3^2 x_k^2$, $k > 3$ or $x_1 x_2^2 x_i x_j^2$, $2 < i < j$ is a factor of a if $n \ge 2$ or $x_1 x_2 x_3^2$, or $x_1 x_2^2 x_i$, $i > 2$ is a factor of a if $n = 1$. Then

$$a' = x_{\sigma(2)} x_{\sigma(3)}^2 \cdots x_{\sigma(n)}^{2^{n-2}} x_{\sigma(n+2)} x_{\sigma(n+3)}^2 \cdots x_{\sigma(n+m)}^{2^{m-2}} \in \mathbf{P}^{d'}(n+m-2)$$

where $d' = 2^{m-1} - 1 + 2^{n-1} - 1$. By the induction hypothesis a' is inadmissible so by Theorem 1.3 a is also inadmissible. This completes the proof of the theorem.

In the next section we obtain a procedure for counting the number of distinct monomials that can be obtained from a_n and b_m by means of permutations of their product as outlined in Theorem 2.3.

3. Order Preserving Permutations

Let m, n with $m \ge n$ be a pair of positive integers and let $A = \{1, 2, 3, \ldots, m\}$ and let $B = \{1, 2, 3, \ldots, n\}$ be ordered subsets of $\mathbf{N}$ consisting of the first m and n elements respectively. We shall say that a permutation $a_1, a_2, a_3, \ldots, a_{m+n}$, of the sequence

$$1, 2, 3, \ldots, m, 1, 2, 3, \ldots, n \tag{1}$$

is **order preserving** if for all $a_i, a_j \in A$, $i < j$ whenever $a_i < a_j$ and for all $a_r, a_s \in B$, $r < s$ whenever $a_r < a_s$. Let $\mathcal{C}$ denote the family of all distinct order preserving permutations of the sequence (1). We claim that:

Lemma 3.1

$$|\mathcal{C}| = \binom{m+n}{m}\left(1 - \frac{n}{m+1}\right).$$

Proof. Let m, n with $m \ge n$ be a pair of positive integers. We first note that

$$\binom{m+n}{m}\left(1 - \frac{n}{m+1}\right) = \binom{m+n}{m} - \binom{m+n}{m+1}.$$

We know that independent of duplicates there are $\binom{m+n}{m}$ order preserving permutations of the sequence (1). We therefore need only show that of these $\binom{m+n}{m}$ permutations there, altogether, are $\binom{m+n}{m+1}$ duplicate permutations.

We note further that $\binom{m+n}{m+1} = \sum_{i=1}^{n} \binom{m+n-i}{m}$. The statement of the lemma therefore holds for all $m \ge 1$ when $n = 1, 2$ since the set of all order preserving permutations of the sequence $1,1,2,3,\ldots,m$ has only one duplicate so that indeed

$$|\mathcal{C}|=\binom{m+1}{m}-\binom{m}{m} \text{ when } n=1.$$

If $n=2$, then we have:

$$|\mathcal{C}|=\binom{m+2}{m}-\binom{m+1}{m}-\binom{m}{m}$$

since every order preserving permutation $a_1,a_2,a_3,\ldots,a_{m+2}$ of the sequence $1,2,1,2,3,\ldots,m$ has $\binom{m+1}{m}=m+1$ duplicates if $a_1=a_2=1$, while it has $\binom{m}{m}=1$ duplicate if $a_1=1$ and $a_2=2$. Thus the formula is true for all $m\geq 1$ when $n=1,2$.

Proceeding by induction on n, assume $n>2$ and that the formula holds for all pairs of integers m, r whenever $r<n$ and $m\geq r$. We prove that the formula holds for the pairs of integers n, m with $m\geq n$.

We first prove the lemma in the case $m=n$. The statement of the lemma becomes

$$|\mathcal{C}|=\binom{2n}{n}-\sum_{i=1}^{n}\binom{2n-i}{n}$$

if $m=n$. In this case $A=B=\{1,2,3,\ldots,n\}$. For purpose of making distinction represent the integers in $j\in A$ by j_A and those in $j\in B$ by j_B. Let

$$a_1,a_2,a_3,\ldots,a_{2n-1},a_{2n}$$

be an order preserving permutation of the sequence

$$1,2,3,\ldots,n,1,2,3,\ldots,n \tag{2}$$

The number of order preserving permutations of (2) of the form

$$a_1,a_2,a_3,\ldots,a_{2n-2},a_{2n-1},n_A \tag{3}$$

is equal to $\binom{2n-1}{n}$ and, by induction, there altogether are

$$\sum_{i=1}^{n-1}\binom{2n-1-i}{n}=\binom{2n-1}{n+1}$$

duplicates. Add to these duplicates the duplicates to the sequences of the form $a_1,a_2,a_3,\ldots,a_{2n-2},a_{2n-1},n_B$ we have that the total number of duplicates is

$$\binom{2n-1}{n+1}+\binom{2n-1}{n}=\binom{2n}{n+1}=\sum_{i=1}^{n}\binom{2n-i}{n}$$

as required.

Finally we prove the formula is true in the cases $m>n$. Consider the set of all order preserving permutations of the sequence

$$1,2,3,\ldots,n,1,2,3,\ldots,m \tag{4}$$

The set splits into categories of sequences:

(1) $a_1,a_2,a_3,\ldots,a_{m+n-1},n$

(2) $a_1,a_2,a_3,\ldots,a_{m+n-2},n,m$

(3) $a_1,a_2,a_3,\ldots,a_{m+n-3},n,m-1,m$

(4) $a_1,a_2,a_3,\ldots,a_{m+n-4},n,(m-2),m-1,m$

⋮ ⋮ ⋮

$(m-n+1)$ $a_1,a_2,a_3,\ldots,a_{2n-1},n,n+1,\ldots,m-1,m$

for all $s, 1\leq s<m-n+1$.

By induction, the total number of duplicates in each such category is

$$\sum_{i=1}^{n-1}\binom{m+n-s-i}{m-s+1}=\binom{m+n-s}{m-s+2} \quad \text{for } 1\leq s<m-n+1$$

and

$$\sum_{i=1}^{n}\binom{2n-i}{n}=\binom{2n}{n+1} \quad \text{for } s=m-n+1.$$

Thus the total number of duplicates of the set of all order preserving permutations of the sequences of the form (4) is

$$\sum_{s=1}^{m-n}\binom{m+n-s}{m-s+2}+\binom{2n}{n+1}=\binom{m+n}{m+1}$$

as required.

4. Concluding Remarks

It follows from Lemma 3.1 that if $n\leq m$ and a_n, b_m are the monomials given in Theorem 2.3, then the number of permutation representatives of the monomial $a_n b_m$ that are admissible is $\binom{m+n}{m}\left(1-\frac{n}{m+1}\right)$.

The numbers $\binom{m+n}{m}\left(1-\frac{n}{m+1}\right)$ are known to form the Catalan triangle which appears in the OEIS as A009766.

The result of Lemma 3.1 has been shown by Mothebe and Phuc [3] to be closely related to the problem of determining the density of the prime numbers and twin primes in the sequence of natural numbers.

Statement of Competing Interests

The author has no competing interests.

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