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Optimal Portfolios of an Insurer and a Reinsurer with Proportional Reinsurance through Exponential Utility Maximization under Constant Elasticity of Variance Model

Silas A. Ihedioha*, Danat N. Tanko, Dominic P. Shie

Department of Mathematics, Plateau State University Bokkos, P.M.B 2012 Jos, Plateau state, Nigeria
*Corresponding author: silasihedioha@yahoo.com

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Abstract This work studied optimal portfolios of an insurer and a reinsurer under proportional reinsurance and exponential utility preference, aiming at obtaining the optimal strategies for both the insurer and the reinsurer and determined the condition that would warrant reinsurance according to the proportional reinsurance chosen by the insurer and accepted by the reinsurer. The insurer and the reinsurer invested in a market where the price processes of the risky asset adopted constant elasticity of variance (CEV) model and their surplus processes approximated by stochastic differential equations (SDEs). Hamilton-Jacobi-Bellman equations (HJB) were derived and closed form solutions obtained, giving the optimal values of the insurer's and the reinsurer's portfolio. Obtained also was the condition for proportional reinsurance.

Keywords: *exponential utility maximization, Hamilton-Jacobi-Bellman equations (HJB), insurer and reinsurer, optimal strategies, stochastic differential equations (SDEs)*

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1. Introduction

Insurance is one of the social sciences which essentially is designed for risk taking. This process of risk taking entails the pooling together of resources of many individuals.

People are exposed to infinite number of risks that may affect their persons or their properties and so take insurance cover transferring those risks to an insurer at a premium (fixed cost). Many authors have made contributions in this area. From Gu, [1], the contribution that investment and reinsurance are two important ways for insurers to balance their profit and risk, is made. He also opined that reinsurance is a transaction whereby one insurance company agrees to indemnify another insurance company against all or part of the loss that the latter sustains under a policy or policies that it has issued.

Many literatures have studied terminal wealth through utility maximization. Among these is the contribution by Brown [2]. The study gave explicit solution for a firm to maximize the exponential utility of terminal wealth and minimize the probability of ruin with its surplus process given by Lundberg risk model.

Hipp and Plum [3] contributed that in the area of different claim sizes of insurers and the ruin probability.

Wang et al. [4] found that the risk of insurance cannot be avoided by singularly investing in the bond and other assets the in market having applied martingale method to study the optimal portfolio selection for insurer using mean-variance criterion and expected constant absolute risk aversion (CARA) utility maximization.

Promislow and Young [5] found that the reinsurance comes up in different forms.

Bauerle [6] studied quota-share reinsurance on investment considering proportional reinsurance. He minimized the expected quadratic distance of the terminal value over a positive constant and successfully solved the related mean-variance problem.

Zhibin and Bayraktar [7] derived the explicit expressions for the optimal strategies and value function having considered the problem of optimal reinsurance - investment problem under constant elasticity of variance stock market for Jump- diffusion risk model.

Yang and Jiaqin [8] studied the optimal investment-consumption-insurance with random parameter. In their work, they discuss an optimal investment, consumption, and life insurance purchase problem for a wage earner in a complete market with Brownian information. They assumed that the parameter governing the market model and the wage earner, including the interest rate, appreciation rate, volatility, force of mortality, premium-insurance ratio, income and discount rate, were all random processes adapted to the Brownian motion filtration.

Deng et al [9] studied the optimal proportional reinsurance and investment for a constant elasticity of variance model under variance principle. They assumed that the insurer's surplus process followed a jump-diffusion process where the insurer could purchase proportional reinsurance from the reinsurer via the variance principle and invest in a risk-free asset and a risky asset whose price is modeled by a CEV model and obtained the techniques of stochastic control theory, closed-form expression, for the value functions and optimal strategies.

Osu et al. [10] studied on the survival of insurance company's investment with consumption under power and exponential utility functions and obtained the optimal strategies in which they discovered that both utility functions yielded results that are alike.

Jianwei [11] considered the optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model and derived the explicit solution for the power and exponential utility function in two different periods (before and after retirement).

Ihedioha and Osu [12] studied the optimal portfolio of an insurer and a reinsurer under proportional reinsurance and power utility preference in which the insurer's and the reinsurer's surplus processes were approximated by Brownian motion with drift.

Li et. al. [13] studied a time consistent reinsurance investment strategy for a mean-variance insurer under stochastic interest rate model and inflation risk. They derived the time-consistent reinsurance-investment strategies and corresponding value function for the mean-variance problem explicitly. They also studied the optimal investment problem for an insurer and a reinsurer under the proportional reinsurance model.

Hipp and Plum [3] studied optimal investment for insurer and proved the existence of a smooth solution. They applied verification theorem to obtain explicit solutions to some cases with exponential claim size distribution and numerical results in a case with Pareto claim size.

Zhou and Cai [14] made thier contribution in optimal dynamic risk control for insurers with state-dependent income, investigating the optimal forms of dynamic reinsurance policies among a class of general reinsurance strategies.

In this study, we consider the case of an insurer and a reinsurer to assess the impact of proportional reinsurance and determine the condition that would warrant reinsurance according to the optimal reinsurance proportion chosen by the insurer where the insurer's and the reinsurer's surplus processes were approximated by constant elasticity of variance (CEV) model, and the insurer could purchase proportional reinsurance from the reinsurer.

Ito's lemma will used to obtain the Hamilton-Jacob-Bellman (HJB) equations and there solutions give the optimized values of the insurer's and reinsurer's optimal investment in the risky asset and the value of the discount rate that would warrant reinsurance. The investment cases that will be considered are where the insurer and the reinsurer traded two assets; the riskless (bond) asset and the risky (stock) asset.

2. The Model Formulation the Model

The constant elasticity of variance (CEV) model is one-dimensional diffusion process that solves a stochastic differential equation (SDE). It is a natural extension of the geometric Brownian motion (GBM). The constant elasticity of variance model was originally proposed by Cox and Ross [15] as an alternative diffusion process for European option pricing. Compared to Geometric Brownian Motion (GBM), we see that the advantages of the constant elasticity of variance (CEV) model are that the volatility rate has correlation with risky asset price and can explain the empirical bias such as volatility smile.

The constant elasticity of variance (CEV) model is giving as;

$$dS(t) = S(t)\left[\mu dt + \beta S(t)^{\gamma}\, dZ^{(2)}(t)\right], \tag{1}$$

where

μ is a long term rate of return, γ is the elasticity parameter satisfying $\gamma > 0$, $\beta S^{\gamma}(t)$ is the volatility, and $Z^{(2)}(t)$ is a standard Brownian motion .

Remark: Note that when the elasticity parameterγ equals zero, the constant elasticity of variance (CEV) model reduces to Geometric Brownian motion.

2.1. The Model

Suppose the claim process $C(t)$ of an insurance company is described by;

$$dC(t) = adt - bdZ^{(1)} \tag{2}$$

where a and b are positive constant and $Z^{(1)}(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, f, (f_t);\ t > 0)$.

Assuming also that the premium rate is

$$c = (1+\theta)a \tag{3}$$

where $\theta > 0$ is the security risk premium.

Applying (3) in (2), the surplus process of the insurer is now given as

$$dR(t) = cdt - dC(t) = a\theta dt + bdZ^{(1)}(t). \tag{4}$$

We assume that the insurance company has the permission to purchase proportional reinsurance to reduce her risk and pays reinsurance premium continuously at the rate of $(1+\eta)ap(t)$ where $\eta > \theta > 0$ is the safety loading of the reinsurer.

The surplus of the insurance company is then given as

$$dR_I(t) = (\theta - \eta p(t))adt + b(1 - p(t))dZ^{(1)}(t), \tag{5}$$

and for the reinsurer

$$dR_R(t) = \eta p(t)adt + bp(t)dZ^{(1)}(t). \tag{6}$$

We also assume that both the insurer and the reinsurer invest their surpluses in the same market consisting of two assets; a risky asset (stock) and a riskless asset (bond). Let the prices of the risky and riskless asset be $S(t)$ and $B(t)$ respectively, then the price of the risky asset follows the constant elasticity of variance given as the stochastic differential equation

$$dS(t) = S(t)\left[\mu dt + \beta S^{\gamma}(t) dZ^{(2)}(t)\right], \quad \gamma > 0, 0 \le \beta \le 1. \tag{7}$$

where μ denotes the appreciation rate of the risky asset and the $\beta S^{\gamma}(t)$ its volatility. $Z^{(2)}(t)$ is another standard Brownian motion defined on a complete probability space. The dynamics of the price of the riskless asset is given by the equation

$$dB(t) = kB(t); B(0) = 1, \tag{8}$$

where k is a constant.

Let $W_I(t)$ and $W_R(t)$ be the total money amount the insurer and reinsurer have for investment and their investments in the risky are $\pi_I(t)$ and $\pi_R(t)$ respectively, then their investments on the riskless assets are $[W_I(t) - \pi_I(t)]$ and $[W_R(t) - \pi_I(t)]$ respectively.

Corresponding to the policy π, the admissible strategies $[\,p(t)\,, \pi_I(t)]$ and $[\,p(t), \pi_R(t)\,]$, the wealth processes of the insurer and the reinsurer evolve according to the stochastic differential equations (SDEs);

$$dW_I^{\pi} = \pi_I(t)\frac{dS(t)}{S(t)} + \left[W_I(t) - \pi_I(t)\right]\frac{dB(t)}{B(t)} + dR_I(t), \tag{9}$$

for the insurer, and

$$dW_R^{\pi}(t) = \pi_R(t)\frac{dS(t)}{S(t)} + \left[W_R(t) - \pi_R(t)\right]\frac{d\mathrm{B}(t)}{\mathrm{B}(t)} + dR_R(t), \tag{10}$$

for the reinsurer.

Substituting the expression for $\frac{dS(t)}{S(t)}$, $\frac{d\mathrm{B}(t)}{\mathrm{B}(t)}$, $dR_I(t)$, and $dR_R(t)$ making use of equations (7), (8), (5), and (6) respectively in equations (9) and (10), we obtain,

$$\begin{aligned} dW_I^{\pi}(t) &= \pi_I(t)\left[\mu dt + \beta S^{\gamma}(t) dZ^{(2)}(t)\right] \\ &+ \left[W_I(t) - \pi_I(t)\right]kdt \\ &+ \left(\theta - \eta p(t)\right)adt + b(1 - p(t)dZ^{(1)}(t) \end{aligned} \tag{11}$$

for the insurer and

$$\begin{aligned} dW_R^{\pi}(t) &= \pi_R(t)\left[\mu dt + \beta S^{\gamma}(t) dZ^{(2)}(t)\right] \\ &+ \left[W_R(t) - \pi_R(t)\right]kdt + \eta p(t)adt + bp(t)dZ^{(1)}(t), \end{aligned} \tag{12}$$

for the reinsurer.

Employing the fact that

$$\left.\begin{aligned} (dt)^2 &= (dt)(dZ^{(1)}(t) = (dt)(dZ^{(2)}(t) = 0, \\ \left[dZ^{(1)}(t)\right]^2 &= \left[dZ^{(2)}(t)\right]^2 = dt, \\ \left[dZ^{(1)}(t)\right]&\left[dZ^{(2)}(t)\right] = \rho dt \end{aligned}\right\}, \tag{13}$$

the quadratic variation of the wealth process of the insurer is

$$\left(dW_I^{\pi}(t)\right)^2 = \begin{bmatrix} \beta^2 \pi_I^2(t) S^{2\gamma}(t) + 2\beta b\left(1 - p(t)\right) S^{\gamma}(t) \pi_I(t)\rho \\ + b^2\left(1 - p(t)\right)^2 \end{bmatrix}, \tag{14}$$

and that of the reinsurer given as

$$\left(dW_I^{\pi}(t)\right)^2 = \begin{bmatrix} \pi_R^2(t)\beta^2 S^{2\gamma}(t) \\ +2\rho\beta b S^{\gamma}(t)\pi_R(t)p(t) \\ +b^2 p^2(t) \end{bmatrix} dt \tag{15}$$

Suppose the insurer and the reinsurer both have exponential utility preferences given as

$$U(w) = \frac{-e^{-\phi w}}{\phi}, \ \ \phi > 0, \tag{16}$$

then the investors (the insurer and the reinsurer) problem can therefore be written as

$$V(T, w) = Max_{\pi} E^{(t,w)}\left[U\left(W^{\pi}\right)\right] \tag{17}$$

subject to;

$$\begin{aligned} dW_I^{\pi}(t) &= \pi_I(t)\left[\mu dt + \beta S^{\gamma}(t) dZ^{(2)}(t)\right] \\ &+ \left[W_I(t) - \pi_I(t)\right]kdt \\ &+ \left(\theta - \eta p(t)\right)adt + b(1 - p(t)dZ^{(1)}(t), \end{aligned}$$

in the case of the insurer, and

$$\begin{aligned} dW_R^{\pi}(t) &= \pi_R(t)\left[\varpi dt + \beta S^{\gamma}(t) dZ^{(2)}(t)\right] \\ &+ \left[W_R(t) - \pi_R(t)\right]kdt + \eta p(t)adt + bp(t)dZ^{(1)}(t) \end{aligned}$$

for the case of the reinsurer.

3. The Optimization Programme

The optimal strategies and the value functions of the insurer's and the reinsurer's portfolios are obtained in this section.

3.1. The Case of the Insurer

We derive the Hamilton-Jacobi-Bellman (HJB) partial differential equation starting with the Bellman equation

$$V(w, T) = Max_{\pi} E\left[V(w', T)\right] \tag{18}$$

where w', denotes the wealth of the insurer at time T and equation (18) can be written as:

$$Max_{\pi} E\left[V(w, t + \Delta t, T)\right] - V(w, t; T) = 0. \tag{19}$$

Dividing both side of the equation by Δt and taking the limit as Δt tends to zero, gives the Bellman equation

$$Max_{\pi} E\left(\frac{dV}{\Delta t}\right)=0. \tag{20}$$

Applying the Ito's lemma which state that

$$dV=\frac{\partial V}{\partial w}dt+\frac{\partial V}{\partial w}dw+\frac{1}{2}\frac{\partial^2}{\partial w^2}(dw)^2, \tag{21}$$

and substituting for $dW_I^{\pi}(t)$ and $< dW_I^{\pi}(t) >$ in (21) using (11) and (14) respectively yields the stochastic differential equation (S D E)

$$dV=\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial w}\begin{Bmatrix}\pi_I(t)\left[\mu dt+\beta S^{\gamma}(t)dZ^{(2)}(t)\right]\\+\left[W_I(t)-\pi_I(t)\right]kdt\\+\left(\theta-\eta p(t)\right)adt\\+b(1-p(t)dZ^{(1)}(t)\end{Bmatrix} \tag{22}$$
$$+\frac{1}{2}\frac{\partial^2 v}{\partial w^2}\begin{Bmatrix}\beta^2\pi_I^2(t)S^{2\gamma}(t)\\+2\beta b\left(1-p(t)\right)S^{\gamma}(t)\pi_I(t)\rho\\+b^2\left(1-p(t)\right)^2\end{Bmatrix}dt.$$

Applying (23) to the Bellman equation (20) and taking expectation, we get

$$\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial w}\begin{Bmatrix}\mu\pi_I(t)dt+\left[W_I(t)-\pi_I(t)\right]kdt\\+\left(\theta-\eta p(t)\right)adt\end{Bmatrix}$$
$$+\frac{1}{2}\frac{\partial^2 v}{\partial w^2}\begin{Bmatrix}\beta^2\pi_I^2(t)S^{2\gamma}(t)\\+2\beta b\left(1-p(t)\right)S^{\gamma}(t)\pi_I(t)\rho\\+b^2\left(1-p(t)\right)^2\end{Bmatrix}dt \tag{23}$$

where

$$E\left(dZ^{(1)}(t)\right)=E(dZ^{(2)}(t)=0. \tag{24}$$

Equation (23) is the required Hamilton-Jacobi-Bellman equation.

The homogeneity of the objective function, the restriction and the terminal condition, leads to the conjecture that the value function V must be linear to $\frac{-e^{-\phi w}}{\phi}$. Therefore let

$$V(w,t,T)=g(t,T)\left(\frac{-e^{-\phi w}}{\phi}\right), \tag{25}$$

be such a function such that at the terminal date T,

$$g(T,T)=1. \tag{26}$$

Then

$$\frac{\partial V}{\partial t}=\frac{-e^{-\phi w}}{\phi}g;' \frac{\partial V}{\partial w}=e^{-\phi W_I}g;\frac{\partial^2 V}{.\partial w^2}=-\phi e^{-\phi W_I}g. \tag{27}$$

Using (27) in (23), the HJB equation becomes

$$\frac{-e^{-\phi W_I}}{\phi}g'$$
$$+e^{-\phi W_I}g\begin{Bmatrix}\mu\pi_I(t)dt+\left[W_I(t)-\pi_I(t)\right]kdt\\+\left(\theta-\eta p(t)\right)a\end{Bmatrix} \tag{28}$$
$$-\frac{\phi e^{-\phi W_I}}{2}g\begin{Bmatrix}\beta^2\pi_I^2(t)S^{2\gamma}(t)\\+2\beta b\left(1-p(t)\right)S^{\gamma}(t)\pi_I(t)\rho\\+b^2\left(1-p(t)\right)^2\end{Bmatrix}=0.$$

To obtain the optimal investment strategy $\pi_I^*(t)$ of $\pi_I(t)$, we differentiate (28) with respect to $\pi_I(t)$ to obtain

$$(\mu-k)-\frac{\phi}{2}\begin{bmatrix}2\pi_I^2(t)\beta^2S^{2\gamma}(t)\\+2\beta\rho b\left(1-p(t)\right)S^{\gamma}(t)\pi_I\end{bmatrix}=0, \tag{29}$$

from which we get

$$\pi_I^*(t)=\frac{(\mu-k)}{\phi\beta^2S^{2\gamma}(t)}-\frac{b\rho(1-p(t))}{\beta S^{\gamma}(t)}. \tag{30}$$

This is the insurer's optimal investment in the risky asset (stock) that is horizon dependent.

Also differentiating equation (28) with respect to $p(t)$, we obtain

$$-a\eta+\varnothing b^2\left(1-p(t)\right)+\beta b\rho S^{\gamma}(t)\pi_I(t)=0, \tag{31}$$

and the optimal reinsured proportion

$$p^*(t)=1+\frac{\beta\rho S^{\gamma}(t)\pi_I(t)}{\varnothing b}-\frac{\eta a}{\phi b^2} \tag{32}$$

which is also dependent on time.

To obtain the solution of the HJB equation (28) we replace $\pi_I(t)$ and $p(t)$ with their corresponding optimal values $\pi_I^*(t)$ and $p^*(t)$ as in (30) and (32) respectively, to get,

$$\frac{-e^{-\phi W_I}}{\phi}g'+e^{-\phi W_I}g\begin{Bmatrix}\mu\pi_I^*(t)dt+\left[W_I(t)-\pi_I^*(t)\right]kdt\\+\left(\theta-\eta p^*(t)\right)a\end{Bmatrix}$$
$$-\frac{\phi e^{-\phi W_I}}{2}g\begin{Bmatrix}\beta^2\pi_I^{*2}(t)S^{2\gamma}(t)\\+2\beta b\left(1-p^*(t)\right)S^{\gamma}(t)\pi_I^*(t)\rho\\+b^2\left(1-p^*(t)\right)^2\end{Bmatrix}=0, \tag{33}$$

that simplifies to

$$\frac{-1}{\phi}g'+\begin{Bmatrix}\begin{Bmatrix}\mu\pi_I^*(t)dt+\left[W_I(t)-\pi_I^*(t)\right]kdt\\+\left(\theta-\eta p^*(t)\right)a\end{Bmatrix}\\-\frac{\phi}{2}\begin{Bmatrix}\beta^2\pi_I^{*2}(t)S^{2\gamma}(t)\\+2\beta b\left(1-(t)\right)S^{\gamma}(t)\pi_I^*(t)\rho\\+b^2\left(1-p^*(t)\right)^2\end{Bmatrix}\end{Bmatrix}g=0. \tag{34}$$

Equation (34) further simplifies to

$$g' + \xi(t,w) g = 0, \tag{35}$$

where

$$\xi(t) = -\varnothing \left\{ \begin{matrix} \left\{ \begin{matrix} \mu \pi_I^*(t) dt + \left[W_I(t) - \pi_I^*(t) \right] kdt \\ + \left(\theta - \eta p^*(t) \right) a \end{matrix} \right\} \\ -\frac{\phi}{2} \left\{ \begin{matrix} \beta^2 \pi_I^{*2}(t) S^{2\gamma}(t) \\ +2\beta b \left(1 - p^*(t)\right) S^\gamma(t) \pi_I^*(t) \rho \\ +b^2 \left(1 - p^*(t)\right)^2 \end{matrix} \right\} \end{matrix} \right\}. \tag{36}$$

Assuming t is the dominating variable, as implied in our choice of $g\ (t,T)$, equation (35) becomes

$$g' + \xi(t) g = 0. \tag{37}$$

From (40) we have

$$\int_t^T \frac{g'}{g} d\tau = h \int_t^T \xi(\tau) d\tau, \tag{38}$$

where

$$h = -1. \tag{39}$$

The evaluation (38) gives

$$g(t,T) = g(T,T) \exp[-h \int_t^T \phi(\tau) d\tau]. \tag{40}$$

Applying (26) in (40), we get

$$g(t,T) = \exp[-h \int_t^T \phi(\tau) d\tau]. \tag{41}$$

This implies that the horizon dependent solution to the insurance company's investment problem is

$$V(w,t;T) = \frac{-e^{-\phi W_I}}{\phi} \exp[-h \int_t^T \phi(\tau) d\tau]. \tag{42}$$

The optimal value of the insurer's portfolio through exponential utility maximization is as obtained in (42) above.

3.2. The Case of the Reinsurer

We substitute for $dW_R^\pi(t)\ and\ < dW_R^\pi(t) >$ in (21) to obtain

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial w} \left\{ \begin{matrix} \pi_R(t) \left[\mu dt + \beta S^\gamma(t) dZ^{(2)}(t) \right] \\ + \left[W_R(t) - \pi_R(t) \right] kdt \\ + \eta p(t) adt + bp(t) dZ^{(1)}(t) \end{matrix} \right\} \tag{43}$$

$$+ \frac{1}{2} \frac{\partial^2 V}{w^2} \left\{ \left[\begin{matrix} \pi_R^2(t) \beta^2 S^{2\gamma}(t) \\ +2\rho\beta b S^\gamma(t) \pi_R(t) p(t) \\ +b^2 p^2(t) \end{matrix} \right] dt \right\}.$$

Using (43) in the Bellman equation (20) and taking expectation we get

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial w} \left\{ \mu \pi_R(t) + \left[W_R(t) - \pi_R(t) \right] k + \eta p(t) a \right\}$$
$$+ \frac{1}{2} \frac{\partial^2 V}{w^2} \left\{ \left[\begin{matrix} \pi_R^2(t) \beta^2 S^{2\gamma}(t) \\ +2\beta b \rho S^\gamma(t) \pi_R(t) \\ +b^2 p^2(t) \end{matrix} \right] \right\} = 0. \tag{44}$$

Applying (25)-(27) to (44), we obtain the new HJB equation

$$\frac{-1}{\phi} g' + g \left\{ \mu \pi_R(t) + \left[W_R(t) - \pi_R(t) \right] k + \eta p(t) a \right\}$$
$$-\frac{\phi}{2} g \left\{ \left[\begin{matrix} \pi_R^2(t) \beta^2 S^{2\gamma}(t) \\ +2\rho\beta b S^\gamma(t) \pi_R(t) p(t) \\ +b^2 p^2(t) \end{matrix} \right] \right\} = 0. \tag{45}$$

The differentiation of (45) with respect to $\pi_R(t)$ gives

$$(\mu - k) - \phi \left[\begin{matrix} \pi_R(t)(t) \beta^2 S^{2\gamma}(t) \\ +\beta b \rho S^\gamma(t) p(t) \end{matrix} \right] = 0, \tag{46}$$

which when simplified gives the optimal investment value $\pi_R^*(t)$ of $\pi_R(t)$ as

$$\pi_R^*(t) = \frac{(\mu - k)}{\phi \beta^2 S^{2\gamma}(t)} - \frac{\rho b p(t)}{\beta S^\gamma(t)}. \tag{47}$$

This is the reinsurer's optimal investment strategy for investing in the risky asset.

Also, differentiating equation (45) with respect to $p\ (t)$ gives

$$g\eta a - \frac{\phi}{2} g \left[2\beta b \rho p^\gamma(t) \pi_R(t) + 2b^2 p(t) \right] = 0, \tag{48}$$

from which we obtain the optimal proportion reinsured as

$$p^*(t) = \frac{\eta a}{\phi b^2} - \frac{\rho \beta S(t) \pi_R(t)}{b}. \tag{49}$$

This is the optimal proportion the reinsurer accepted for reinsurance.

Replacing $\pi_R(t)$ and $p(t)$ in (45) with their corresponding optimal values $\pi_R^*(t)$ and p* (t) from (47) and (49) respectively, we obtain

$$\frac{-e^{-\phi W_R}}{\phi} g' + e^{-\phi W_I} g \left[\begin{matrix} kW_I(t) + (\mu - k) \pi_R^*(t) \\ +\eta p^*(t) a \end{matrix} \right]$$
$$+ \frac{-\phi e^{-\phi W_I}}{2} g \left[\begin{matrix} \pi^{*2}{}_R(t) \beta^2 S^{2\gamma}(t) \\ +2\beta b \rho S^\gamma(t) p^*(t) \pi_R(t) \\ +b^2 p^{*2}(t) \end{matrix} \right] = 0, \tag{50}$$

which gives

$$g' + \left\{ \begin{matrix} -\phi \left[kW_R(t) + (\mu - k) \pi_R^*(t) + \eta p^*(t) a \right] \\ + \frac{-\phi^2}{2} \left[\begin{matrix} \pi^{*2}{}_R(t) \beta^2 p^{2\gamma}(t) \\ +2\beta b \rho S^\gamma(t) p^*(t) \pi_R(t) \\ +b^2 p^{*2}(t) \end{matrix} \right] \end{matrix} \right\} g = 0. \tag{51}$$

Equation (51) then reduces to

$$\frac{g'}{g} = \xi(t) \quad (52)$$

where

$$\xi(t) = \begin{Bmatrix} -\phi\left[kW_R(t) + (\mu - k)\pi_R^*(t) + \eta p^*(t)a\right] \\ +\frac{-\phi^2}{2}\begin{bmatrix} \pi^{*2}{}_R(t)\beta^2 p^{2\gamma}(t) \\ +2\beta b\rho S^{\gamma}(t)p^*(t)\pi_R(t) \\ +b^2 p^{*2}(t) \end{bmatrix} \end{Bmatrix}. \quad (53)$$

From (52), we have

$$\int_t^T \frac{g'(\tau)}{g(\tau)} d\tau = \int_t^T \xi(\tau) d\tau. \quad (54)$$

The evaluation of (54) and application of the terminal condition (26) yields

$$g(t,T) = e^{-\int_t^T \xi(t)d\tau}. \quad (55)$$

Therefore that the horizon dependent solution to the reinsurer's investment problem is

$$V(w,t,T) = -\frac{e^{-\phi w}}{\phi} e^{-\int_t^T \phi(\tau)d\tau}, \phi \neq 0. \quad (56)$$

3.3. The Equality of the Insurer's and the Reinsurer's Strategies for the Reinsured Proportion

Here we find the condition under which the proportion reinsured by the insurer equals the amount accepted to be insured by the reinsurer.

Therefore, we equate the values of $p^*(t)$ in both cases and find the value of the discount factor ϕ.

$$p^*(t) = \frac{\eta a}{\phi b^2} + \frac{\beta\rho S^{\gamma}(t)\pi_I(t)}{b} = \frac{\eta a}{b^2} - \frac{\beta\rho S^{\gamma}(t)\pi_R(t)}{b}. \quad (57)$$

That is,

$$(\frac{1}{\phi} - 1)\frac{\eta a}{b^2} = -\frac{\beta\rho S^{\gamma}(t)}{b}[\pi_I(t) + \pi_R(t)], \quad (58)$$

and

$$\frac{1}{\phi} - 1 = -\frac{b\beta\rho S^{\gamma}(t)}{\eta a}\left[\pi_I(t) + \pi_R(t)\right],$$

from which,

$$\phi = \frac{\eta a}{\eta a - b\beta\rho S^{\gamma}(t)\left[\pi_I(t) + \pi_R(t)\right]}. \quad (59)$$

Clearly, the optimal policies that maximize the expected exponential utility and value functions for both the Insurer and the Reinsurer are horizon dependent.

3.4. Findings

1. The case of the insurer: we find out that the optimal policy to maximize the expected exponential utility at the terminal time T is to invest at each time,

$$\pi_I^*(t) = \frac{(\mu - k)}{\phi\beta^2 S^{2\gamma}(t)} - \frac{b\rho(1 - p(t))}{\beta S^{\gamma}(t)},$$

of the available wealth.

2. The case of the reinsurer: This investment strategy also shows that the optimal policy to maximize the expected exponential utility at the terminal time T is to invest at each time,

$$\pi_R^*(t) = \frac{(\mu - k)}{\phi\beta^2 S^{2\gamma}(t)} - \frac{\rho b p(t)}{\beta S^{\gamma}(t)}.$$

4. Conclusion

In this work, we consider the optimal investment problem for both an insurer and a reinsurer. In the model, the basic claim process is assumed to follow the constant elasticity of variance model and the insurer can purchase proportional reinsurance from the reinsurer. Both the insurer and the reinsurer are allowed to invest in a risk-free (bond) asset and a risky (stock) asset. Their aim is to maximize the expected exponential utility of terminal wealth.

Optimal strategies are obtained solving the corresponding HJB equations. In particular, the optimal investment strategies and the reinsured proportions for both the insurer and the reinsurer are horizon dependent. Obtained also is the condition under which proportional reinsurance is possible.

References

[1] Gu M., Yang Y., Li S., & Zhang J. (2010). Constant elasticity of variance model for proportional reinsurance and investment strategies. Insurance: Mathematics and Economics Vol. 46 (3) 580-587.

[2] Browne S. (1995). Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Mathematics of operations Research, Vol.20, No.4, 937-958.

[3] Hipp C. & Plum. M. (2000). Optimal investment for insurers, Insurance: Mathematics and Economics, Vol.27 No2, 215-228.

[4] Wang Z, Xia J. & Zhang L. (2007). Optimal investment for an insurer: The martingale approach. Insurance Mathematics and Economics, Vol 40 No 2, 322-334.

[5] Promislow S. D. & Young V. R. (2005). Minimizing the Probability of Ruin When Claims Follow Brownian Motion with Drift. North American Actuarial Journal 9(3) 110-128.

[6] B¨auerle N. (2005). Benchmark and mean-variance problems for insurers. Mathematical Methods of Operations Research, 62(1), 159-165.

[7] Zhibin L, & Bayraktar E. (2014). Optimal reinsurance and investment with un- observable claim size and intensity. Insurance Mathematics and Economics, 55, 156-166.

[8] Yang S. & Jiaqin W. (2014). Optimal investment-consumption-insurance with random parameters, Scandinavian Actuarial Journal, 2016:1, 37-62.

[9] Deng Y., Zhou J. & Huang Y. (2015). Optimal proportional reinsurance and investment for a constant elasticity of variance model under variance principle. Acta Mathematica Scientia Volume 35, Issue 2, March 2015, Pages 303-312.

[10] Osu, Bright O., Ihedioha, Silas A. ,& Adindu, J. I. (2014). On the survival of insurancecompany's investment with consumption under power and exponential utility functions: American journal of Applied Mathematics. Vol. 2, No. 1, pp. 8-13.

[11] Jianwei G. (2009).Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model. Insurance: Mathematics and Economics, Vol. 45,1, 9-18.

[12] Ihedioha S. A. & Osu, B. O. (2015) .Optimal Probability of Survival of an Insurer and a Reinsurer under Proportional Reinsurance and Power Utility Preference. International Journal of Innovation in Science and Mathematics Volume 3, Issue 6, ISSN (Online): 2347-9051.

[13] Li D., Rong X., & Zhao H. (2015). Time consistent reinsurance-investment strategy for a Mean-variance insurer under stochastic interest rate model and Inflation risk. surance: Mathematics and Economics 64, 28-44.

[14] Zhou M. & Jun C. (2014). Optimal dynamic risk control for insurers with state-dependent income. J. Appl. Probability. 51, no. 2, 417-435.

[15] Cox J C and Ross S A. (1976). The valuation of options for alternative stochastic processes. Journal of Financial Economics, 3(2):145-166.