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Asymptotic Property of Mediant for Lagrange's Mean Value Theorem

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Abstract In this paper, we mainly discuss the asymptotic property of mediant for m $(m \geq 1)$ order Lagrange 's mean value theorem, and obtain the general results for this problem: $\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{2}$, $\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{m}\sqrt[n-m]{\frac{\sum_{k=0}^{m-1}(-1)^k C_m^k (m-k)^n}{m!C_n^m}}$.

Keywords: *Lagrange's mean value theorem, mediant, asymptotic property*

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1. Introduction

Differential mean value theorem is an important theorem in advanced mathematics, over the years, there are many papers concerning the intermediate point of differential mean value theorem, and many related results have been obtained, such as the Lagrange mean value theorem in the literature, the authors studied the second-order, third-order and fourth-order gradual properties of middle value theorem in some cases, For example, for the fourth order mean value theorem, the authors got the following conclusions:

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{2},$$

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{4}\sqrt[n-4]{\frac{4^n-4\times 3^n+3\times 2^{n+1}-4}{n(n-1)(n-2)(n-3)}}.$$

In the literature [7], the authors studied the five order situation, and then further generalized the result of the above. Inspired from some research in the literature, and the results given in the similarity, we should naturally consider this question for any order of mean value theorem. In this article, we used the combination mathematics knowledge, we further study the asymptotic property of mediant for any order's mean value theorem, and obtain the general results for this problem, as a result, we generalize the corresponding results of other authors.

2. Some Lemms

In order to prove our main results, we first need the following two lemms.

Lemma 2.1. ([4]) Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and has a derivative of order n at the interior points of the interval $[a,x]$, then there exists a point $\varepsilon\in(a,x)$ such that

$$\sum_{k=0}^{n}(-1)^k C_n^k f\left(x-\frac{k(x-a)}{n}\right)=\left(\frac{x-a}{n}\right)^n f^{(n)}(\varepsilon). \quad (2.1)$$

Lemma 2.2. ([2]) For any $n\in\mathbb{N}$, we have:

$$\sum_{k=0}^{n}(-1)^k C_n^k (n-k)^r=\begin{cases}0 & 0\leq r<n\\ n! & r=n\\ \frac{n(n+1)}{2}n! & r=n+1\end{cases}$$

Corollary 2.3. For any $n\in\mathbb{N}$, we have:

$$\sum_{k=0}^{m-1}(-1)^k C_m^k\left(\frac{m-k}{m}\right)^m=\frac{m!}{m} \quad (2.2)$$

$$\sum_{k=0}^{m-1}\frac{(-1)^k C_m^k\left(\frac{m-k}{m}\right)^m}{\frac{1}{m-k}\cdot\frac{(m+1)!}{m^m}}=\frac{m}{2} \quad (2.3)$$

Proof. We can obtain these two inequalities by simple computation, here we omit the details.

3. Main Results

In this section, we will prove our main results in this paper.

Theorem 3.1. Let $f:[a,b]\to\mathbb{R}$ be continuous on interval $[a,b]$ and has a derivative of order m at the interior points of the interval (a,b). If f has a derivative of order $m+1$ at the point a and $f^{m+1}(\mathrm{a})\neq 0$, then for any $x\in(a,b)$, there exists a point $\varepsilon\in(a,x)$ satisfies (2.1) and

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{2}.$$

Proof. For any $x\in(a,b)$, since $f(t)$ is continuous on $[a,b]$ and has a derivative of order m at the interior points of the interval (a,b), then from (1.1):

$$\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k\, f\Big(x-\frac{k(x-a)}{m}\Big) = \left(\frac{x-a}{m}\right)^m f^{(m)}(\varepsilon),\ \varepsilon\in(a,x). \tag{3.1}$$

Now, construct $F(x)$ such that:

$$F(x) = \frac{\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k\, f\Big(x-\frac{k(x-a)}{m}\Big)-\left(\frac{x-a}{m}\right)^m f^{(m)}(a)}{\left(\frac{x-a}{m}\right)^{m+1}} \tag{3.2}$$

where $x\in(a,b)$. Using (3.1), we can obtain

$$F(x)=\frac{(\frac{x-a}{m})^m f^{(m)}(\varepsilon)-(\frac{x-a}{m})^m f^{(m)}(a)}{(\frac{x-a}{m})^{m+1}} = \frac{f^{(m)}(\varepsilon)-f^{(m)}(a)}{(\frac{x-a}{m})} = m\cdot\frac{f^{(m)}(\varepsilon)-f^{(m)}(a)}{\varepsilon-a}\cdot\frac{\varepsilon-a}{x-a}.$$

Now, let $x\to a$, we have

$$\lim_{x\to a}F(x)=m\cdot f^{(m+1)}(a)\cdot\lim_{x\to a}\frac{\varepsilon-a}{x-a}. \tag{3.3}$$

On the other hand, applying Hopital's rule to (3.2):

$$\lim_{x\to a}F(x) = \lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k\, f(x-\frac{k(x-a)}{m}) -(\frac{x-a}{m})^m f^{(m)}(a)\right)'}{\left((\frac{x-a}{m})^{m+1}\right)'}$$

$$== \lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k (\frac{m-k}{m}) f'(\frac{(m-k)}{m}x+\frac{ka}{m}) -(\frac{x-a}{m})^{m-1} f^{(m)}(a)\right)}{\frac{m+1}{m}\cdot(\frac{x-a}{m})^m}$$

Now, applying Hopital's rule to $F(x)$ for n times:

$$\lim_{x\to a}F(x) = \lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k (\frac{m-k}{m})^m f^{(m)}(\frac{(m-k)}{m}x+\frac{ka}{m}) -\frac{m!}{m^m}\cdot f^{(m)}(a)\right)}{\frac{(m+1)!}{m^m}\cdot(\frac{x-a}{m})}$$

$$= \lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k (\frac{m-k}{m})^m f^{(m)}(\frac{(m-k)}{m}x+\frac{ka}{m}) -\frac{m!}{m^m}\cdot f^{(m)}(a)\right)}{\frac{(m+1)!}{m^m}\cdot(\frac{x-a}{m})}$$

$$= \lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k (\frac{m-k}{m})^m f^{(m)}(\frac{(m-k)}{m}x+\frac{ka}{m}) -\frac{m!}{m^m}\cdot f^{(m)}(a)\right)}{\frac{(m+1)!}{m^m}\cdot(\frac{1}{m-k})\cdot[(\frac{(m-k)}{m}x+\frac{ka}{m})-a]}. \tag{3.4}$$

By using Corolly 2.3 and (3.4), we can botian the following

$$\lim_{x\to a}F(x)=\Big(\sum_{k=0}^{m-1}\frac{(-1)^k \mathrm{C}_m^k(\frac{m-k}{m})^m}{\frac{(m+1)!}{m^m}\cdot(\frac{1}{m-k})}\Big)f^{(m+1)}(a). \tag{3.5}$$

Using Corollary 2.3 again, then from (3.5),

$$\lim_{x\to a}F(x)=\frac{m}{2}\cdot f^{(m+1)}(a). \tag{3.6}$$

It follows from (3.6) and (3.6) that

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{2}.$$

This completes the proof.

Theorem 3.2. Let $f:[a,b]\to\mathbb{R}$ be continuous on interval $[a,b]$ and has a derivative of order n $(n\geq m+2)$ at the interior points of the interval (a,b). If $f^{(n)}(t)$ is continuous at the point a and $f'(\mathrm{a})=f''(\mathrm{a})=\cdots=f^{(n-1)}(\mathrm{a})=0$, $f^{m+1}(\mathrm{a})\neq 0$, then for any $x\in(a,b)$, there exists a point $\varepsilon\in(a,x)$ satisfies (2.1) and

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{m}\sqrt[n-m]{\frac{\sum_{k=0}^{m-1}(-1)^k \mathrm{C}_m^k(m-k)^n}{m!\mathrm{C}_n^m}}.$$

Proof. Construct $G(x)$ such that

$$G(x)=\frac{\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k\, f\left(x-\frac{k(x-a)}{m}\right)}{\left(\frac{x-a}{m}\right)^n},\ x\in(a,b). \tag{3.7}$$

Using (3.1), we can obtain

$$G(x)=\frac{\left(\frac{x-a}{m}\right)^{m} f^{(m)}(\varepsilon)}{\left(\frac{x-a}{m}\right)^{n}}, \varepsilon\in(a,x). \qquad (3.8)$$

Applying Taylor's formula to $f^{(m)}(\varepsilon)$, we have

$$f^{(m)}(\varepsilon),=\frac{f^{(n)}(\varepsilon_1)}{(n-m)!}(\varepsilon-a)^{n-m}, \varepsilon_1\in(a,\varepsilon). \qquad (3.9)$$

Now, applying (3.9) to (3.8),

$$G(x)=\frac{\frac{f^{(n)}(\varepsilon_1)(\varepsilon-a)^{n-m}}{(n-m)!}}{(\frac{x-a}{m})^{n}}$$

$$=m^{n-m}\cdot\frac{f^{(n)}(\varepsilon_1)}{(n-m)!}\cdot\left(\frac{\varepsilon-a}{x-a}\right)^{n-m}.$$

Let $x\to a$, we get

$$\lim_{x\to a}G(x)=\frac{m^{n-m}}{(n-m)!}\cdot f^{(n)}(a)\cdot\lim_{x\to a}\left(\frac{\varepsilon-a}{x-a}\right)^{n-m}. \qquad (3.10)$$

On the other hand, applying Hopital’s rule to (3.7) for n times:

$$\lim_{x\to a}G(x)=\lim_{x\to a}\frac{\left(\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k f\left(x-\frac{k(x-a)}{m}\right)\right)'}{\left(\left(\frac{x-a}{m}\right)^{n}\right)'}$$

$$=\lim_{x\to a}\frac{\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k \left(\frac{m-k}{m}\right)^{n} f^{(n)}\left(x-\frac{k(x-a)}{m}\right)}{\frac{1}{m^n}\cdot n!} \qquad (3.11)$$

$$=\frac{m^n\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k \left(\frac{m-k}{m}\right)^{n} f^{(n)}(a)}{n!}.$$

It follows from (3.5) and (3.6) that

$$\lim_{x\to a}\frac{\varepsilon-a}{x-a}=\frac{1}{m}\sqrt[n-m]{\frac{\sum_{k=0}^{m}(-1)^k \mathrm{C}_m^k (m-k)^n}{m!\mathrm{C}_n^m}}.$$

This completes the proof.

Remark 3.3. If we put $m=2,3,4,5$ in our Theorem 3.1 and Theorem 3.2, then we can some related results in the literature, thus, we obtain the general results for this problem.

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