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Family of Functional Inequalities for the Uniform Measure

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Abstract We consider on the interval $[-1,1]$ the heat semigroup $(P_t)_{t\geq 0}$ generated by the Legendre operator $L:=\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}$ acting on the Hilbert space $\mathbb{L}^2\left([-1,1],\mu\right)$ with respect to the uniform measure $\mu(dx):=\frac{dx}{2}$. By means of a simple method involving some semigroup techniques, we describe a large family of optimal integral inequalities with the Poincaré and logarithmic Sobolev inequalities as particular cases.

Keywords: *heat semigroup, legendre operator, spectral gap, poincaré inequality, sobolev inequality, logarithmic sobolev inequality, φ-entropy inequality*

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1. Introduction

Gross' logarithmic Sobolev inequality [2] states that for all smooth functions f on $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} f^2\log f^2 d\gamma_d-\left(\int_{\mathbb{R}^d} f^2 d\gamma_d\right)\log\left(\int_{\mathbb{R}^d} f d\gamma_d\right) \leq 2\left(\int_{\mathbb{R}^d}|\nabla f|^2 d\gamma_d\right), \quad (1)$$

white $d\gamma_d$ denote the normalized Gaussian measure on $\mathbb{R}^d : d\gamma_d(x)=\left(\sqrt{2\pi}^{-d}\right)\exp\left(-\|x\|^2/2\right)$. In this Gaussian context, the Poincaré inequality(spectral gap inequality) is given by:

$$\int_{\mathbb{R}^d} f^2 d\gamma_d-\left(\int_{\mathbb{R}^d} f d\gamma_d\right)^2 \leq \int_{\mathbb{R}^d}|\nabla f|^2 d\gamma_d. \quad (2)$$

In 1989, W.Bekner [2] derived a family of generalized Poincaré inequalities that yield a sharp interpolation between Poincaré inequality and logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^d} f^2 d\gamma_d-\int_{\mathbb{R}^d}\left(e^{tL} f\right)d\gamma_d \leq \left(1-e^{-2t}\right)\int_{\mathbb{R}^d}|\nabla f|^2 d\gamma_d,\ \textit{for all } t\geq 0, \quad (3)$$

where L is the Ornstein-Uhlenbeck operator: $L:=\Delta - x\nabla$. Recently, A.Bentaleb, S.Fahlaoui and A.Hafidi proposed in [[3], Section 2] a generalized of the inequality 3 and obtained the following inequality: for all smooth function f on $\mathbb{R}^d$,

$$\mathbb{E}nt_\tau^{\psi}(f):=\int_{\mathbb{R}^d}\psi(f)d\gamma_d-\int_{\mathbb{R}^d}\psi(P_\tau f)d\gamma_d \leq \frac{1-e^{-2t}}{2}\int_{\mathbb{R}^d}\psi''|\nabla f|^2 d\gamma_d$$

where $\psi\in C^\infty\left(\mathbb{R}^+\right)$ and ψ, $\frac{-1}{\psi''}$ are strictly convex, $\psi\left(0^+\right)=0$. Similar researches on this kind of inequalities for general probability measure generated by diffusion have been done by many authors (see, for instance, [4,10]).

The purpose of this paper is to present a family of integral inequalities on the interval [-1,1] which provide interpolation between the Sobolev and Poincaré inequalities (see Theorem 1 below).

2. Preliminaries

In order to keep the paper reasonably self-contained, we summarize in this section the basic notion that will be used in this work.

We consider the Legendre operator L on the interval $I:=[-1,1]$ defined by:

$$L:=\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx},\ (x\in I),$$

acting on acting on functions of class C^2. The Hilbert space $L^2(I,\mu)$, with respect to the probability measure $\mu(dx) := \frac{dx}{2}$. The space $L^2(I,\mu)$ admits orthogonal basis for the Legendre polynomials $(G_k)_{k\in\mathbb{N}}$ defined by the following generating series:

$$\frac{1}{\sqrt{\left(1-2xt+t^2\right)}} = \sum_{k=0}^{\infty} t^k G_k.$$

It is Known that the Legendre polynomials are eigenvectors for the operator $-L$:

$$-LG_k = k(1+k)G_k.$$

In fact, the distribution μ is symmetrizing for L and the sequence $\left(-k(k+1), Vect(G_k)_{k\in\mathbb{N}}\right)$ forms the spectral decomposition of the minimal self-adjoint extention of this operator on $L^2(I,\mu)$. With the help of an integration by parts, it is easily seen that

$$\begin{aligned}&\forall f,g\in C^2(I),\\ &\int (Lf)\,g\,d\mu = \int f\,(Lg)\,d\mu = -\int \Gamma(f,g)\,d\mu,\end{aligned} \tag{4}$$

where Γ is the positive symmetric bilinear form defined by:

$$\Gamma(f,g)(x) = \left(1-x^2\right) f'(x)\,g'(x).$$

An important consequence of property (4) is:

$$\forall f\in C^2(I), \int Lf\,d\mu = 0,$$

which expresses the invariance of the measure μ.

By means of the above mentioned properties of the operator L, essentially the one concerning the symmetry with respect to μ, we deduce the existence of a semigroup of operator $(P_t)_{t\geq 0}$ generated by L acting on $L^2(I,\mu)$ by:

$$P_t G_k = e^{-k(k+1)t} G_k,\ \forall k\in\mathbb{N}, \tag{5}$$

and such that:

(1) P_t is a contraction in all spaces $L^p(I,\mu)$ $(1\leq p\leq +\infty)$;

(2) P_t is symmetric $\int (P_t f)\,g\,d\mu = \int f\,(P_t g)\,d\mu$, $\forall f,g\in C^2(I)$;

(3) P_t is positive and $P_t 1 = 1$.

According to (5), $P_t = e^{tL}$ and P_t is ergodic:

$P_t f$ tends to $\int f\,d\mu$ μ-almost everywhere as $t\to +\infty$.

The commutation relation between the action of the operator L and the derivation is given as:

$$\frac{d}{dx}L = \tilde{L}\left(\frac{d}{dx}\right) - 2\frac{d}{dx},$$

where $\tilde{L}$ is the operator associated to the family of Jacobi polynomials of second kind:

$$\tilde{L} := \left(1-x^2\right)\frac{d^2}{dx^2} - 4x\frac{d}{dx}.$$

This commutation formula translates for the semigroup $(P_t)_{t\geq 0}$ by:

$$\frac{d}{dx}P_t = e^{-2t}\tilde{P}_t\left(\frac{d}{dx}\right), \tag{6}$$

where $\tilde{P}_t$ designates the heat semigroup generated by $\tilde{L}$. Notice that $\tilde{P}_t$ is symmetric (and so invariant) with respect to the probability measure $\tilde{\mu}(dx) := \frac{3}{4}\left(1-x^2\right)dx.$ The generator L satisfies the following dissipativity formula:

$$\int \left(-\tilde{L}f\right) g\,d\mu = \int \left(1-x^2\right) f'g'\,d\tilde{\mu}, \tag{7}$$

f, g being sufficiently smooth on I. We emphasize that $\tilde{L}$ may be obtained as the projection of the Laplacian on the unit sphere S^4 and $\tilde{\mu}$ is obtained as the projection of the normalized Lebesgue measure on S^4. For $p\in[1,+\infty[$, let $D_p(L)$ denote the domain of the generator L of $(P_t)_{t\geq 0}$. In virtue of density of $C^2(I)$ in $D_2(L)$, we may extend formula (4) to $D_2(L)$.

3. The Main Result

Our objective in this section is to establish a family of integral inequalities On $I=[-1,1]$ which provide interpolation between the Sobolev and Poincaré inequalities. For $p\in[1,+\infty[$, we adopt the notation

$$L_p^+(I,\mu) = \left\{ f\in L^p(I,\mu); \exists\varepsilon>0, f\geq\varepsilon \right\}.$$

Let $\varphi:\mathbb{R}^+\to\mathbb{R}$ be a strictly convex function such that $\varphi(0)=0$, we define the φ-entropy functional $\mathbb{E}nt_t^{\psi}$ of $f\in L_p^+(I,\mu)$ by:

$$\mathbb{E}nt_t^{\varphi}(f) = \int \varphi(f)\,d\mu - \int \varphi(P_t f)\,d\mu,\ p\in[0,+\infty].$$

The quantity $\mathbb{E}nt_t^{\psi}(f)$ is always nonnegative since P_t is invariant for the probability measure μ. By the ergodic property of the semigroup,

$$\mathbb{E}nt_\infty^{\varphi}(f) := \int \varphi(f)\,d\mu - \varphi\left(\int f\,d\mu\right).$$

When $\varphi(x)=x^2$, $\mathbb{E}nt_\infty^{\varphi}(f)$ coincides with the classical notion of variance,

$$\mathbb{E}nt_\infty^{\varphi}(f) := Var(f) = \int f^2\,d\mu - \left(\int f\,d\mu\right)^2.$$

When $\varphi(x) = x\log x$,

$$\mathbb{E}nt_{\infty}^{\varphi}(f) := \mathbb{E}nt(f) = \int f \log f d\mu - \int f d\mu \log\left(\int f d\mu\right).$$

In The sequel, we shall restrict ourself to the following class C of real functions $\varphi \in C^{\infty}\left(\mathbb{R}^{+}\right) : \varphi \in C$ mean that $\varphi(0) = 0,$ φ'' is strictly positive on $\mathbb{R}^{+}$ and

$$\frac{25}{16}(\varphi''')^2 \leq \varphi''\psi^{(IV)} \ on\ \mathbb{R}^{+}.$$

Having in our disposal enough machinery, we are now ready to prove the following estimate of φ -entropy functional $\mathbb{E}nt_t^{\varphi}$:

Theorem. 1. *Let* $\varphi \in C.$ *Then, for all function* $L_{+}^{\infty}(I,\mu) \cap D_2(L)$ *and* $t \in [0,+\infty],$

$$\mathbb{E}nt_t^{\varphi}(f) \leq \frac{1}{4}\left(1-e^{-4t}\right)\int \varphi''(f)\Gamma(f,f)d\mu. \qquad (8)$$

Moreover, the numeric constant at the right hand side of inequality (8) is best. To illustrate this theorem, let analyze some practical applications. The most important examples of the class C in our mind are:

$$\varphi_p = \frac{-x^{\frac{2}{p}} + x}{p-2} \ for \ p \in [1,34], \ p \neq 2$$

and

$$\varphi_2 = \frac{1}{2} x \log x,$$

which corresponds to the limiting case of φ_p as $p \to +\infty.$ If $\varphi = \varphi_p,$ inequality (8), written for $t = +\infty,$ describes the Sobolev inequality: for all $p \in [1,34](p \neq 2)$ and for all functions $f \in L_{\infty}^{+}(\mu,I) \cap D_2(L),$

$$\frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \leq \frac{1}{2}\int \Gamma(f,f)d\mu. \qquad (9)$$

For $\varphi = \varphi_2$ and $t = +\infty$ inequality (8) is exactly the Sobolev Logarithmic. Replacing f positive by f^2, we get

$$\mathbb{E}nt\left(f^2\right) \leq \frac{1}{2}\int \Gamma(f,f)d\mu, \qquad (10)$$

$$\forall f \in L_{\infty}^{+}(\mu,I) \cap D_2(L).$$

Taking into account that

$$\int \Gamma(|f|,|f|)d\mu \leq \int \Gamma(f,f)d\mu,$$

and using the fact that set of bounded functions in $C^2(I)$ is dense in $D_2(L),$ we can extend inequalities (9) and (10) to $D_2(L).$ this last inequality (10) is equivalent to the hypercontractive estimate for the semigroup $(P_t)_{t\geq 0}$:

Whenever $1 < p < q < +\infty$ and $t > 0$ satisfy $e^{4t} \geq \frac{q-1}{p-1},$ then , for all functions $f \in L^p(\mu,I),$

$$\|P_t f\|_q \leq \|f\|_p .$$

In other words, P_t maps $L^p(\mu,I)$ in $L^q(\mu,I)(q > p)$ with norm one.

Proof. By the Fubini theorem it follows from the definition of $\mathbb{E}nt_t^{\varphi}(f)$ that for any $t > 0,$

$$\begin{aligned}
\mathbb{E}nt_t^{\varphi}(f) &= -\int \varphi(P_t f) - \varphi(P_0 f), d\mu \\
&= \int_0^t \frac{d}{ds}\left[\int \varphi(P_s f) - \varphi(P_0 f), d\mu\right]ds \\
&= \int_0^t \left(\int -(LP_s f)\varphi'(P_s f)\, d\mu\right)ds \\
&= \int_0^t \left(\int \left(1-x^2\right)(P_s f)'^2 \varphi''(P_s f)\, d\mu\right)ds \\
&= \int_0^t e^{-4s}\left(\int \left(1-x^2\right)\left(\tilde{P}_s f'\right)^2 \varphi''(P_s f)\, d\mu\right)ds.
\end{aligned}$$

The last tow equalities follow from the dissipativity property (4), respectively. An integration by part over the time variables yields

$$\begin{aligned}
\mathbb{E}nt_t^{\varphi}(f) &= -\frac{1}{4}e^{-4t}\int \left(1-x^2\right)\left(\tilde{P}_t f'\right)^2 \varphi''(P_t f)\, d\mu \\
&+ \frac{1}{4}\int \left(1-x^2\right) f'^2 \varphi''(f)\, d\mu \\
&+ \frac{1}{4}\int_0^t e^{-4s}\frac{d}{ds}\left(\int \left(1-x^2\right)\left(\tilde{P}_s f'\right)^2 \varphi''(P_s f)\, d\mu\right)ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^t \frac{d}{ds}\left[\int \left(1-x^2\right)\left(\tilde{P}_s f'\right)^2 \varphi''(P_s f)\, d\mu\right]ds \\
&= \int \left(1-x^2\right)\left(\tilde{P}_t f'\right)^2 \varphi''(P_t f)\, d\mu \\
&- \int \left(1-x^2\right) f'^2 \varphi''(f)\, d\mu,
\end{aligned}$$

we get

$$\begin{aligned}
\mathbb{E}nt_t^{\varphi}(f) &= \frac{1}{4}\left(1-e^{-4t}\right)\int \left(1-x^2\right) f'^2 \varphi''(f) d\mu \\
&+ \frac{1}{4}\int_0^t \left(e^{-4s} - e^{-4t}\right)\frac{d}{ds}\left[\int \left(1-x^2\right)\left(\tilde{P}_s f'\right)^2 \varphi''(P_s f)\, d\mu\right]ds.
\end{aligned}$$

Now,

$$\begin{aligned}
&e^{-4s}\frac{d}{ds}\left[\int \left(1-x^2\right)\left(\tilde{P}_s f'\right)^2 \varphi''(P_s f)\, d\mu\right]ds \\
&= 2\int \left(1-x^2\right)\tilde{L}(P_s f)' \varphi''(P_s f)(P_s f)'\, d\mu \\
&+ \int \left(1-x^2\right) L(P_s f)' \varphi'''(P_s f)(P_s f)'^2\, d\mu.
\end{aligned}$$

Applying successively (4) and (7), the first integral in this sum is reduced to:

$$-2\int\left(1-x^2\right)^2\left(P_s f\right)''^2\varphi''\left(P_s f\right)d\mu$$
$$-2\int\left(1-x^2\right)^2\left(P_s f\right)''\left(P_s f\right)'^2\varphi'''\left(P_s f\right)\left(P_s f\right)'\,d\mu,$$

while the second integral is equal to:

$$-2\int\left(1-x^2\right)^2\left(P_s f\right)'^2\left(P_s f\right)''\varphi'''\left(P_s f\right)d\mu$$
$$-\int\left(1-x^2\right)^2\left(P_s f\right)'^4\varphi''''\left(P_s f\right)d\mu$$
$$+2\int x\left(1-x^2\right)\left(P_s f\right)'^3\varphi''\left(P_s f\right)d\mu.$$

Replacing x by $\frac{-\tilde{L}(x)}{4}$, and invoking again the dissipativity formula (7), the last member in the preceding sum becomes:

$$\frac{3}{2}\int\left(1-x^2\right)^2\left(P_s f\right)'^2\left(P_s f\right)''\varphi'''\left(P_s f\right)d\mu$$
$$+\frac{1}{2}\int\left(1-x^2\right)^2\left(P_s f\right)'^4\varphi''''\left(P_s f\right)d\mu.$$

As a consequence, after reassembling the terms, we find:

$$\frac{\mathbb{E}nt_t^{\varphi}(f)}{1-e^{-4t}}$$
$$=\frac{1}{4}\int\left(1-x^2\right)f'^2\varphi''(f)\,d\mu \tag{11}$$
$$-\frac{1}{4}\int_0^t\frac{1-e^{-4(t-s)}}{1-e^{-4t}}\times\left(\int\left(1-x^2\right)^2\xi(s,f,\varphi)d\mu\right)ds$$

with

$$\xi(s,f,\varphi)$$
$$=2f_s''^2\varphi''(f_s)+\frac{5}{2}f_s'^2f_s''\varphi'''(f_s)+\frac{1}{2}f_s'^4\varphi''''(f_s)$$
$$=2\left[f_s''\sqrt{\varphi''(f_s)}+\frac{5}{8}\frac{f_s'^2\varphi'''(f_s)}{\sqrt{\varphi''(f_s)}}\right]^2$$
$$+\frac{1}{2}\frac{f_s'^4}{\varphi''(f_s)}\left(\varphi''''(f_s)\varphi''(f_s)-\frac{25}{16}\left(\varphi'''(f_s)\right)^2\right)$$

where we have posed $f_s=P_s f$. The of $\xi(s,f,\varphi)$ then allows us to exhibit the desired inequality (8) from (11).

It remains to show that the numeric constant $\frac{1}{4}\left(1-e^{-4t}\right)$ at the right hand side of inequality (8) is optimal. As usual, let us consider $c\in\left]0,+\infty\right[$ such that $\varphi''(c)>0$. If f is replaced by $c+\varepsilon f$ in (8), and we pass to limit as ε tends to 0^+, we easily recover the Poincaré inequality with best constant:

$$\forall t\in[0,+\infty],\int f^2d\mu-\int\left(P_t f\right)^2d\mu$$
$$\le\frac{\left(1-e^{-4t}\right)}{2}\int\Gamma(f,f)d\mu,$$

which completes the proof.

We close this paper by the following concluding remarks:

Of course letting $t=+\infty$, inequality (8) in Theorem 1 gives rise to:

$$\mathbb{E}nt_\infty^{\varphi}(f)\le\frac{1}{4}\int\Gamma(f,f)d\mu. \tag{12}$$

Moreover, it's easy to observe that (8) provides a smooth nonincreasing interpolation for inequality (12):

$$\mathbb{E}nt_\infty^{\varphi}(f)\le\frac{\mathbb{E}nt_t^{\varphi}(f)}{1-e^{-4t}}\le\frac{1}{4}\int\Gamma(f,f)d\mu.$$

By (11), we point out at once that, if $\frac{25}{16}\left(\varphi'''\right)^2\le\varphi''\psi^{(IV)}$, the equality holds in (8) if and only if f is constant. In particular, inequalities (9) and (10) do not admit nonconstant extremal functions.

Corollary. 1. *Let* $\varphi\in C_\infty$. *Then for all nonnegative smooth function* $f\,[0,1]\to\left[0,+\infty\right[$,

$$\mathbb{E}nt_{\mu 1}^{\varphi}(f)=\int_0^1\varphi\left(f(x)\right)x-\varphi\left(\int_0^1 f(x)\,x\right)$$
$$\le\frac{1}{4}\int_0^1 x(1-x)\psi''(f)\,f'^2dx. \tag{13}$$

Moreover, this inequality is optimal.

Proof. We note in the sequel by μ the uniform measure on $[0,1]$.

Let $f\in\mathcal{C}^1([0,1])$ positive function. We consider the function g defined on the interval [−1, 1] by:

$$g(x)=f\left(\frac{x+1}{2}\right).$$

We can apply the theorem 1 to g for the measure μ, we obtain

$$\mathbb{E}nt_{\mu_1}^{\varphi}(f)=\mathbb{E}nt_{\mu}^{\varphi}(g)$$
$$\le\frac{1}{4}\int\left(1-x^2\right)\varphi''(g)\,g'^2d\mu \tag{14}$$
$$=\frac{1}{4}\int_0^1 x(1-x)\varphi''\left(f(x)\right)f'(x)^2dx.$$

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