Modelling of Multivariate Stationary Time Series Using Rational Approximation of the Spectral Density Function and Wavelet Coherence

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Abstract This paper presents a new method for the modelling of multivariate stationary time series by applying multiple input-single output transfer function noise model, rational approximation of spectral density function and wavelet coherence. Parameter estimation process is simple and the number of parameters needs to be estimated is very less, is the main advantage of this method. The method is verified by simulation studies and it is also applied to model US hog data with five component series.

Keywords: *multivariate stationary time series, spectral density functions, the rational approximation of spectral density functions, wavelet coherence*

1. Introduction

Multivariate time series (MTS) data arises from different fields of life including physical sciences, medicine, finance, engineering, business, etc. Modeling multivariate stationary time series data effectively is im- portant for many decision-making activities. MTS analysis refers to the modeling of one output series in terms of n input series by explaining the interactions and co-movements among the group of input series. The reliability of time series model will be increased when the output series is modeled in terms of more than one input series which seriously causes the variations in the output variable. In this paper, we consider individual time series as components of a vector time series and analyze them jointly.

The study of MTS analysis got a new momentum in 1950's. Whittle [24] derived the least squares estimation equations for a non-deterministic stationary multiple process. In 1957, Quenouille [17] summa- rized the work of multivariate time series analysis. Akaike [1], Anderson [2], Box G.E.P and Jenkins. G.M [3], Brillinger. D.R [4], Lutkepohl [12], Liu. L.M and Hanssens. D.M. [13], Parzen. E [15], Richard. H. Jones [18], Reinsel [19], Tiao G.C and Box G E P [21], Tiao, G.C and R.S.Tsay [22] are some of them who made contributions to the field of MTS analysis. Priestly. M.B [16], Carter. C.G [6], Daubechies.I [8], Farge.M [9], Koopmann. L.H [11] are few people who studied wavelet and coherence in detail.

Time series is a single realization of the stochastic process. Multivariate time series is a series of observations X_{it} where $i = 1, 2, ..., n$ and $t = 1, 2, ..., T$ made sequentially through time. X_{it} represents observations at time t of the ith component time series. That is if $X_{1t}, X_{2t}, ..., X_{nt}$ are component time series then

$$X_t = (X_{1t}, X_{2t}, ..., X_{nt})'$$ denote the multivariate time series vector at time t. The concept of rational approximation of spectral density function [10] for modeling univariate and bivariate stationary time series is developed in [10]. In this paper, this method is extended for the analysis of n components stationary time series. That is, a robust method is developed for estimating the model parameters of multiple input-single output transfer function noise model by applying rational approximation of spectral density function [10] and wavelet coherence [20].

This paper is organized as follows: Section 1 is the introduction. Section 2 deals with the theoretical pre-requisites needed to model MTS. The new method developed for the identification of multiple input- single output transfer function noise model is presented in Section 3, which is the main contribution of this paper. Section 4 explains the algorithm for the developed method and also shows the flow chart. Section 5 gives application of the developed method to the simulated data and also to the real world multivariate stationary time series data. Conclusion and scope for future work is presented in Section 6.

2. Theoretical Prerequisites

The rational approximation of spectral density function of a stationary time series is developed by Jessy [10].

Under this method, the rational form of the spectral density function of a stationary time series which is represented by an ARMA(p,q) model is obtained by applying the concept of Pade's approximation theory [5,14]. The spectral density function $s(\lambda)$, is defined as the Fourier transform of the auto-correlation function ρ_k, $k = 0,1,2,...$ and is given by

$$s(\lambda) = \sum_{k=-\infty}^{\infty} \rho_k e^{-i\lambda k} \qquad (1)$$

where $\lambda \in [-\pi, \pi]$. Since ρ_k is an even function, $\rho_k = \rho_{-k}$ for $k = 1,2,....$ and $\rho_0 = 1$. So equation (1) becomes,

$$s(\lambda) = 1 + 2\sum_{k=1}^{\infty} \rho_k \cos(\lambda k) \qquad (2)$$

The rational form of the spectral density function of an ARMA(p,q) model, $\phi_p(B)X_t = \theta_q(B)a_t$ is

$$s(\lambda) = \frac{\sigma_a^2 \left[\begin{array}{l} 1 + \sum_{i=1}^{q}\theta_i^2 \\ + 2\sum_{i=1}^{q}\left(-\theta_i + \sum_{j=1}^{q-i}\theta_j\theta_{j+i}\right)\cos(i\lambda) \end{array}\right]}{\sigma_x^2 \left[\begin{array}{l} 1 + \sum_{i=1}^{q}\phi_i^2 \\ + 2\sum_{i=1}^{q}\left(-\phi_i + \sum_{j=1}^{p-i}\phi_j\phi_{j+i}\right)\cos(i\lambda) \end{array}\right]}. \qquad (3)$$

Applying Pade's approximation theory [5,14] and rational approximation of spectral density function [10], obtain the estimated unique ARMA (p,q) model,

$$\tilde{X}_t = \hat{\phi}_1\tilde{X}_{t-1} + \cdots + \hat{\phi}_p\tilde{X}_{t-p} - \hat{\theta}_1 a_{t-1} - \cdots - \hat{\theta}_q a_{t-q}. \qquad (4)$$

The analysis of a bivariate stationary time series is explained in [10]. In the bivariate case, there will be two-time series, one is the input series and other as the corresponding output series, and method is as follows. Given the stationary time series Y_t and X_t and the transfer function noise model is

$$Y_t = A(B)X_{t-b} \qquad (5)$$

where $A(B) = \dfrac{\omega(B)}{\delta(B)}$, $\omega(B)$ and $\delta(B)$ are polynomials in B, the back shift operator (B such that $B(X_t) = X_{t-1}$.). Then the spectral density function of (5) can be written as

$$f_{YY}(\lambda) = k\left|A(e^{i\lambda})\right|^2 f_{XX}(\lambda) \qquad (6)$$

where $f_{YY}(\lambda)$ and $f_{XX}(\lambda)$ are the spectral density functions of the univariate time series Y_t and X_t respectively. From equation (6) we get,

$$k\left|A(e^{i\lambda})\right|^2 = \frac{f_{YY}(\lambda)}{f_{XX}(\lambda)}. \qquad (7)$$

The relation in equation (7) is used to estimate the transfer function A(B) explained in [10] and is given by,

$$A(B) = \frac{\omega_0 + \omega_1 B + \cdots + \omega_r B^r}{1 + \delta_1 B + \cdots + \delta_s B^s}. \qquad (8)$$

Then the estimated transfer function noise model is

$$\hat{Y}_t = \hat{A}(B)X_{t-b} + \hat{n}_t. \qquad (9)$$

Obtain the observed noise series,

$$\hat{n}_t = \hat{Y}_t - \hat{A}(B)X_t. \qquad (10)$$

Now estimate the ARMA model for n_t using the R-spec procedure [10], taking $\hat{n}_t$ as the observed series. Let the ARMA model be $\hat{n}_t = \dfrac{\hat{\theta}(B)}{\hat{\phi}(B)}a_t$. Then the transfer function noise model for bivariate time series estimated is given by $\hat{Y}_t = \hat{A}(B) + \hat{\phi}^{-1}(B)\hat{\theta}(B)a_t$.

$$i.e., \hat{Y}_t = \frac{\hat{\omega}_0 + \hat{\omega}_1 B + \cdots + \hat{\omega}_r B^r}{1 + \hat{\delta}_1 B + \cdots + \hat{\delta}_s B^s}X_{t-b} + \hat{\phi}^{-1}(B)\hat{\theta}(B)a_t. \quad (11)$$

3. Method

The new method of estimating multiple input-single output transfer function noise model rely on the concept of rational approximation of spectral density function [10] and wavelet coherence [20].

Coherence [6] is a measure of correlation in frequency. In time series analysis, coherence is a measure of correlation between two time series at each frequency, and it is denoted by $K_{XY}^2(\lambda)$. In this paper, wavelets is used to estimate coherence due to its time-frequency localized property.

A wavelet is a 'small wave'. A wavelet grows and decays essentially in a limited time period. A real-valued function $\psi(.)$ defined over the real axis $(-,+)$ and satisfying two properties,

$$1) \int_{-\infty}^{\infty} \psi(u)du = 0$$

$$2) \int_{-\infty}^{\infty} \psi^2(u)du = 1$$

is called a wavelet. Wavelet transform Coherence [6] is a method for analyzing the coherence and phase lag between two-time series as a function of both time and frequency. So wavelet coherence [20] can be taken as a localized correlation coefficient in time-frequency space.

The multiple input-single output transfer function noise model is of the form,

$$Y_t = A_1(B)X_{1t-b_1} + \cdots + A_n(B)X_{nt-b_n} + n_t \qquad (12)$$

where $X_{1t}, \cdots, X_{nt}$ are pairwise uncorrelated. i.e., $E[X_{it}X_{jt}] = 0$ for $i \neq j$ and $A_1(B), \cdots, A_n(B)$ are ratios of polynomials in B which are called transfer functions. Then the spectral density function of [10] is given by

$$f_{YY}(\lambda) = \left| A_1\left(e^{i\lambda}\right) \right|^2 f_{X_1 X_1}(\lambda) + \cdots$$
$$+ \left| A_n\left(e^{i\lambda}\right) \right|^2 f_{X_n X_n}(\lambda) \tag{13}$$

and the cross spectral densities will be given by,

$$\left. \begin{aligned} f_{X_1 Y}(\lambda) &= A_1\left(e^{i\lambda}\right) f_{X_1 X_1}(\lambda) \\ &\cdots\cdots \\ f_{X_n Y}(\lambda) &= A_n\left(e^{i\lambda}\right) f_{X_n X_n}(\lambda) \end{aligned} \right\} \tag{14}$$

which gives;

$$\left. \begin{aligned} A_1\left(e^{i\lambda}\right) &= \frac{f_{X_1 Y}(\lambda)}{f_{X_1 X_1}(\lambda)} \\ &\cdots\cdots \\ A_n\left(e^{i\lambda}\right) &= \frac{f_{X_n Y}(\lambda)}{f_{X_n X_n}(\lambda)} \end{aligned} \right\}. \tag{15}$$

Now to obtain the coefficients of $A_i\left(e^{i\lambda}\right)$, $i=1,2,\cdots,n$, estimate the rational form of $\left| f_{X_i Y}(\lambda) \right|^2$, $i=1,2,\cdots,n$ and $\left| f_{X_i X_i}(\lambda) \right|^2$, $i=1,2,\cdots,n$ using the R-spec (Rational form of spectral density function) procedure [10] and then equate the coefficients in the equation,

$$\left| A_i\left(e^{i\lambda}\right) \right|^2 = \frac{\left| f_{X_i Y}(\lambda) \right|^2}{\left| f_{X_i X_i}(\lambda) \right|^2}. \tag{16}$$

Now;

$$K_{XY}^2(\lambda) = \frac{\left| f_{XY}(\lambda) \right|^2}{f_{XX}(\lambda) f_{YY}(\lambda)} \tag{17}$$

is the Coherence[6] (squared coherency) between X and Y. Equations (16) and (17) gives equation (18).

$$\left| A_i\left(e^{i\lambda}\right) \right|^2 = K_{X_i Y}^2(\lambda) \frac{f_{XY}(\lambda)}{f_{X_i X_i}(\lambda)} \text{ for i=1,2,.....,n.} \tag{18}$$

I.e.,

$$\left. \begin{aligned} \left| A_1\left(e^{i\lambda}\right) \right|^2 &= K_{X_1 Y}^2(\lambda) \frac{f_{XY}(\lambda)}{f_{X_1 X_1}(\lambda)} \\ &\cdots\cdots \\ \left| A_n\left(e^{i\lambda}\right) \right|^2 &= K_{X_n Y}^2(\lambda) \frac{f_{XY}(\lambda)}{f_{X_n X_n}(\lambda)} \end{aligned} \right\}. \tag{19}$$

In matrix form, equation (19) can be written as

$$[D] = [K][F] \tag{20}$$

where $[D] = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix}$, each $D_i = \left| A_i\left(e^{i\lambda}\right) \right|^2$ for $i=1,2,...,n$

$$[K] = \begin{bmatrix} K_1 & 0 & 0 & ... & 0 \\ 0 & K_2 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & K_n \end{bmatrix},$$

each $K_i = K_{X_i Y}^2(\lambda)$, for $i=1,2,...,n$

$$[F] = \begin{bmatrix} F_1 \\ F_2 \\ ... \\ F_n \end{bmatrix},$$

each $F_i = \dfrac{f_{XY}(\lambda)}{f_{X_i X_i}(\lambda)}$, for $i=1,2,...,n.$

By solving equation (19), we obtain the estimates of the transfer functions $A_1(B),\cdots,A_n(B)$. But for solving equation (19), we need the squared coherences $K_1,...,K_n$ which are estimated using method of wavelet anlaysis. Now for each $i'=1,2,...,n$, $\quad A_i'(B) = \dfrac{\left(\omega_{0i'} + \omega_{1i'} B + \cdots + \omega_{ri'} B^r\right)}{\left(1 + \delta_{1i'} B + \cdots + \delta_{si'} B^s\right)}$

and

$$\left| A_i\left(e^{i\lambda}\right) \right|^2 = \frac{\sum_{i=0}^{r}\omega_{ii'}^2 + \cdots + 2\omega_{0i'}\omega_{ri'}\cos(r\lambda)}{\left(1 + \sum_{i=1}^{s}\delta_{ii'}^2 + \cdots + 2\delta_{si'}\cos(s\lambda)\right)}. \tag{21}$$

Now

$$f_{YY}(\lambda) = \frac{2\left(a_0 + a_1\cos(\lambda) + \cdots + a_{l_1 i'}\cos(l_1\lambda)\right)}{1 + b_1\cos(\lambda) + \cdots + b_{k_1}\cos(k_1\lambda)}$$

$$f_{X_i' X_i'}(\lambda) = \frac{2\left(a_{0i'} + a_{1i'}\cos(\lambda) + \cdots + a_{l_2 i'}\cos(l_2\lambda)\right)}{\left(1 + b_{1i'}\cos(\lambda) + \cdots + b_{k_2 i'}\cos(k_2\lambda)\right)}$$

and

$$\frac{f_{YY}(\lambda)}{f_{X_i' X_i'}(\lambda)} = \frac{\left(A_{0i'} + \cdots + A_{Li'}\cos(L\lambda)\right)}{\left(B_{0i'} + \cdots + B_{Ki'}\cos(K\lambda)\right)} \tag{22}$$

where each $L = l_1 + k_2 = r$ and $K = l_2 + k_1 = s$. Here $f_{YY}(\lambda)$ and $f_{X_i' X_i'}(\lambda)$ are called rational form spectral density function of the time series Y and X_i. Now using equations (21) and (22), we get the following equations;

$$\left. \begin{aligned} A_{0i'} &= \frac{\sum_{i=0}^{r}\omega_{ii'}^2}{K_{X_{i'} Y}^2(\lambda)} \\ A_{ji'} &= \frac{2\sum_{i=0}^{r-j}\omega_{ii'}\omega_{i+j,i'}}{K_{X_{i'} Y}^2(\lambda)} \end{aligned} \right\} \text{ for } j=1,2,...,r \tag{23}$$

$$\left. \begin{aligned} B_{0i'} &= \frac{1 + \sum_{i=0}^{s}\omega_{ii'}^2}{K_{X_{i'} Y}^2(\lambda)} \\ B_{ji'} &= \frac{2\sum_{i=0}^{s-j}\delta_{ii'}\delta_{i+j,i'} + \delta_{ii'}}{K_{X_{i'} Y}^2(\lambda)} \end{aligned} \right\} \text{ for } j=1,2,...,s. \tag{24}$$

The above equations hold for $i' = 1, 2, ..., n$. Now applying the method of iterations [5,10] to the system of equations (23) and (24), we get all values of $\omega_i's$ and $\delta_i's$. Thus, we calculated $\hat{A}_1(B), ..., \hat{A}_n(B)$, the estimated transfer functions and the estimated transfer function noise model is,

$$\hat{Y}_t = \hat{A}_1(B) X_{1t-b_1} + \cdots + \hat{A}_n(B) X_{nt-b_n} + \hat{n}_t. \quad (25)$$

Then the noise series is obtained using,

$$\hat{n}_t = \hat{Y}_t - \sum_{i=1}^{n} \hat{A}_i(B) X_{it-b_i}. \quad (26)$$

Now estimate the ARMA model representing the noise series n_t using the R-spec procedure [10]. Then the noise

model will be of the form, $\hat{\phi}(B) n_t = \hat{\theta}(B) a_t$

$$\text{ie } n_t = \hat{\phi}^{-1}(B) \hat{\theta}(B) a_t. \quad (27)$$

Then using the equations (25) and (27), we obtain the multivariate time series model as

$$\hat{Y}_t = \hat{A}_1(B) X_{1t-b_1} + \cdots + \hat{A}_n(B) X_{nt-b_n}$$
$$+ \hat{\phi}^{-1}(B) \hat{\theta}(B) a_t \quad (28)$$

and the variance of the white noise σ_a^2 that we obtain along with the noise model.

4. Algorithm and Flow Chart of the Developed Method

Flow Chart
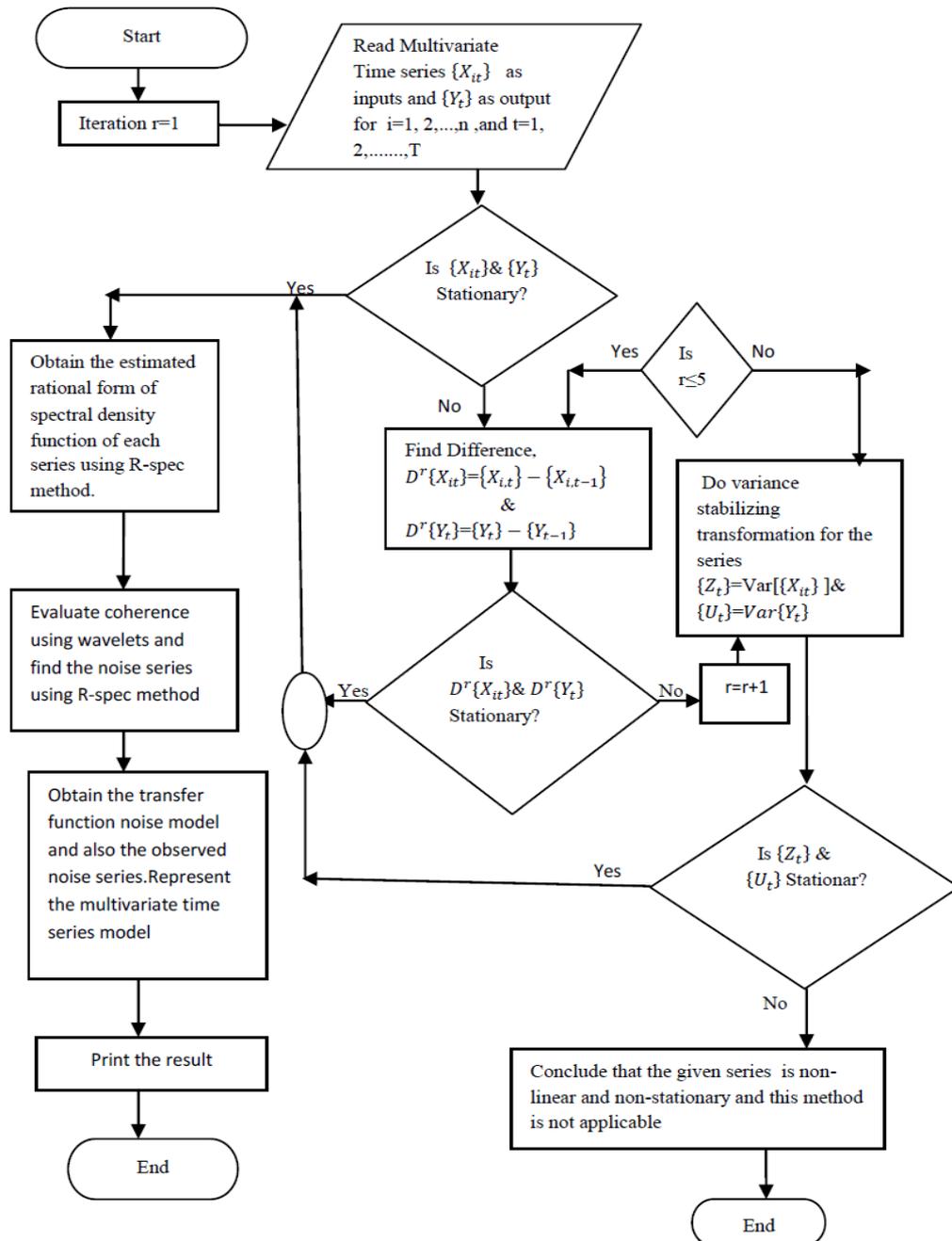
1. Choose a multivariate time series and take $\{X_{it}\}$ as inputs and Y_t as output for $i = 1, 2, ..., n$ and $t = 1, 2, ..., T$

2. Then check stationarity of $\{X_{it}\}$ and $\{Y_t\}$. If all of them are stationary then go to step 3. Otherwise, go to step 7.

3. Obtain the initial rational forms of the spectral density function of all the series using the R-spec procedure [5,10].

4. Evaluate coherence using wavelets and thus estimate the coefficients of the transfer functions $\hat{A}_i(B)$ which give the estimated transfer function noise model represented in equation (25).

5. Find noise series and estimate the ARMA model representing the noise series using the R-spec procedure [5,10].

6. Obtain the multivariate stationary time series model by combining the estimated transfer function model and the noise series.

7. If $\{X_{it}\}'$ and $\{Y_t\}$ are not stationary, then find the difference $D_X^1 = X_{it} - X_{i,t-1}$ and $D_Y^1 = Y_t - Y_{t-1}$ and then check D_X^1 and D_Y^1 are stationary or not. If it is stationary, then go to step 3. Otherwise go to step 8.

8. If D_X^1 and D_Y^1 are not stationary, then continuously differencing D_X^n and D_Y^n till n=5 and check stationarity at each time. If D_X^1 and D_Y^1 are stationary within $n \le 5$ then go to step 3. Otherwise, go to step 9.

9. If D_X^n and D_Y^n are not stationary then for $n \ge 5$, do variance stabilizing transformation for the series and again check D_X^n and D_Y^n are stationary or not. If D_X^n and D_Y^n are stationary then go to step 3. Otherwise, go to step 10.

10. If D_X^n and D_Y^n are not stationary then conclude that the given series is non-linear and non-stationary and this method is not applicable.

5. Application - Numerical Results and Comparison

The developed method is verified using simulation studies and it is applied to US hog data[17] (US hog data studied by Quenouille from 1867 to 1948).

5.1. Simulation Study Results

The method was applied to simulated multivariate time series data and the results obtained are given in Table 5.1.

Table 5.1: For verifing the method developed, a number time series were simulated as input series and output series and estimated the parameters of multiple-input single output transfer function noise model with various parameter combinations by applying the developed method. Table 5.1 shows eleven selected examples from the simulated studies. Also estimated Mean absolute percentage error (MAPE)

Table 5.1. Parameter Estimation of Simulated Time Series Models

Example	Inputed Parameter Values	Estimated Parameter Values	Error	MAPE
1	$\omega_{01} = 0.16, \omega_{11} = -0.027, \delta_{11} = 0.867$	$\hat{\omega}_{01} = 0.228, \hat{\omega}_{11} = 1.05, \hat{\delta}_{11} = 0.66$	$Er\omega_{01} = 0.068, Er\omega_{11} = 1.07, Er\delta_{11} = 0.207$	0.8093
2	$\omega_{01} = 0.78, \omega_{11} = -0.31, \delta_{11} = 0.7$	$\hat{\omega}_{01} = 1.05, \hat{\omega}_{11} = 1.07, \hat{\delta}_{11} = 0.67$	$Er\omega_{01} = 0.27, Er\omega_{11} = 1.38, Er\delta_{11} = 0.033$	0.4788
3	$\omega_{01} = 0.8, \omega_{11} = -0.4, \delta_{11} = 0.87$	$\hat{\omega}_{01} = 1.09, \hat{\omega}_{11} = 1.1, \hat{\delta}_{11} = 0.668$	$Er\omega_{01} = 0.29, Er\omega_{11} = 1.51, Er\delta_{11} = 0.202$	0.2165
4	$\omega_{01} = 0.328, \omega_{11} = -0.727, \delta_{11} = 0.78$	$\hat{\omega}_{01} = 0.67, \hat{\omega}_{11} = -1.02, \hat{\delta}_{11} = 0.6681$	$Er\omega_{01} = 0.342, Er\omega_{11} = -0.2930, Er\delta_{11} = -.0.1120$	0.7016
5	$\omega_{01} = 0.685, \omega_{11} = -0.056, \delta_{11} = 0.618$	$\hat{\omega}_{01} = 0.0.09, \hat{\omega}_{11} = -0.86, \hat{\delta}_{11} = 0.6681$	$Er\omega_{01} = -0.595, Er\omega_{11} = -0.804, Er\delta_{11} = 0.0501$	0.6252
6	$\omega_{01} = 0.824, \omega_{11} = -0.29, \delta_{11} = 0.684$	$\hat{\omega}_{01} = 1.04, \hat{\omega}_{11} = 1.07, \hat{\delta}_{11} = 0.66$	$Er\omega_{01} = 0.216, Er\omega_{11} = 1.36, Er\delta_{11} = -0.024$	0.9705
	$\omega_{02} = 0.62, \omega_{12} = -0.44, \delta_{12} = 0.324$	$\hat{\omega}_{02} = 0.1187, \hat{\omega}_{12} = 0.69, \hat{\delta}_{12} = 0.707$	$Er\omega_{02} = -0.5013, Er\omega_{12} = 1.13, Er\delta_{12} = 0.383$	
7	$\omega_{01} = 0.56, \omega_{11} = -0.209, \delta_{11} = 0.8$	$\hat{\omega}_{01} = 1.07, \hat{\omega}_{11} = 1.08, \hat{\delta}_{11} = 0.6$	$Er\omega_{01} = 0.51, Er\omega_{11} = 1.289, Er\delta_{11} = 0.2$	0.5819
	$\omega_{02} = 0.87, \omega_{12} = -0.64, \delta_{12} = 0.589$	$\hat{\omega}_{02} = -0.1725, \hat{\omega}_{12} = 0.6857, \hat{\delta}_{12} = 0.7071$	$Er\omega_{02} = -1.0425, Er\omega_{12} = 1.3257, Er\delta_{12} = 0.118$	
8	$\omega_{01} = 0.424, \omega_{11} = -0.619, \delta_{11} = 0.684$	$\hat{\omega}_{01} = 1.057, \hat{\omega}_{11} = 1.078, \hat{\delta}_{11} = 0.6681$	$Er\omega_{01} = 0.633, Er\omega_{11} = 1.697, Er\delta_{11} = -0.016$	0.4942
	$\omega_{02} = 0.622, \omega_{12} = -0.144, \delta_{12} = 0.824$	$\hat{\omega}_{02} = -0.1445, \hat{\omega}_{12} = 0.6922, \hat{\delta}_{12} = 0.7$	$Er\omega_{02} = -0.8065, Er\omega_{12} = 0.836, Er\delta_{12} = 0.124$	
9	$\omega_{01} = 0.44, \omega_{11} = -0.28, \delta_{11} = 0.31$	$\hat{\omega}_{01} = 1.086, \hat{\omega}_{11} = 1.09, \hat{\delta}_{11} = 0.668$	$Er\omega_{01} = 0.646, Er\omega_{11} = 1.37, Er\delta_{11} = 0.358$	0.7827
	$\omega_{02} = 0.57, \omega_{12} = -0.66, \delta_{12} = 0.42$	$\hat{\omega}_{02} = -0.2061, \hat{\omega}_{12} = 0.676, \hat{\delta}_{12} = 0.7$	$Er\omega_{02} = -0.7761, Er\omega_{12} = 1.336, Er\delta_{12} = 0.28$	
	$\omega_{03} = 0.38, \omega_{13} = -0.15, \delta_{13} = 0.84$	$\hat{\omega}_{03} = 0.3558, \hat{\omega}_{13} = 0.5381, \hat{\delta}_{13} = 0.698$	$Er\omega_{03} = -0.0242, Er\omega_{13} = 0.6881, Er\delta_{13} = -0.142$	
10	$\omega_{01} = 0.12, \omega_{11} = -0.04, \delta_{11} = 0.67$	$\hat{\omega}_{01} = 1.0566, \hat{\omega}_{11} = 1.07, \hat{\delta}_{11} = 0.668$	$Er\omega_{01} = 0.936, Er\omega_{11} = 1.11, Er\delta_{11} = -0.002$	0.7827
	$\omega_{02} = 0.34, \omega_{12} = -0.14, \delta_{12} = 0.73$	$\hat{\omega}_{02} = 0.1385, \hat{\omega}_{12} = 0.6934, \hat{\delta}_{12} = 0.707$	$Er\omega_{02} = -0.2015, Er\omega_{12} = -0.2015, Er\delta_{12} = -0.023$	
	$\omega_{03} = 0.42, \omega_{13} = -0.15, \delta_{13} = 0.12$	$\hat{\omega}_{03} = 0.1834, \hat{\omega}_{13} = -0.4903, \hat{\delta}_{13} = 0.69$	$Er\omega_{03} = -0.2366, Er\omega_{13} = -0.3403, Er\delta_{13} = 0.57$	
11	$\omega_{01} = 0.77, \omega_{11} = -0.27, \delta_{11} = 0.81$	$\hat{\omega}_{01} = 1.08, \hat{\omega}_{11} = 1.09, \hat{\delta}_{11} = 0.6681$	$Er\omega_{01} = 0.31, Er\omega_{11} = 1.36, Er\delta_{11} = -0.1419$	0.3313
	$\omega_{02} = 0.47, \omega_{12} = -0.56, \delta_{12} = 0.64$	$\hat{\omega}_{02} = -0.2069, \hat{\omega}_{12} = 0.6762, \hat{\delta}_{12} = 0.707$	$Er\omega_{02} = -0.6769, Er\omega_{12} = 1.2362, Er\delta_{12} = 0.067$	
	$\omega_{03} = 0.54, \omega_{13} = -0.25, \delta_{13} = 0.38$	$\hat{\omega}_{03} = 0.3564, \hat{\omega}_{13} = 0.5386, \hat{\omega}_{13} = 0.6986$	$Er\omega_{03} = -0.1836, Er\omega_{13} = 0.7886, Er\delta_{13} = 1.0786$	

Simulated studies from No.1 to No.5 are examples of bivariate time series models. That is one input series $\{X_t\}$ and one output series $\{Y_t\}$ and its transfer function noise model are simulated by applying the method above. The model is given by

$$i.e., Y_t = \frac{\omega_{01} + \omega_{11}B}{1 + \delta_{11}B} X_t + n_t. \tag{29}$$

Examples from No.6 to No.8 are illustrating about simulated transfer function noise models of two inputs series $\{X_{1t}\}$, $\{X_{2t}\}$ and one output series $\{Y_t\}$ and the model is given by

$$i.e., Y_t = \frac{\omega_{01} + \omega_{11}B}{1 + \delta_{11}B} X_{1t} + \frac{\omega_{02} + \omega_{12}B}{1 + \delta_{12}B} X_{2t} + n_t. \tag{30}$$

Examples from No.9 to No.11 pointing towards the simulated studies of three input series $\{X_{1t}\}$, $\{X_{2t}\}$, $\{X_{3t}\}$ and one output series $\{Y_t\}$ and its transfer function noise model is given by

$$i.e., Y_t = \frac{\omega_{01} + \omega_{11}B}{1 + \delta_{11}B} X_{1t} + \frac{\omega_{02} + \omega_{12}B}{1 + \delta_{12}B} X_{2t}$$
$$+ \frac{\omega_{03} + \omega_{13}B}{1 + \delta_{13}B} X_{2t} + n_t. \tag{31}$$

5.2. Analysis of US Hog Data from 1867 to 1948

The method is applied to model US hog data [17]. Estimated transfer function parameters are given in Table 5.2.

US hog data consists of 5 component series namely, hog number, hog price, corn price, corn supply and wage rates, each having 82 observations. Here hog price- $\{X_1\}$ (Figure 2), corn price- $\{X_2\}$ (Figure 3), corn supply- $\{X_3\}$ (Figure 4) and wage rates- $\{X_4\}$ (Figure 5) are taken as inputs(component time series) and hog number $\{Y_t\}$ (Figure 1) as output for the multiple-input transfer function model expressed in equation (12). Here corn supply $\{X_3\}$ is omitted for model building as its contribution is not significant indeed (Figure 18). Then find auto-correlations and partial auto-correlations of all inputs (Figure 10, Figure 11, Figure 14, Figure 15, Figure 19, Figure 20) and output (Figure 6 & Figure 7). First order differenced series in each of these series are found to be stationary (Figure 8, Figure 12, Figure 16, Figure 22) as the differenced autocorrelation of each of the series terminates sudden. Also calculated partial auto-correlations of all differenced series (Figure 9, Figure 13, Figure 17, Figure 22). Then applied the method above and estimated the parameters of multiple-input single-output transfer function noise model. Also estimated Mean absolute percentage error (MAPE).
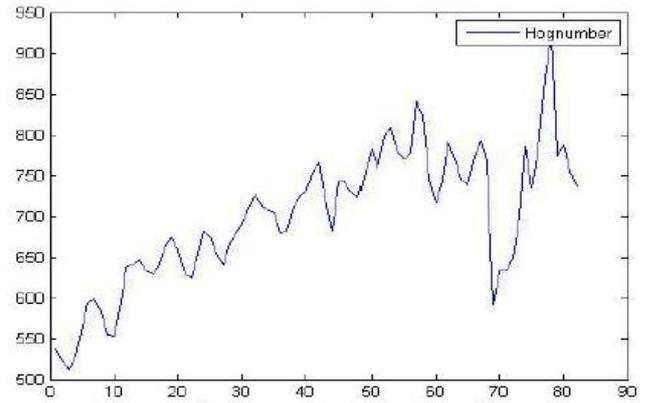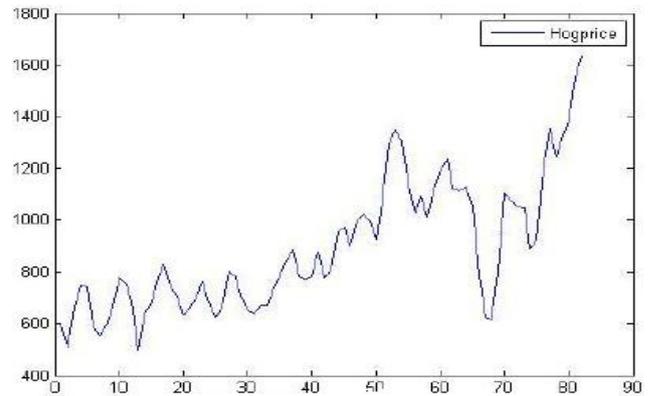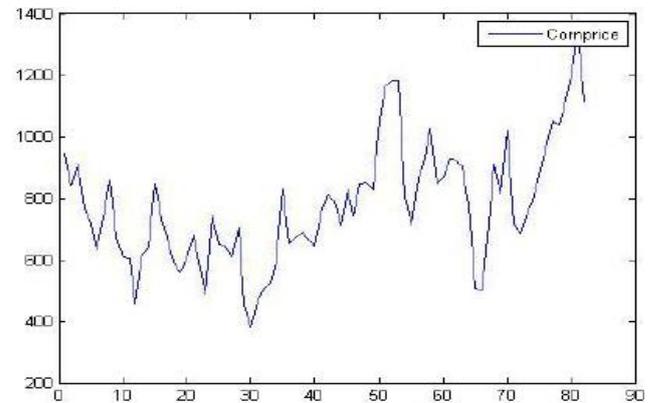

Figure 1. Hognumber

Figure 2. Hogprice

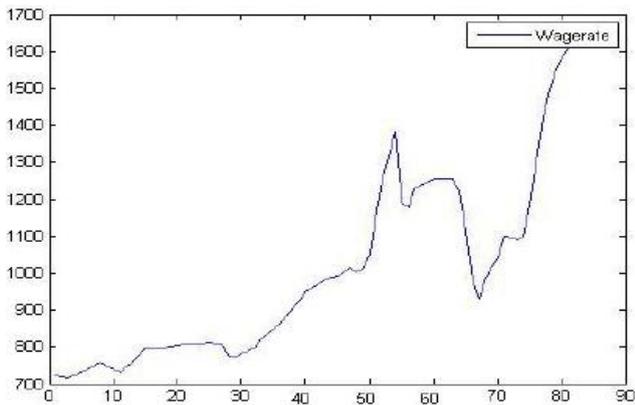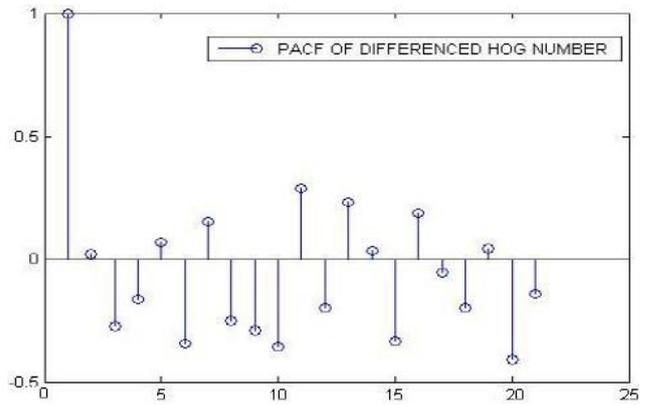**Figure 3.** Cornprice
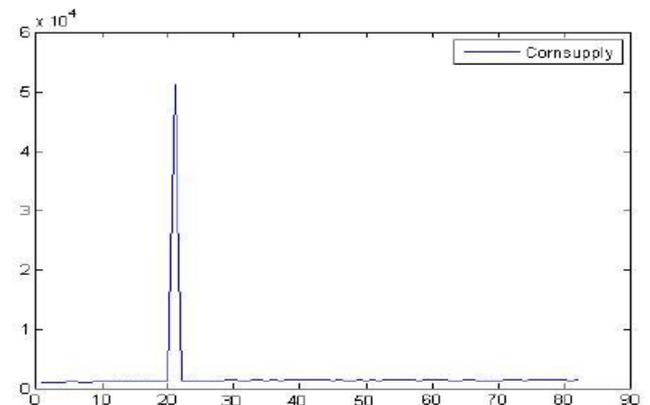
Figure 4. Cornsupply

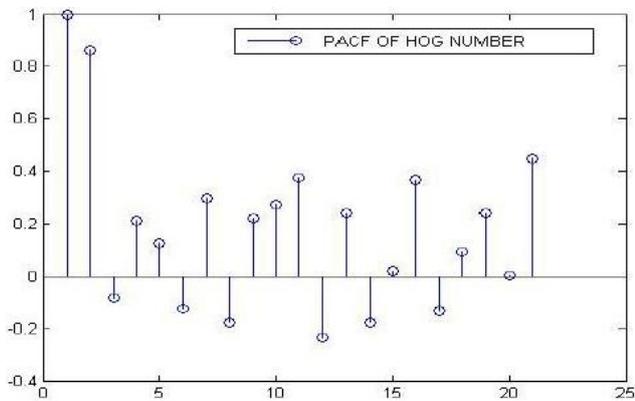**Figure 5.** Wagerates

Figure 6. AUTO-CORRELATION OF HOG NUMBER DATA

Figure 7. PARTIAL-AUTOCORRELATION OF HOG NUMBER DATA

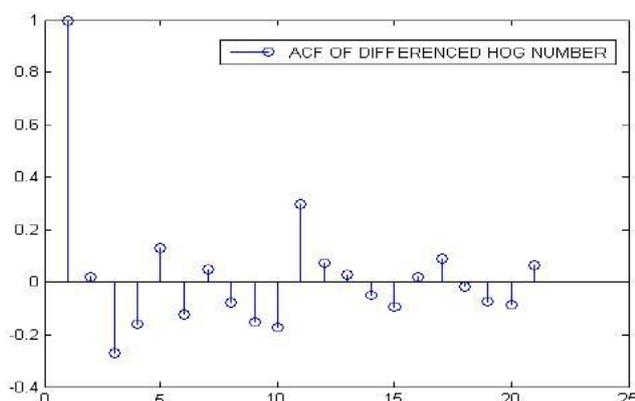**Figure 8.** AUTO-CORRELATION OF DIFFERENCED HOG NUMBER DATA

Figure 9. PARTIAL-AUTOCORRELATION OF DIFFERENCED HOG NUMBER DATA

Figure 10. AUTO-CORRELATION OF HOG PRICE

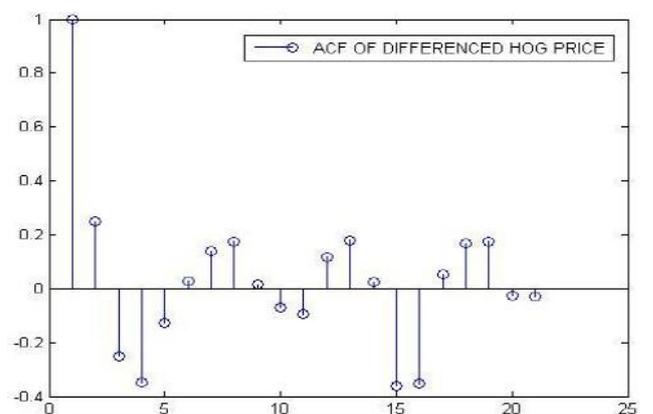**Figure 11.** PARTIAL-AUTOCORRELATION OF HOG PRICE

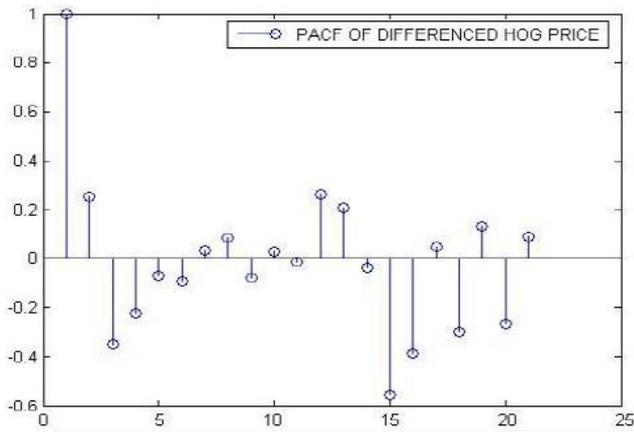**Figure 12.** AUTO-CORRELATION OF DIFFERENCED HOG PRICE

Figure 13. PARTIAL-AUTOCORRELATION OF DIFFERENCED HOG PRICE

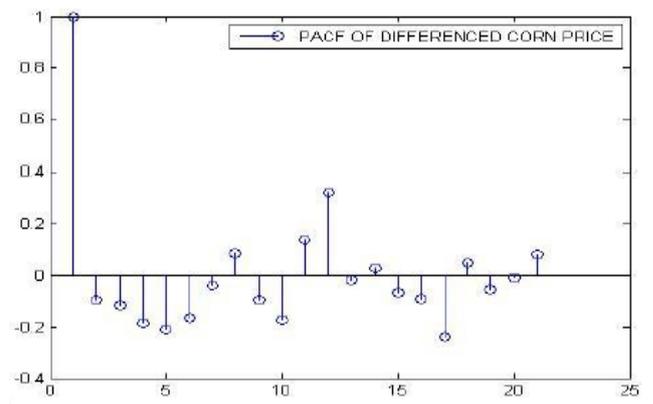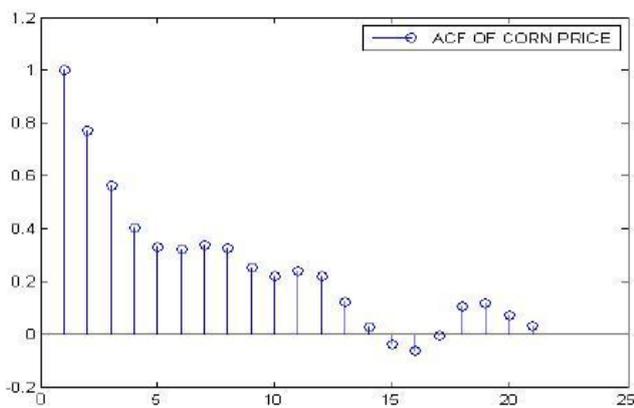**Figure 17.** PARTIAL-AUTOCORRELATION OF DIFFERENCED CORN PRICE

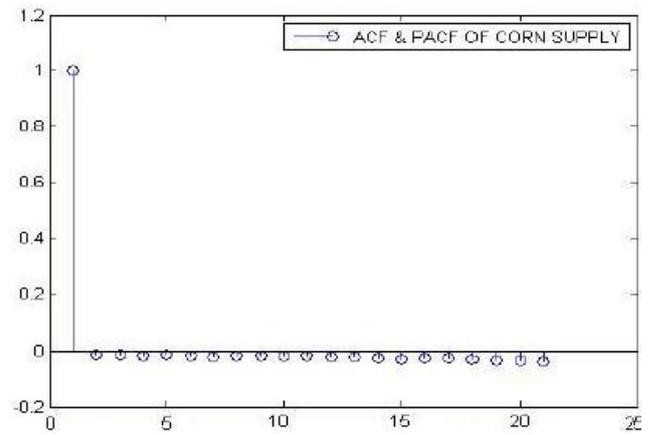**Figure 14.** AUTO-CORRELATION OF CORN PRICE

Figure 18. AUTO-CORRELATION & PARTIAL-AUTOCORRELATION OF CORN SUPPLY
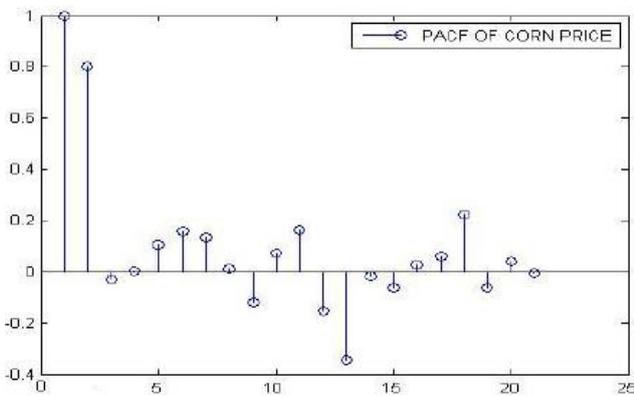

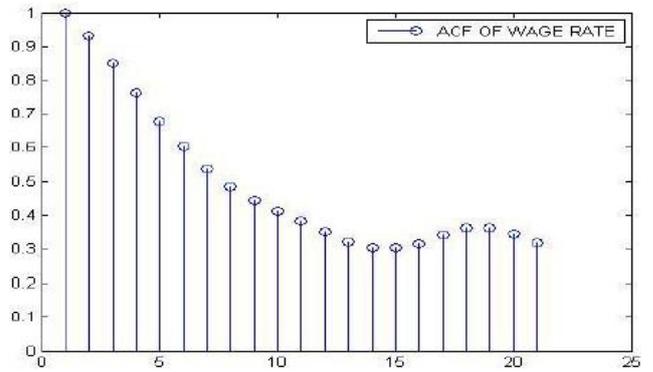**Figure 15.** PARTIAL-AUTOCORRELATION OF CORN PRICE

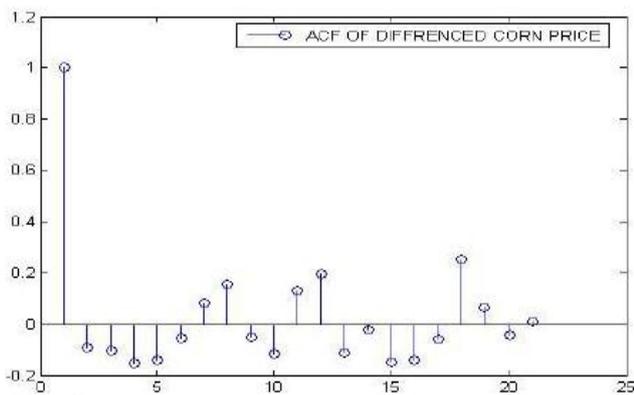**Figure 19.** AUTO-CORRELATION OF WAGE RATE

Figure 16. AUTO-CORRELATION OF DIFFERENCED CORN PRICE

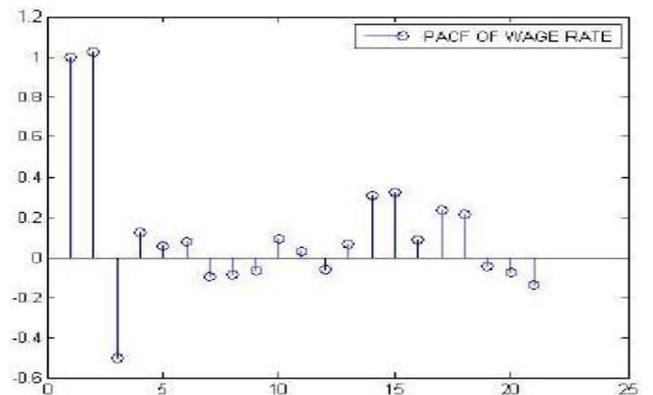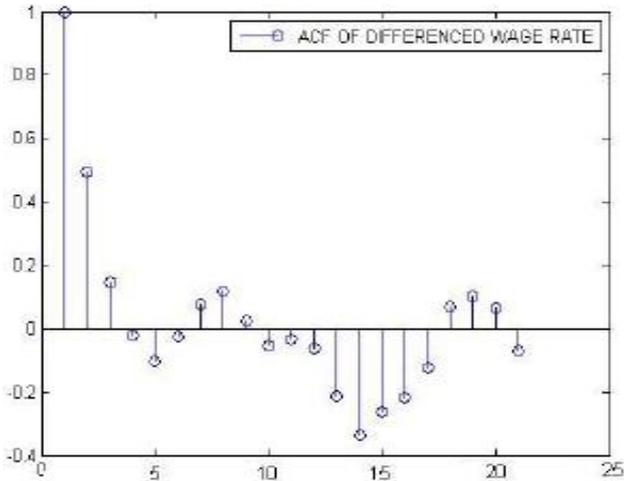**Figure 20.** PARTIAL-AUTO-CORRELATION OF WAGE RATE

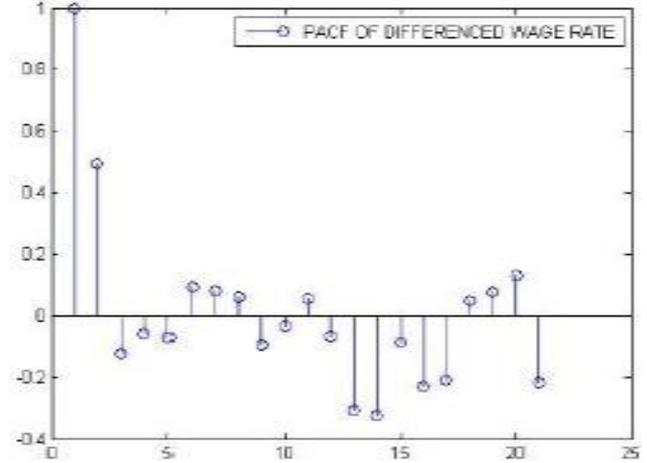**Figure 21.** AUTO-CORRELATION OF DIFFERENCED WAGE RATE

Figure 22. PARTIAL-AUTO-CORRELATION OF DIFFERENCED WAGE RATE

Table 5.2. Parameter Estimation of US Hog Data

Estimated Transfer function Parameters of US Hog data			
Hog Price	Corn Price	Wage Rates	MAPE
$\hat{\omega}_{01} = 0.01, \hat{\omega}_{11} = 0.71, \hat{\delta}_{11} = 0.71$	$\hat{\omega}_{02} = -0.01, \hat{\omega}_{12} = -0.33, \hat{\omega}_{22} = 0.63, \hat{\delta}_{12} = 0.68$	$\hat{\omega}_{03} = -0.01, \hat{\omega}_{13} = -0.34, \hat{\omega}_{23} = 0.62, \hat{\delta}_{13} = 0.84$	0.7038

Table 5.2: shows the estimated parameters of multiple-input transfer function noise model by applying the developed method. Mean Absolute Percentage Error (MAPE) is also mentioned.

The model estimated for n_t is given by

$$\hat{n}_t = a(t) + 0.0874 a(t-1) \tag{32}$$

Therefore the final model estimated for the US hog data is given by,

$$\hat{Y}_t = \frac{(0.01 + 0.71B)}{(1 + 0.71B)} X_1(t)$$
$$+ \frac{(-0.01 - 0.33B + 0.63B^2)}{(1 + 0.68B)} X_2(t) \tag{33}$$
$$+ \frac{(-0.01 - 0.34B + 0.62B^2)}{(1 + 0.84B)} X_3(t) + n_t$$

where $\hat{n}_t$ is given in equation (32).

6. Computational Complexity Analysis

The Computational complexity analysis of the matlab programs of simulated studies and US hog data are given below.

7. Conclusion and Future Work

A method for modeling stationary multivariate time series by estimating the parameters of multiple-input single-output transfer function noise model using rational approximation of spectral density function [10] and wavelet coherence [20] is developed in this paper. The method is verified using simulated data and also applied to US hog data [17]. Applying this method to more real world time series models and exploring more mathematical properties of this model is one part of future work.

Algorithms	Time Complexity	Space Complexity	Execution Complexity
Simulation Study 1	$O(N^3)$	$O(N^2)$	1.554742 seconds
Simulation Study 2	$O(N^3)$	$O(N^2)$	2.010651 seconds
Simulation Study 3	$O(N^3)$	$O(N^2)$	2.069510 seconds
Simulation Study 4	$O(N^3)$	$O(N^2)$	1.806636 seconds
Simulation Study 5	$O(N^3)$	$O(N^2)$	1.671667 seconds
Simulation Study 6	$O(N^3)$	$O(N^2)$	3.057166 seconds
Simulation Study 7	$O(N^3)$	$O(N^2)$	2.739439 seconds
Simulation Study 8	$O(N^3)$	$O(N^2)$	2.168736 seconds
Simulation Study 9	$O(N^3)$	$O(N^2)$	4.092139 seconds
Simulation Study 10	$O(N^3)$	$O(N^2)$	3.774437 seconds
Simulation Study 11	$O(N^3)$	$O(N^2)$	3.282044 seconds
US Hog Data	$O(N^2)$	$O(N)$	22.376459 seconds

8. Appendix

8.1. Matlab Code

```
%R−spec Procedure for the five series Us Hog data from 1867 to 1948
tic;
Y= load('Hognumber.m');
n=length(Y);
acf= autocorr(Y);
plot(acf);
stem(acf);
title('SAMPLE AUTO−CORRELATION OF HOG NUMBER DATA');
figure,acf;
pacf=parcorr(Y);
stem(pacf);
title('SAMPLE PARTIAL AUTO CORRELATION OF HOG NUMBER DATA');
figure,pacf;
Y1=diff(Y);
n1=length(Y1);
acf1=autocorr(Y1);
plot(acf1);
stem(acf1);
title('SAMPLE AUTO−CORRELATION OF DIFFERENCED HOG NUMBER DATA');
figure,acf1;
pacf1=parcorr(Y1);
stem(pacf1);
title('SAMPLE PARTIAL AUTOCORRELATION OF DIFFERENCED HOG NUMBER DATA');
figure,pacf1;
x1 = load('Hogprice.m');
n2=length(x1);
acf2=autocorr(x1);
plot(acf2);
stem(acf2);
title('SAMPLE AUTO−CORRELATION OF HOG PRICE DATA');
figure,acf2;
pacf2=parcorr(x1);
stem(pacf2);
title('SAMPLE PARTIAL AUTO CORRELATION OF HOG PRICE DATA');
figure,pacf2;
X1=diff(x1);
n3=length(X1);
acf3=autocorr(X1);
plot(acf3);
stem(acf3);
title('SAMPLE AUTO−CORRELATION OF DIFFERENCED HOG PRICE DATA');
figure,acf3;
pacf3=parcorr(X1);
stem(pacf3);
title('SAMPLE PARTIAL AUTOCORRELATION OF DIFFERENCED HOG PRICE DATA');
figure,pacf3;
x2= load('Cornprice.m');
n4=length(x2);
acf4=autocorr(x2);
plot(acf4);
stem(acf4);
title('SAMPLE AUTO−CORRELATION OF CORN PRICE DATA');
figure,acf4;
pacf4=parcorr(x2);
stem(pacf4);
title('SAMPLE PARTIAL AUTO CORRELATION OF CORN PRICE DATA');
figure,pacf4;
X2=diff(x2);
n5=length(X2);
```

```
acf5=autocorr(X2);
plot(acf5);
stem(acf5);
title('SAMPLE AUTO−CORRELATION OF DIFFERENCED CORN PRICE DATA');
figure,acf5;
pacf5=parcorr(X2);
stem(pacf5);
title('SAMPLE PARTIAL AUTOCORRELATION OF DIFFERENCED CORN PRICE DATA');
figure,pacf5;
X3 = load('Cornsupply.m');
n6=length(X3);
acf6=autocorr(X3);
plot(acf6);
stem(acf6);
title('SAMPLE AUTO−CORRELATION OF CORN SUPPLY DATA');
figure,acf6;
pacf6=parcorr(X3);
stem(pacf6);
title('SAMPLE PARTIAL AUTO CORRELATION OF CORN SUPPLY DATA');
figure,pacf5;
x4 = load('Wagerates.m');
n7=length(x4);
acf7=autocorr(x4);
plot(acf7);
stem(acf7);
title('SAMPLE AUTO−CORRELATION OF WAGE RATES DATA');
figure,acf7;
pacf7=parcorr(x4);
stem(pacf7);
title('SAMPLE PARTIAL AUTO CORRELATION OF WAGE RATES DATA');
figure,pacf7;
X4=diff(x4);
n8=length(X4);
acf8=autocorr(X4);
plot(acf8);
stem(acf8);
title('SAMPLE AUTO−CORRELATION OF DIFFERENCED WAGE RATES DATA');
figure,acf8;
pacf8=parcorr(X4);
stem(pacf8);
title('SAMPLE PARTIAL AUTOCORRELATION OF DIFFERENCED WAGE RATES DATA');
figure,pacf8;
for k=[−1:0.1:1]
la=k*pi;
a0=0.5;
a1=acf1(2);
T01=a0+a1*cos(la);
end
for k=[−1:0.1:1]
la=k*pi;
b1=−2*acf1(2)/(1+acf1(3));
a01=0.5*(1+acf1(2)*b1);
T10=a01/(1+b1*cos(la));
end
for k=[−1:0.1:1]
la=k*pi;
b11=−2*acf1(3)/(acf1(2)+acf1(4));
a02=0.5*(1+acf1(2)*b11);
a11=0.5*(2*acf1(2)+b11*(1+acf1(3)));
T11=(a02+a11*cos(la))/(1+b11*cos(la));
end
for i=1:20
for k=[−1:0.1:1]
```

```
la=k*pi;
f(i)=(0.5+sum(acf1(i+1)*cos(i*la)));
E1(i)=f(i)−T01;
E2(i)=f(i)−T10;
E3(i)=f(i)−T11;
end
end
E11=abs(E1);
Error1=max(E11);
E12=abs(E2);
Error2=max(E12);
E13=abs(E3);
Error3=max(E13);
M=min([Error1(:);
Error2(:); Error3(:)]);
for k=[−1:0.1:1]
la=k*pi; a03=0.5;
a12=acf3(2);
t01=a03+a12*cos(la);
end
for k=[−1:0.1:1]
la=k*pi;
b12=−2*acf3(2)/(1+acf3(3));
a04=0.5*(1+acf3(2)*b12);
t10=a04/(1+b12*cos(la));
end
for k=[−1:0.1:1]
la=k*pi;
b13=−2*acf3(3)/(acf3(2)+acf3(4));
a05=0.5*(1+acf3(2)*b13);
a13=0.5*(2*acf3(2)+b13*(1+acf3(3)));
t11=(a05+a13*cos(la))/(1+b13*cos(la));
end
for i=1:20
for k=[−1:0.1:1]
la=k*pi;
f(i)=(0.5+sum(acf3(i+1)*cos(i*la)));
e1(i)=f(i)−t01;
e2(i)=f(i)−t10;
e3(i)=f(i)−t11;
end
end
e11=abs(e1);
Error11=max(e11);
e12=abs(e2);
Error21=max(e12);
e13=abs(e3);
Error31=max(e13);
M1=min([Error11(:); Error21(:); Error31(:)]);
for k=[−1:0.1:1]
la=k*pi;
a014=0.5;
a111=acf5(2);
t012=a014+a111*cos(la);
end
for k=[−1:0.1:1]
la=k*pi;
b110=−2*acf5(2)/(1+acf5(3));
a015=0.5*(1+acf5(2)*b110);
t102=a015/(1+b110*cos(la));
end
for k=[−1:0.1:1]
la=k*pi;
```

```
b111=−2*acf5(3)/(acf5(2)+acf5(4));
a016=0.5*(1+acf5(2)*b111);
a112=0.5*(2*acf5(2)+b111*(1+acf5(3)));
t112=(a016+a112*cos(la))/(1+b111*cos(la));
end
for i=1:20
for k=[−1:0.1:1]
la=k*pi;
f(i)=(0.5+sum(acf5(i+1)*cos(i*la)));
ec1(i)=f(i)−t012;
ec2(i)=f(i)−t102;
ec3(i)=f(i)−t112;
end
end
ec11=abs(ec1);
Error12=max(ec11);
ec22=abs(ec2);
Error22=max(ec22);
ec33=abs(ec3);
Error32=max(ec33);
M2=min([Error12(:); Error22(:); Error32(:)]);
for k=[−1:0.1:1]
la=k*pi;
a017=0.5;
a113=acf6(2);
t013=a017+a113*cos(la);
end
for k=[−1:0.1:1]
la=k*pi;
b112=−2*acf6(2)/(1+acf6(3));
a018=0.5*(1+acf6(2)*b112);
t103=a018/(1+b112*cos(la));
end
for k=[−1:0.1:1]
la=k*pi;
b113=−2*acf6(3)/(acf6(2)+acf6(4));
a019=0.5*(1+acf6(2)*b113);
a114=0.5*(2*acf6(2)+b113*(1+acf6(3)));
t113=(a019+a114*cos(la))/(1+b113*cos(la));
end
for i=1:20
for k=[−1:0.1:1]
la=k*pi;
f(i)=(0.5+sum(acf6(i+1)*cos(i*la)));
es1(i)=f(i)−t013;
es2(i)=f(i)−t103;
es3(i)=f(i)−t113;
end
end
es11=abs(es1);
Error13=max(es11);
es22=abs(es2);
Error23=max(es22);
es33=abs(es3);
Error33=max(es33);
M3=min([Error13(:); Error23(:); Error33(:)]);
for k=[−1:0.1:1]
la=k*pi;
a020=0.5;
a115=acf8(2);
t014=a020+a115*cos(la);
end
for k=[−1:0.1:1]
```

```
la=k*pi;
b114=−2*acf8(2)/(1+acf8(3));
a021=0.5*(1+acf8(2)*b114);
t104=a021/(1+b114*cos(la));
end
for k=[−1:0.1:1]
la=k*pi;
b115=−2*acf8(3)/(acf8(2)+acf8(4));
a022=0.5*(1+acf8(2)*b115);
a116=0.5*(2*acf8(2)+b115*(1+acf8(3)));
t114=(a022+a116*cos(la))/(1+b115*cos(la));
end
for i=1:20
for k=[−1:0.1:1]
la=k*pi;
f(i)=(0.5+sum(acf8(i+1)*cos(i*la)));
ew1(i)=f(i)−t014;
ew2(i)=f(i)−t104;
ew3(i)=f(i)−t114;
end
end
ew11=abs(ew1);
Error14=max(ew11);
ew22=abs(ew2);
Error24=max(ew22);
ew33=abs(ew3);
Error34=max(ew33);
M4=min([Error14(:);Error24(:); Error34(:)]);
A01=a0;
A11=a1;
A21=(a1*b13);
B01=a03;
B11=a12;
syms W01 W11
[W01,W11]=solve(W01^2+W11^2−K11*A01==0,2*W01*W11−K11*A11==0);
w01=double(W01(1));
w11=double(W11(1));
A02=a0;
A12=(a0*b111+a1);
A22=(a1*b111);
B02=a016; B12=(a112);
syms W02 W12 W22
[W02,W12,W22]=solve(W02^2+W12^2+W22^2−K12*A02==0,2*W02*W12+2*W12*W22−K12*A12==0,2*W02*W2
2−K12*A22==0);
w02=double(W02(1));
w12=double(W12(1));
w22=double(W22(1));
A04=(a0);
A14=(a0*b114+a1);
A24=(a1*b114);
B04=a021;
syms W04 W14 W24
[W04,W14,W24]=solve(W04^2+W14^2+W24^2−K14*A04==0,2*W04*W14+2*W14*W24−K14*A14==0,2*W04*W2
4−K14*A24==0);
w04=double(W04(1));
w14=double(W14(1));
w24=double(W24(1))
for t=10:80
for n=0:1:8
for i=0:1:6
ycap(t)=(−1)^n*w01*delta11^i*X1(t−i)+(−1)^n*w11*delta11^i*X1(t−(i+1))
+(−1)^n*w02*delta12^i*X2(t−i)+(−1)^n*w12*delta12^i*X2(t−(i+1))+
```

```
(−1)^n*w22*delta12^i*X2(t−(i+2))+(−1)^n*w04*delta14^i*X4(t−i)+
(−1)^n*w14*delta14^i*X4(t−(i+1))+(−1)^n*w24*delta14^i*X4(t−(i+2));
end
end
end
for t=10:80
noise(t)=Y1(t)−ycap(t);
end
plot(noise);
stem(noise)
title('NOISE OF MULTIVARIATE DATA');
figure,noise;
acf11=autocorr(noise);
plot(acf11);
stem(acf11);
title('SAMPLE AUTO−CORRELATION OF NOISE SERIES');
figure,acf11;
n12=length(noise);
pacf11=parcorr(noise);
plot(pacf11);
stem(pacf11);
title('SAMPLE  PARTIAL AUTO−CORRELATION OF NOISE SERIES');
figure,pacf11;
fprintf('\n\n PARTICAL AUTOCORRELATION OF NOISE SERIES \n');
for t=3:80
D11(t)=(ycap(t)−Y1(t))^2;
D12=sum(D11(t));
RMSE=sqrt(D12/80);
end
plot(RMSE);
stem(RMSE);
title('ROOT MEAN SQUARE ERROR');
fprintf('\n\n The value of RMSE=%f',RMSE);
for t=3:80
D(t)=abs((ycap(t)−Y1(t))/Y1(t));
D1=sum(D(t));
MAPE=(D1/n1)*100;
end
plot(MAPE);
stem(MAPE);
title('MEAN ABSOLUTE PERCENTAGE ERROR');
fprintf('\n\n The value of MAPE=%f',MAPE);
toc;
```

References

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