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Generalized (ψ,φ)-weak Contractions In 0-complete Partial Metric Spaces

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Abstract In this paper, we prove some common fixed point theorems in 0-complete partial metric spaces. Our results extend and generalize many existing results in the literature. Some examples are included which show that the generalization is proper.

Keywords: *partial metric space, weak contraction, fixed point*

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1. Introduction and Preliminaries

Partial metric spaces were introduced by Matthews in [9] as a part of the study of denotational semantics of dataow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation [10,11,12,13,14].

Definition 1. [9] *A partial metric on a nonempty set X is a function $p: X\times X\to\mathbb{R}^+$ such that for all $x,y,z\in X$,*

(pms1) $x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y)$,

(pms2) $p(x,x)\leq p(x,y)$,

(pms3) $p(x,y)=p(y,x)$,

(pms4) $p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$.

The pair (X,p) is called a partial metric space.

If p is a partial metric on X, then the function $p^s: X\times X\to\mathbb{R}^+$ given by $p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$ is a metric on X. Each partial metric p on X inroduces a T_0 topology τ_p on X which has as a base the family of open balls $D_p(x,\varepsilon)=\{c\in X : p(x,c)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

Definition 2. [9] Let (X,p) be a partial metric space, and let $\{x_n\}$ be any sequence in X and $x\in X$. Then

(a) a sequence $\{x_n\}$ is convergent to x with respect to τ_p, if $\lim_{n\to\infty} p(x_n,x)=p(x,x)$;

(b) a sequence $\{x_n\}$ is a Cauchy sequence in (X,p) if $\lim_{n,m\to\infty} p(x_n,x_m)$ existsand is finite;

(c) (X,p) is called complete if for every Cauchy sequence $\{x_n\}$ in X there exists $x\in X$ such that $\lim_{n,m\to\infty} p(x_n,x_m)=p(x,x)$.

In 2010, Romaguera proved in [4-Theorem 2.3] that a partial metric space (X,p) is 0-complete if and only if every p^s-Caristi mapping on X has a fixed point. Since then several papers have dealt with fixed point theory for single-valued and multi-valued operators in 0-complete partial metric space (see [1-8] and references therein).

Definition 3. [4] *Let (X,p) be a partial metric space. A sequence $\{x_n\}$ in X is called a 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)=0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_p to a point $x\in X$ such that $p(x,x)=0$.*

Remark 1. [15,16] *Let (X,p) be a partial metric space. If $p(x_n,z)\to p(z,z)=0$ as $n\to\infty$, then $p(x_n,y)\to p(z,y)$ as $n\to\infty$ for all $y\in X$.*

Lemma 1. [2] *Let (X,p) be a partial metric space and let $\{y_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} p(y_{n+1},y_n)=0. \qquad (1.1)$$

If $\{y_{2n}\}$ is not 0-Cauchy sequence in (X,p), then there exists $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k>n_k>k$ and the following sequences tend to ε^+ as $k\to\infty$:

$$p\left(y_{2m_k},y_{2n_k}\right),p\left(y_{2m_k},y_{2n_{k+1}}\right),$$
$$p\left(y_{2m_{k-1}},y_{2n_k}\right),p\left(y_{2m_{k-1}},y_{2n_{k+1}}\right). \quad (1.2)$$

Definition 4. [17] *Let f and g be self maps of a set X. If* $w=fx=gx$ *for some* $x\in X$*, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair f, g of self maps is weakly compatible if they commute at their coincidence points.*

Proposition 1. [17] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.*

2. Main Results

Denote by Ψ the set of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(ψ_i) ψ is continuous nondecreasing;

(ψ_{ii}) $\psi(t)<t$ for all $t>0$ and $\psi(0)<0$.

Denote by Φ the set of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(φ_i) φ is a lower semi-continuous functions;

(φ_{ii}) $\varphi(t)<t$ for all $t>0$ and $\varphi(0)=0$.

Theorem 1. *Let* (X,p) *be a 0-complete partial metric spaces. Suppose mappings* $f,g:X\to X$ *satisfy*

$$\psi\left(p(fx,fy)\right)\le\psi\left(M(x,y)\right)-\varphi\left(M(x,y)\right) \quad (2.1)$$

where $\psi\in\Psi$ *and* $\varphi\in\Phi$ *and*

$$M(x,y)=\max\left\{\begin{array}{l}p(gx,gy),p(gx,fy),(gy,fy),\\ \dfrac{p(gx,fy)+p(gy,fx)}{2}\end{array}\right\} \quad (2.2)$$

for all $x,y\in X$*. If the range of g contains the range of f and f (X) or g (X) is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point z and* $p(v,v)=0=p(fz,fz)=p(gz,gz)$.

Proof. First, we prove that f and g have a unique point of coincidence (if it exists). If $c_1\in X$ with $fa_1=ga_1=c_1$ and $c_2\in X$ with $fa_2=ga_2=c_2$, we assume c1 6= c2. Using (2.1) and (2.2), we have

$$\begin{aligned}\psi\left(p(c_1,c_2)\right)&=\psi\left(p(fa_1,fa_2)\right)\\ &\le\psi\left(\max\left\{\begin{array}{l}p(ga_1,ga_2),p(ga_1,fa_1),p(ga_2,fa_2),\\ \dfrac{p(ga_1,fa_2)+p(ga_2,fa_1)}{2}\end{array}\right\}\right)\\ &-\varphi\left(\max\left\{\begin{array}{l}p(ga_1,ga_2),p(ga_1,fa_1),p(ga_2,fa_2),\\ \dfrac{p(ga_1,fa_2)+p(ga_2,fa_1)}{2}\end{array}\right\}\right)\\ &=\psi\left(\max\left\{\begin{array}{l}p(c_1,c_2),p(c_1,c_1),p(c_2,c_2),\\ \dfrac{p(c_1,c_2)+p(c_2,c_1)}{2}\end{array}\right\}\right)\\ &-\varphi\left(\max\left\{\begin{array}{l}p(c_1,c_2),p(c_1,c_1),p(c_2,c_2),\\ \dfrac{p(c_1,c_2)+p(c_2,c_1)}{2}\end{array}\right\}\right)\\ &=\psi\left(p(c_1,c_2)\right)-\varphi\left(p(c_1,c_2)\right)\quad\text{(by }(pms2))\\ &<\psi\left(p(c_1,c_2)\right),\end{aligned}$$

which is a contradiction. Thus $p(c_1,c_2)$, that is, $c_1=c_2$. Thus, the point of coincidence of f and g is unique (if it exists).

We construct a sequence $\{y_n\}\subset X$ as follows:

Let $x_0\in X$. Choose a point $x_1\in X$ such that $fx_0=gx_0=y_1$. This can be done, as the range of g contains the range of f. Continuing in the same way, having chosen $x_n\in X$, we get $x_{n+1}\in X$ such that $fx_n=gx_{n+1}=y_n$ (say). Therefore, we get the sequence $\{y_n\}=\{gx_{n+1}\}$ such that $fx_n=gx_{n+1}=y_n$ for all $x\in\mathbb{N}$. Consider the two possible cases:

(i) $p(y_{n+1},y_n)=0$ for some $n\in\mathbb{N}$.

In this case $fx_n=gx_n=y_n$ is a point of coincidence and then the proof is finished.

(ii) $p(y_{n+1},y_n)=0$ for every $n\in\mathbb{N}$.

From (2.1) and (2.2), using properties of functions ψ and φ, we obtain

$$\begin{aligned}\psi\left(p(y_{n+1},y_n)\right)&=\psi\left(p(fx_{n+1},fx_n)\right)\\ &\le\psi\left(M(x_{n+1},x_n)\right)-\varphi\left(M(x_{n+1},x_n)\right)\\ &\le\psi\left(M(x_{n+1},x_n)\right)\end{aligned}$$

which implies that

$$p(y_{n+1},y_n)\le M(x_{n+1},x_n).$$

Then, we have

$$\begin{aligned}&M(x_{n+1},x_n)\\ &=\max\left\{\begin{array}{l}p(gx_{n+1},gx_n),p(gx_{n+1},fx_{n+1}),p(gx_n,fx_n),\\ \dfrac{p(gx_{n+1},fx_n)+p(gx_n,fx_{n+1})}{2}\end{array}\right\}\\ &=\max\left\{\begin{array}{l}p(y_n,y_{n-1}),p(y_n,y_{n+1}),p(y_{n-1},y_n),\\ \dfrac{p(y_n,y_n)+p(y_{n-1},y_{n+1})}{2}\end{array}\right\}\\ &\le\max\left\{\begin{array}{l}p(y_n,y_{n-1}),p(y_n,y_{n+1}),\\ \dfrac{p(y_{n-1},y_n)+p(y_n,y_{n+1})}{2}\end{array}\right\}\\ &=\max\left\{p(y_n,y_{n-1}),p(y_n,y_{n+1})\right\}.\end{aligned}$$

If $p(y_n,y_{n+1})>p(y_n,y_{n-1})$, then $M(x_{n+1},x_n)=p(y_n,y_{n+1})>0$. Furthermore, it implies that

$$\psi\left(p\left(y_{n+1}, y_n\right)\right) \le \psi\left(p\left(y_{n+1}, y_n\right)\right) - \varphi\left(p\left(y_{n+1}, y_n\right)\right)$$

which is a contradiction. Therefore, we have

$$p\left(y_{n+1}, y_n\right) \le M\left(x_{n+1}, x_n\right) \le p\left(y_n, y_{n-1}\right). \quad (2.3)$$

It follows from (2.3) that the sequence $\left\{p\left(gx_{n+1}, gx_n\right)\right\}$ is nonincreasing. Therefore,

$$\lim_{n\to\infty} p\left(y_{n+1}, y_n\right) = \lim_{n\to\infty} M\left(x_{n+1}, x_n\right) = p^* \ge 0.$$

Letting $n \to \infty$ in inequality

$$\psi\left(p\left(y_{n+1}, y_n\right)\right) \le \psi\left(M\left(x_{n+1}, x_n\right)\right) - \varphi\left(M\left(x_{n+1}, x_n\right)\right)$$

we obtain $\psi\left(p^*\right) \le \psi\left(p^*\right) - \varphi\left(p^*\right)$ and $p^* = 0.$ Thus

$$\lim_{n\to\infty} p\left(y_{n+1}, y_n\right) = 0. \quad (2.4)$$

We next prove that $\left\{fx_n\right\} = \left\{gx_{n+1}\right\} = \left\{y_n\right\}$ is a 0-Cauchy sequence in the space $\left(X, p\right)$. It is sufficient to show that $\left\{fx_{2n}\right\}$ is a 0-Cauchy sequence. Suppose the opposite. Then using Lemma 1, we see that there exist $\varepsilon > 0$ and two sequences $\left\{m_k\right\}$ and $\left\{n_k\right\}$ of positive integers and sequences

$$\begin{aligned} &p\left(y_{2m_k}, y_{2n_k}\right), p\left(y_{2m_k}, y_{2n_{k+1}}\right), \\ &p\left(y_{2m_{k-1}}, y_{2n_k}\right), p\left(y_{2m_{k-1}}, y_{2n_{k+1}}\right). \end{aligned} \quad (2.5)$$

all tend to ε^+, when $k \to \infty$. Using (2.1) and (2.2), we get that

$$\begin{aligned} &\psi\left(p\left(y_{2m_k}, y_{2n_{k+1}}\right)\right) = \psi\left(p\left(fx_{2m_k}, fx_{2n_{k+1}}\right)\right) \\ &\le \psi\left(\max\left\{\begin{array}{l} p\left(gx_{2m_k}, gx_{2n_{k+1}}\right), p\left(gx_{2m_k}, fx_{2m_k}\right), \\ p\left(gx_{2n_{k+1}}, fx_{2n_{k+1}}\right), \\ \dfrac{p\left(gx_{2m_k}, fx_{2n_{k+1}}\right) + p\left(gx_{2n_{k+1}}, fx_{2m_k}\right)}{2} \end{array}\right\}\right) \\ &-\varphi\left(\max\left\{\begin{array}{l} p\left(gx_{2m_k}, gx_{2n_{k+1}}\right), p\left(gx_{2m_k}, fx_{2m_k}\right), \\ p\left(gx_{2n_{k+1}}, fx_{2n_{k+1}}\right), \\ \dfrac{p\left(gx_{2m_k}, fx_{2n_{k+1}}\right) + p\left(gx_{2n_{k+1}}, fx_{2m_k}\right)}{2} \end{array}\right\}\right) \\ &= \psi\left(\max\left\{\begin{array}{l} p\left(y_{2m_{k-1}}, y_{2n_k}\right), p\left(y_{2m_{k-1}}, y_{2m_k}\right), \\ p\left(y_{2n_k}, y_{2n_{k+1}}\right), \\ \dfrac{p\left(y_{2m_{k-1}}, y_{2n_{k+1}}\right) + p\left(y_{2n_k}, fx_{2m_k}\right)}{2} \end{array}\right\}\right) \\ &-\varphi\left(\max\left\{\begin{array}{l} p\left(y_{2m_{k-1}}, y_{2n_k}\right), p\left(y_{2m_{k-1}}, y_{2m_k}\right), \\ p\left(y_{2n_k}, y_{2n_{k+1}}\right), \\ \dfrac{p\left(y_{2m_{k-1}}, y_{2n_{k+1}}\right) + p\left(y_{2n_k}, fx_{2m_k}\right)}{2} \end{array}\right\}\right). \end{aligned} \quad (2.6)$$

Using (2.4) and (2.5), we obtain

$$\max\left\{\begin{array}{l} p\left(y_{2m_{k-1}}, y_{2n_k}\right), p\left(y_{2m_{k-1}}, y_{2m_k}\right), \\ p\left(y_{2n_k}, y_{2n_{k+1}}\right), \\ \dfrac{p\left(y_{2m_{k-1}}, y_{2n_{k+1}}\right) + p\left(y_{2n_k}, fx_{2m_k}\right)}{2} \end{array}\right\} \to \varepsilon^+ \quad as \quad k \to \infty.$$

Letting $k \to \infty$ in (2.6), we get that $\psi\left(\varepsilon\right) \le \psi\left(\varepsilon\right) - \varphi\left(\varepsilon\right)$ which is a contradiction if $\varepsilon > 0.$

This show that $\left\{fx_{2n}\right\}$ is a 0-Cauchy sequence in the space $\left(X, p\right)$ and $\left\{fx_n\right\}$ is a 0-Cauchy sequence in the space $\left(X, p\right)$.

If $g\left(X\right)$ is closed in $\left(X, p\right)$ then there exist $z, v \in X$ such that $v = gz$ and

$$\lim_{n\to\infty} p\left(y_n, v\right) = \lim_{n,m\to\infty} p\left(y_n, y_m\right) = p\left(v, v\right) = 0.$$

Now, putting $x = x_n,$ $y = z,$ $gz = v$ and $y_n = fx_n = gx_{n+1}$ in (2.1) and (2.2) we have

$$\begin{aligned} &\psi\left(p\left(fx_n, fz\right)\right) \\ &\le \psi\left(\max\left\{\begin{array}{l} p\left(gx_n, gz\right), p\left(gx_n, fx_n\right), p\left(gz, fz\right), \\ \dfrac{p\left(gx_n, fz\right) + p\left(gz, fx_n\right)}{2} \end{array}\right\}\right) \quad (2.7) \\ &-\varphi\psi\left(\max\left\{\begin{array}{l} p\left(gx_n, gz\right), p\left(gx_n, fx_n\right), p\left(gz, fz\right), \\ \dfrac{p\left(gx_n, fz\right) + p\left(gz, fx_n\right)}{2} \end{array}\right\}\right). \end{aligned}$$

Letting $k \to \infty$ in (2.7) and by Remark 1, we obtain

$$\psi\left(p\left(gz, fz\right)\right) \le \psi\left(p\left(gz, fz\right)\right) - \varphi\left(p\left(gz, fz\right)\right)$$

This implies $p\left(gz, fz\right) = 0,$ that is, $gz = fz.$ Hence, f and g have a unique point of coincidence. By Proposition 1, f and g have a unique common fixed point.

When $f\left(X\right)$ is closed set in $\left(X, p\right)$ the proof similar.

Corollary 1. *Let* $\left(X, p\right)$ *be a* 0*-complete partial metric spaces. Suppose mapping* $f : X \to X$ *satisfy*

$$\psi\left(p\left(fx, fy\right)\right) \le \psi\left(M\left(x, y\right)\right) - \varphi\left(M\left(x, y\right)\right) \quad (2.8)$$

where $\psi \in \Psi$ *and* $\varphi \in \Phi$ *and*

$$M(x,y)=\max\left\{\begin{array}{l} p(x,y),\, p(x,fx),\, p(y,fy), \\ \dfrac{p(x,fy)+p(y,fx)}{2} \end{array}\right\} \quad (2.9)$$

for all $x, y \in X$. *Then f has a unique fixed point* $v \in X$ *and* $p(v,v)=0$.

Proof. Taking $g = I_X$ (the identity mapping of X), along the lines of the proof of Theorem 1, we get the desired results. In view of the analogy, we skip the details of the proof.

Corollary 2. *Let* (X,p) *be a* 0-*complete partial metric spaces*. *Suppose mappings* $f, g : X \to X$ *satisfy*

$$P(fx,fy) \le M(x,y) - \varphi(M(x,y)) \quad (2.10)$$

where $\varphi \in \Phi$ *and*

$$M(x,y)=\max\left\{\begin{array}{l} p(gx,gy),\, p(gx,fx),\, p(gy,fy), \\ \dfrac{p(gx,fy)+p(gx,fx)}{2} \end{array}\right\} \quad (2.11)$$

for all $x, y \in X$. *If the range of g contains the range of f and f (X) or g (X) is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point z and*

$$p(v,v)=0=p(fz,fz)=p(gz,gz).$$

Proof. To prove the above corollary it suffices to take $\psi(t)=t$ in Theorem 1.

Corollary 3. *Let (X; p) be a* 0-*complete partial metric spaces. Suppose mapping f : X ! X satisfy*

$$p(fx,fy) \le M(x,y) - \varphi(M(x,y)) \quad (2.12)$$

where $\varphi \in \Phi$ *and*

$$M(x,y)=\max\left\{\begin{array}{l} p(x,y),\, p(x,fx),\, p(y,fy), \\ \dfrac{p(x,fy)+p(y,fx)}{2} \end{array}\right\} \quad (2.13)$$

for all $x, y \in X$. *Then f has a unique fixed point* $v \in X$ *and* $p(v,v)=0$.

Proof. Taking $g = I_X$ in Corollary 2, we have desired results.

Corollary 4. [2] *Let* (X,p) *be a* 0-*complete partial metric spaces. Suppose mappings* $f, g : X \to X$ *satisfy*

$$p(fx,fy) \le kM(x,y) \quad (2.14)$$

where $k \in [0,1)$ *and*

$$M(x,y)=\max\left\{\begin{array}{l} p(gx,gy),\, p(gx,fx),\, p(gy,fy), \\ \dfrac{p(gx,fy)+p(gy,fx)}{2} \end{array}\right\} \quad (2.15)$$

for all $x, y \in X$. *If the range of g contains the range of f and* $f(X)$ *or* $g(X)$ *is a closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point z and* $p(v,v)=0=p(fz,fz)$ $=p(gz,gz)$.

Proof. To prove the above corollary it suffices to take $\varphi(t)=(1-k)t$ in Corollary 2.

Corollary 5. *Let* (X,p) *be a* 0-*complete partial metric spaces. Suppose mapping* $f : X \to X$ *satisfy*

$$p(fx,fy) \le kM(x,y) \quad (2.16)$$

where $k \in [0,1)$ *and*

$$M(x,y)=\max\left\{\begin{array}{l} p(x,y),\, p(x,fx),\, p(y,fy), \\ \dfrac{p(x,fy)+p(y,fx)}{2} \end{array}\right\} \quad (2.17)$$

for all $(x,y) \in X$. *Then f has a unique fixed point* $v \in X$ *and* $p(v,v)=0$.

Corollary 6. [18] *Let* (X,p) *be a* 0-*complete partial metric spaces. Suppose mapping* $f : X \to X$ *and there exist nonnegative constants* b_i *satisfying* $\sum_{i=1}^{5} b_i < 1$ *such that, for each* $x, y \in X$

$$p(fx,fy) \le b_1 p(x,y) + b_2 p(x,fx) + b_3 p(y,fy) + b_4 p(x,fy) + b_5 p(y,fx). \quad (2.18)$$

Then f has a unique fixed point $v \in X$ *and* $p(v,v)=0$.

Corollary 6 is a simple consequence of Corollary 5.

Corollary 7. *Let* (X,p) *be a* 0-*complete partial metric spaces. Suppose mapping* $f : X \to X$ *satisfy*

$$p(fx,fy) \le kp(x,y) \quad (2.19)$$

for each $x, y \in X$ *and* $k \in [0,1)$. *Then f has a unique fixed point* $v \in X$ *and* $p(v,v)=0$.

Proof. It follows from Corollary 6.

Conclusion 1. *1. Our theorems and corolaries which include the corresponding results announced in Boyd and Wong* [19] *(1969), Rhoades* [20] *(1977) as special cases fundamentally improve and generalize the results of Ahmad et al.* [2] *(2012) and Radenović* [18] *(2013).*

2. Taking $b_1 = b_4 = b_5 = 0$ *and* $b_2 + b_3 = \lambda \in \left[0, \frac{1}{2}\right)$ *in Corollary 6, we obtain extension of Kannan Theorem on a 0-complete partial metric spaces.*

3. Taking $b_4 = b_5 = 0$ *and* $b_1 + b_2 + b_3 \in [0,1)$ *in Corollary 6, we obtain extension of Reich Theorem on a* 0-*complete partial metric spaces.*

4. Taking $b_1 = b_2 = b_3 = 0$ *and* $b_4 = b_5 = \lambda \in \left[0, \frac{1}{2}\right)$ *in Corollary 6, we obtain extension of Chatterjea Theorem on a* 0-*complete partial metric spaces.*

Now, we give a example which illustrate Theorem 1.

Example 1. *Let* $X = \{0,1,2,3\}$, *and le* $p : X \to X \to \mathbb{R}^+$ *be defined by* $p(x,y) = \max\{x,y\} + |x-y|$ *for all*

$x, y \in X$. Then, (X, p) is a 0-complete partial metric space. Define $f, g : X \to X$

$$f0 = 0,\ f1 = 0,\ f2 = 0,\ f3 = 1,$$
$$g0 = 0, g1 = 1, g2 = 2, g3 = 3.$$

Take $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$ for each $t \geq 0$.

We distinguish five cases:

***Case 1:** If (x = 0 and y = 0) or (x = 0 and y = 1) or (x = 0 and y = 2) or (x = 1 and y = 1) or (x = 1 and y = 2) or (x = 2 and y = 2), we have*

$$\psi(p(fx, fy)) = 0 \leq \psi(M(x,y)) - \varphi(M(x,y))$$

where

$$M(x,y) = \max\left\{ p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)+p(gy,fx)}{2} \right\}.$$

***Case 2:** If x = 0 and y = 3, we have*

$$\psi(p(f0, f3)) = p(0,1) = 2$$

and

$$\begin{aligned}
&\psi(M(0,3)) - \varphi(M(0,3)) \\
&= \max\left\{ p(g0,g3), p(g0,f0), p(g3,f3), \frac{p(g0,f3)+p(g3,f0)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(g0,g3), p(g0,f0), p(g3,f3), \frac{p(g0,f3)+p(g3,f0)}{2} \right\} \\
&= \max\left\{ p(0,3), p(0,0), p(3,1), \frac{p(0,1)+p(3,0)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(0,3), p(0,0), p(3,1), \frac{p(0,1)+p(3,0)}{2} \right\} \\
&= \max\left\{ 6,0,5,\frac{2+6}{2} \right\} - \frac{1}{2}\max\left\{ 6,0,5,\frac{2+6}{2} \right\} \\
&= 6 - 3 = 3.
\end{aligned}$$

Hence,

$$\psi(p(f0, f3)) = 2 \leq \psi(M(0,3)) - \varphi(M(0,3)) = 3.$$

***Case 3**: If x = 1 and y = 3, we have*

$$\psi(p(f1, f3)) = p(0,1) = 2$$

and

$$\begin{aligned}
&\psi(M(1,3)) - \varphi(M(1,3)) \\
&= \max\left\{ p(g1,g3), p(g1,f1), p(g3,f3), \frac{p(g1,f3)+p(g3,f1)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(g1,g3), p(g1,f1), p(g3,f3), \frac{p(g1,f3)+p(g3,f1)}{2} \right\} \\
&= \max\left\{ p(1,3), p(1,0), p(3,1), \frac{p(1,1)+p(3,0)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(1,3), p(1,0), p(3,1), \frac{p(1,1)+p(3,0)}{2} \right\} \\
&= \max\left\{ 5,2,5,\frac{1+6}{2} \right\} - \frac{1}{2}\max\left\{ 5,2,5,\frac{1+6}{2} \right\} \\
&= 5 - \frac{5}{2} = \frac{5}{2}.
\end{aligned}$$

Thus,

$$\psi(p(f1, f3)) = 2 \leq \psi(M(1,3)) - \varphi(M(1,3)) = \frac{5}{2}.$$

***Case 4:** If x = 2 and y = 3, we have*

$$\psi(p(f2, f3)) = p(0,1) = 2$$

and

$$\begin{aligned}
&\psi(M(2,3)) - \varphi(M(2,3)) \\
&= \max\left\{ p(g2,g3), p(g2,f2), p(g3,f3), \frac{p(g2,f3)+p(g3,f2)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(g2,g3), p(g2,f2), p(g3,f3), \frac{p(g2,f3)+p(g3,f2)}{2} \right\} \\
&= \max\left\{ p(2,3), p(2,0), p(3,1), \frac{p(2,1)+p(3,0)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(2,3), p(2,0), p(3,1), \frac{p(2,1)+p(3,0)}{2} \right\} \\
&= \max\left\{ 4,4,5,\frac{3+6}{2} \right\} - \frac{1}{2}\max\left\{ 4,4,5,\frac{3+6}{2} \right\} \\
&= 5 - \frac{5}{2} = \frac{5}{2}.
\end{aligned}$$

Thus,

$$\psi(p(f2, f3)) = 2 \leq \psi(M(2,3)) - \varphi(M(2,3)) = \frac{5}{2}.$$

Case 5: *If x = 3 and y = 3, we have*

$$\psi(p(f3, f3)) = p(1,1) = 1$$

and

$$\begin{aligned}
&\psi(M(3,3)) - \varphi(M(3,3)) \\
&= \max\left\{ p(g3,g3), p(g3,f3), p(g3,f3), \frac{p(g3,f3)+p(g3,f3)}{2} \right\} \\
&-\frac{1}{2}\max\left\{ p(g3,g3), p(g3,f3), p(g3,f3), \frac{p(g3,f3)+p(g3,f3)}{2} \right\} \\
&= \max\left\{ p(3,3), p(3,1), p(3,1), \frac{p(3,1)+p(3,1)}{2} \right\}
\end{aligned}$$

$$-\frac{1}{2}\max\left\{p(3,3),\, p(3,1),\, p(3,1), \frac{p(3,1)+p(3,1)}{2}\right\}$$
$$=\max\left\{3,5,5,\frac{5+5}{2}\right\}-\frac{1}{2}\max\left\{3,5,5,\frac{5+5}{2}\right\}$$
$$=5-\frac{5}{2}=\frac{5}{2}.$$

Thus,

$$\psi\left(p\left(f3, f3\right)\right)=1\leq\psi\left(M\left(3,3\right)\right)-\varphi\left(M\left(3,3\right)\right)=\frac{5}{2}.$$

It is obvious that all the condition of Theorem 1 is satisfied. Therefore, we apply Theorem 1 and f and g have a unique common fixed point, i.e. 0.

The following is a example which illustrate our results and that the generalizations are proper.

Example 2. *Let* $X=[0,1]\cap\mathbb{Q}$, *and let* $p: X\times X\to\mathbb{R}^{+}$ *be defined by* $p(x,y)=\max\{x,y\}$ *for all* $x,y\in X$. *Then,* (X,p) *is a 0-complete partial metric space, but it is not complete partial metric space. Define* $f,g: X\to X$ *by*

$$fx=\begin{cases}0 & if \quad x=1,\\ \dfrac{x}{5} & otherwise\end{cases} \quad and \quad gx=\begin{cases}1 & if \quad x=1,\\ x & otherwise.\end{cases}$$

Then all the conditions of Theorem 1 are satisfied with $\psi(t)=t$ *and* $\varphi(t)=\frac{t}{5}$ *and f and g have a unique common fixed point, i.e.* 0.

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