

# On Stability of Steady–State Three–Dimensional Flows of an Ideal Incompressible Fluid

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**Abstract** The problem on linear stability of steady–state three–dimensional (3D) flows of an inviscid incompressible fluid, completely filling a volume with a solid boundary, is studied in the absence mass forces. It is proved by the direct Lyapunov method that these flows are absolutely unstable with respect to small 3D perturbations. The a priori exponential estimate from below, which testifies to growth of perturbations under consideration in time, is constructed.

**Keywords:** *an ideal incompressible fluid, steady–state 3D flows, stability, small 3D perturbations, the direct lyapunov method, a priori exponential lower estimate, instability*

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## 1. Introduction

The problem on linear stability of steady–state 3D flows of an inviscid incompressible fluid, entirely filling a vessel with rigid walls, in the absence mass forces [1] continues to be one of classical problems for modern mathematical theory of hydrodynamic stability.

The relevance of this problem is due to the fact that the model of an ideal incompressible fluid is used a kind of perfect kernel for more complex mathematical models of hydrodynamics which take account of those or other physical properties of real liquids in one way or another [2,3,4,5]. Naturally, characteristic features the model of an inviscid incompressible fluid are somehow manifested in all other mathematical models hydrodynamics. Therefore, thorough investigation of stability, in particular, steady–state 3D flows of an ideal incompressible fluid, completely filling a volume with a solid boundary, in the absence mass forces with regard to small 3D perturbations enables us to understand better issues of steady–state flows stability as applied to other hydrodynamic models also.

Unfortunately, as far as it knows the author of this article, the problem on linear stability of steady–state 3D flows of an inviscid incompressible fluid in a vessel with rigid walls, entirely filled with it, and in the absence mass forces has still not received satisfactory solution by either the first (spectral) nor the second (direct) Lyapunov methods [6,7,8].

The fact is that if one try to solve the problem by the spectral method, boundary value problem on finding eigenvalues and eigenfunctions for constitutive ordinary

differential equation with unknown variable coefficients and, most importantly, in area with unknown fixed boundary will occur at intermediate stage, whose resolution is not under force for theoretical approaches developed to date [9-15]. As for the direct Lyapunov method, its use to solve the discussed here problem on stability is hindered by the lack of algorithms for constructing the Lyapunov functionals which grow on time along solutions to the corresponding initial–boundary value problems for small perturbations [1,10,11,16,17].

Nevertheless, as it is familiar to the author of this paper, a few key results are still discovered in the course studying of the problem on linear stability of steady–state 3D flows of an ideal incompressible fluid, completely filling a volume with a solid boundary, in the absence mass forces [1,16,18].

Namely, variational principles for the kinetic energy integral of steady–state 3D flows of an inviscid incompressible fluid, which fills entirely a vessel with rigid walls, in the absence mass forces are constructed in [1,16]. Using these principles, the Lyapunov functionals are calculated as second variations of the kinetic energy integral itself [1] or bundle of the kinetic energy functional and additional integral of motion over the volume, completely occupied by the fluid, from arbitrary function of the Lagrangian coordinates [16]. Unfortunately, in the case of small 3D perturbations, the Lyapunov functionals, created by the authors of articles [1,16], have properties of distinctness/constancy in sign only for states of equilibrium (rest). For the considered in [1,16] steady–state flows, the authors of these papers could not find cases of distinctness/constancy in sign for their Lyapunov functionals (or, in other words, conditions/criteria for linear stability).

Based on results of articles [1,16], the authors of paper [18] have set the goal to show, among other things, instability of steady-state 3D flows of an ideal incompressible fluid, which fills entirely a vessel with a solid boundary, in the absence mass forces with respect to small 3D perturbations. However, they managed to move in this direction though for wide enough, but still only particular class of steady-state flows, constructed the a priori exponential estimate from below on growth of small 3D perturbations of these flows over time with the help of the virial (the Lyapunov functional in the form of integral over the volume, completely filled with the fluid, from square of the Lagrangian displacements field [19]). Unfortunately, the applied by article [18] authors method of constructing the growing in time Lyapunov functional possesses a number of characteristics which exclude possibility of its spread to other steady-state flows. Moreover, as the increment, contained in the constructed by paper [18] authors lower estimate, can take values not on all positive real axle shaft, but only at its specific interval, there is very high probability generally to be empty set for the subclass of small 3D perturbations increasing on time in agreement with this estimate.

In the article, for solving the problem on linear instability of steady-state 3D flows of an inviscid incompressible fluid, entirely filling a vessel with rigid walls, in the absence mass forces with regard to 3D perturbations, it is proposed to use the original analytical method [20] which, firstly, is already well proven itself in the process of research extensive range of linear problems the mathematical theory of hydrodynamic stability states of equilibrium (rest) and steady-state flows of gases, fluids, and plasma [21-32] and, secondly, is free from disadvantages peculiar to the method of paper [18].

The essence of the announced above analytical method [20-32] is algorithmic construction of the Lyapunov functionals which grow over time along solutions to the studied mixed problems for small perturbations. It allows us to get results on both theoretical (at semi-infinite temporal intervals [6,7,8]) and practical (on finite intervals of time [33,34]) instability for states of equilibrium (rest) and steady-state flows of gases, fluids, and plasma with respect to small perturbations. At that, which is very important, it is not required to know explicit form of solutions to the considered initial-boundary value problems for small perturbations.

The structure of construction main body of the presented below article is the following: a) in the second section, the formulation of exact mixed problem is provided, its properties are reported selectively, and its stationary solutions, corresponding to steady-state 3D flows of an ideal incompressible fluid, which fills completely a volume with a solid boundary, in the absence

mass forces, are written in general form; b) in the third section, the formulation of initial-boundary value problem, serving as the result of linearization for exact mixed problem in the vicinity of its stationary solutions, is given, some of its characteristic features are discussed, and the hypothesis about absolute instability of steady-state 3D flows of an inviscid incompressible fluid, entirely filling a vessel with rigid walls, in the absence mass forces with regard to 3D perturbations in linear approximation is set up; c) finally, in the penultimate (fourth) section, this hypothesis is substantiated by direct constructing the a

priori exponential lower estimate on growth of small 3D perturbations in time.

## 2. Formulation of Exact Problem

3D flows of an ideal incompressible fluid, which fills completely a volume  $\tau$  with a solid boundary  $\partial\tau$  (Figure 1), are studied in the absence mass forces.

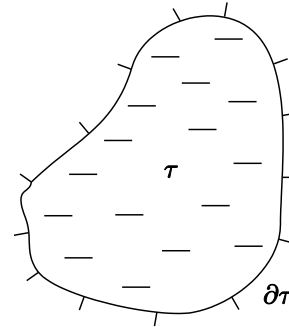


Figure 1. A vessel  $\tau$  with rigid walls  $\partial\tau$

These flows are described by solutions to initial-boundary value problem of the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p, \text{div} \mathbf{u} = 0 \text{ in } \tau \tag{1}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\tau; \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$$

where  $\mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$  is the velocity field;  $p(\mathbf{x}, t)$  is the pressure field;  $\mathbf{x} = (x_1, x_2, x_3)$  is the Cartesian coordinates;  $t$  is the time;  $\mathbf{n} = (n_1, n_2, n_3)$  is the external unit normal to the surface  $\partial\tau$ ;  $\mathbf{u}_0(\mathbf{x}) = (u_{01}, u_{02}, u_{03})$  is the initial velocity field of the fluid. It is assumed that the function  $\mathbf{u}_0$  converts the second and the third ratios of the mixed problem (1) into identities.

The initial-boundary value problem (1) has the kinetic energy integral in the form

$$E \equiv \frac{1}{2} \int_{\tau} u_j u_j d\tau = \text{const} \tag{2}$$

Here  $d\tau \equiv dx_1 dx_2 dx_3$ ; summation from one to three on repeating vector and tensor bottom indices of lower-case Latin letters is fulfilled throughout the paper.

If one act by the differential operator rot on the first equation of the mixed problem (1), it is not difficult to obtain the ratio characterizing development of the vorticity field  $\boldsymbol{\omega}(\mathbf{x}, t) = (\omega_1, \omega_2, \omega_3) \equiv \text{rot} \mathbf{u}$  on time:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \text{rot}(\mathbf{u} \times \boldsymbol{\omega}) \tag{3}$$

It is believed further that the initial-boundary value problem (1) and the equation (3) have exact stationary solutions

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) = (U_1, U_2, U_3), p = P(\mathbf{x}) \tag{4}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{x}) = (\Omega_1, \Omega_2, \Omega_3)$$

which satisfy relations

$$(\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla P, \operatorname{div}\mathbf{U} = 0, \operatorname{rot}(\mathbf{U} \times \boldsymbol{\Omega}) = 0 \quad (5)$$

within the volume  $\tau$  and the condition of impermeability

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad (6)$$

on its solid boundary  $\partial\tau$ . These solutions correspond exactly to steady-state 3D flows of an inviscid incompressible fluid, entirely filling a vessel with rigid walls, in the absence mass forces, whose stability with respect to small 3D perturbations will be considered further.

So, the aim of the following study is to prove absolute linear instability of exact stationary solutions (4)–(6) to the mixed problem (1) and the equation (3) with regard to 3D perturbations.

### 3. Formulation of Linearized Problem

To achieve this goal, linearization of the initial-boundary value problem (1) and the relation (3) is carried out in the neighborhood of exact stationary solutions (4)–(6), leading to mixed problem of the form

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}' \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = -\nabla p', \operatorname{div}\mathbf{u}' = 0 \quad (7)$$

$$\frac{\partial \boldsymbol{\omega}'}{\partial t} = \operatorname{rot}(\mathbf{u}' \times \boldsymbol{\Omega} + \mathbf{U} \times \boldsymbol{\omega}') \text{ in } \tau; \mathbf{u}' \cdot \mathbf{n} = 0 \text{ on } \partial\tau$$

$$\mathbf{u}'(\mathbf{x}, 0) = \mathbf{u}'_0(\mathbf{x})$$

where  $\mathbf{u}'(\mathbf{x}, t) = (u'_1, u'_2, u'_3)$ ,  $p'(\mathbf{x}, t)$ , and  $\boldsymbol{\omega}'(\mathbf{x}, t) = (\omega'_1, \omega'_2, \omega'_3)$  are small 3D perturbations for fields of velocity, pressure, and vorticity;  $\mathbf{u}'_0(\mathbf{x}) = (u'_{01}, u'_{02}, u'_{03})$  is the initial small perturbation for the velocity field of the fluid, which turns the second and the fourth connections of the initial-boundary value problem (7) into identities.

Unfortunately, analogue of the kinetic energy functional for the mixed problem (7) is still not detected.

However, this analogue can be constructed, if one subject solutions to the initial-boundary value problem (7) in addition to special requirement, namely the condition of “isovorticity” [1,16]. This requirement is essentially integral form of the “freezing-in” condition for vortex lines in virtual displacements field of fluid particles and is expressed as equality of velocity circulations along contours resulting from each other while smooth mapping of the vessel  $\tau$  on itself, preserving its volume (Figure 2).

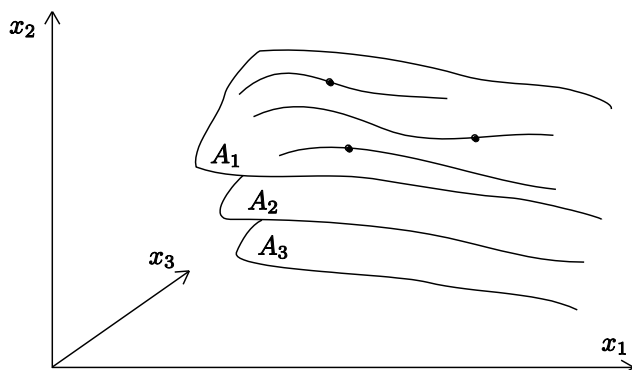


Figure 2. “Isovortical” foliation the phase space of solutions (segments of curves with dots depict sections of trajectories with fluid particles on the sheet  $A_1$ , moving thereon)

“Isovortical” small 3D perturbations (7) can be described accessible to all using the Lagrangian displacements field  $\boldsymbol{\xi}(\mathbf{x}, t) = (\xi_1, \xi_2, \xi_3)$  [19]:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{u}' + \operatorname{rot}(\mathbf{U} \times \boldsymbol{\xi}) \quad (8)$$

Taking into consideration the equation (8), it is not hard to restate the mixed problem (7) in the form

$$\begin{aligned} & \frac{\partial^2 \xi_j}{\partial t^2} + 2U_k \frac{\partial^2 \xi_j}{\partial x_k \partial t} + U_k \frac{\partial}{\partial x_k} \left( U_m \frac{\partial \xi_j}{\partial x_m} \right) \\ & = -\frac{\partial p'}{\partial x_j} - \xi_k \frac{\partial^2 P}{\partial x_k \partial x_j}, \frac{\partial \xi_j}{\partial x_j} = 0, \\ & \omega'_j = \Omega_k \frac{\partial \xi_j}{\partial x_k} - \xi_k \frac{\partial \Omega_j}{\partial x_k} \text{ in } \tau; \\ & \xi_j n_j = 0 \text{ on } \partial\tau; \boldsymbol{\xi}(\mathbf{x}, 0) = \boldsymbol{\xi}_0(\mathbf{x}) \\ & \frac{\partial \boldsymbol{\xi}}{\partial t}(\mathbf{x}, 0) = \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \right)_0(\mathbf{x}) \end{aligned} \quad (9)$$

Here  $\boldsymbol{\xi}_0(\mathbf{x}) = (\xi_{01}, \xi_{02}, \xi_{03})$  is the initial field of the Lagrangian displacements, and  $(\partial \boldsymbol{\xi} / \partial t)_0(\mathbf{x})$  is the initial partial first-order time derivative of the Lagrangian displacements field. It is believed that the second, the third, and the fourth ratios from the system (9) are fulfilled for functions  $\boldsymbol{\xi}_0$  and  $(\partial \boldsymbol{\xi} / \partial t)_0$ .

Analogue of the kinetic energy functional for the initial-boundary value problem (8), (9) represents integral of the form [1]

$$E_1 \equiv \frac{1}{2} \int_{\tau} (u'_j u'_j + \omega'_j e_{jkm} U_k \xi_m) d\tau = \text{const} \quad (10)$$

where  $e_{jkm}$  is the covariant third-order pseudotensor of the weight  $-1$  [35]. It is not difficult to make sure by direct verification that the functional  $E_1$  coincides in form with the second variation  $\delta^2 E$  of integral  $E$  (2) which is calculated in the vicinity of exact stationary solutions (4)–(6) using the condition of “isovorticity” and is recorded in appropriate notations [1,16].

Taking into account the form of functional  $E_1$  (10), it is not hard to conclude that only solutions, which meet equilibrium (rest) states of the considered fluid, namely

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) \equiv 0, p = P(\mathbf{x}) \equiv \text{const}, \boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{x}) \equiv 0$$

will be stable (at that, absolutely stable!) among exact stationary solutions (4)–(6) to the mixed problem (1) and the equation (3) with respect to “isovortical” small 3D perturbations (8), (9). Indeed, the integral  $E_1$  becomes non-negative

$$E_1 = \frac{1}{2} \int_{\tau} u'_j u'_j d\tau = \text{const} \geq 0$$

only in this case, which confirms the aforesaid.

In all other cases, the functional  $E_1$  (10) does not serve neither definite or constant in sign for “isovortical” small 3D perturbations (8), (9), and therefore, exact stationary

solutions (4)–(6) to the initial–boundary value problem (1) and the relation (3), which correspond with steady–state 3D flows of the studied fluid, can come to be absolutely unstable with regard to these perturbations.

For example, the property of fundamental nature

$$\int_{\tau} u'_j u'_j d\tau = \text{const} \geq 0 \tag{11}$$

$$\int_{\tau} \omega'_j e_{jkm} U_k \xi'_m d\tau = \text{const}$$

is inherent to the integral  $E_1$  for quasi–solid rotation states of an ideal incompressible fluid

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) \equiv \boldsymbol{\Omega}_1 \times \mathbf{x}, p = P(\mathbf{x}) \tag{12}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega}(\mathbf{x}) = 2\boldsymbol{\Omega}_1 \equiv \text{const}$$

(here  $P(\mathbf{x})$  is the known scalar function of vector argument). It would seem that nonnegativity the first of functionals (11) must provide absolute stability for quasi–solid rotation states (12) with respect to “isovortical” small 3D perturbations (8), (9). However, this observation is erroneous because the second integral from the system of relations (11) has no specific sign, which, in turn, deprives the functional  $E_1$  properties both distinctness and constancy in sign.

In what follows, the presentation aims to substantiate the hypothesis about absolute instability of exact stationary solutions (4)–(6) to the mixed problem (1) and the equation (3), which meet steady–state 3D flows of the fluid under investigation, with regard to 3D perturbations in linear approximation, and at the same time, the integral  $E_1$  (10) will be applied in the form [18]

$$E_1 \equiv T + T_1 + \Pi = \text{const} \tag{13}$$

where

$$T \equiv \frac{1}{2} \int_{\tau} \left[ \frac{\partial \xi}{\partial t} + (\mathbf{U} \cdot \nabla) \xi \right]^2 d\tau \geq 0$$

$$T_1 \equiv - \int_{\tau} \frac{\partial \xi_j}{\partial t} \xi_k \frac{\partial U_j}{\partial x_k} d\tau, \Pi \equiv - \frac{1}{2} \int_{\tau} \xi_j \xi_k \frac{\partial^2 P}{\partial x_j \partial x_k} d\tau$$

### 4. The Lyapunov Functional

In the interests of further consideration, it is convenient to introduce auxiliary integral of the form [18]

$$M \equiv \int_{\tau} \xi_j \xi'_j d\tau \geq 0 \tag{14}$$

into the study. This functional (according to terminology of article [18], the virial) is volumetric integral over the vessel  $\tau$  from the squared distance  $\xi_j^2$  between locations of the same fluid particles and at the same points of time, but on trajectories of perturbed (8), (9) and on stream–lines of steady–state (4)–(6) flows of an inviscid incompressible fluid, respectively, in the phase space of solutions to the linearized initial–boundary value problem (7).

If one differentiate the functional  $M$  (14) twice by independent variable  $t$  and perform several transformations of the resulting integral using relations (8), (9), and (13), it is not difficult to reach virial equality [19] in the form [18]

$$M'' = 4(T + \Pi) \tag{15}$$

(hereinafter, stroke denotes ordinary time derivative).

Before continuing the description of this paper results, it is logical to focus a little more detail on separate aspects of article [18], already repeatedly cited above.

Namely, the a priori estimate from below

$$M(t) \geq M(0) \exp(-t\sqrt{2\mu}) + \left[ \frac{M'(0)}{+M(0)\sqrt{2\mu}} \right] \times \frac{\sinh t\sqrt{2\mu}}{2\mu} \tag{16}$$

is constructed by the authors of this paper for partial class of exact stationary solutions (4)–(6) to the mixed problem (1) and the equation (3), which is characterized by the relation

$$\xi_j \xi_k \frac{\partial^2 P}{\partial x_j \partial x_k} \leq -\mu \xi_m^2 \tag{17}$$

where  $\mu$  is a positive constant value, with the aid of equality (15). This estimate indicates growth over time (at least, exponential) “isovortical” small 3D perturbations (8), (9) of exact stationary solutions (4)–(6), (17) to the initial–boundary value problem (1) and the equation (3), satisfying initial conditions of the form

$$M(0) \geq 0, M'(0) > -M(0)\sqrt{2\mu} \tag{18}$$

and therefore, absolute instability of steady–state 3D flows (4)–(6), (17) of an ideal incompressible fluid with regard to such perturbations.

At the same time, for subclass of exact stationary solutions (4)–(6) to the mixed problem (1) and the relation (3), which is described by the inequality

$$\xi_j \xi_k \frac{\partial^2 P}{\partial x_j \partial x_k} \leq 0 \tag{19}$$

the authors of article [18] were able to construct only the a priori lower estimate showing just that “isovortical” small 3D perturbations (8), (9) of exact stationary solutions (4)–(6), (19) to the initial–boundary value problem (1) and the equation (3) increase in time, not slower than linearly. On fair recognition of the paper [18] authors, this growth of “isovortical” small 3D perturbations (8), (9) cannot be interpreted in any case as true instability of steady–state 3D flows (4)–(6), (19) of an inviscid incompressible fluid.

As for the remaining exact stationary solutions (4)–(6) to the mixed problem (1) and the relation (3), they have not been studied in article [18] at all.

Unfortunately, the a priori exponential estimate from below (16), constructed by the authors of paper [18], is conditional result which is desperately in need of more rigorous mathematical proof.

The fact is that the inequality (16) is true not only for solutions to the initial–boundary value problem (8), (9), (18), but also for functions which do not serve solutions to this problem. Therefore, the authors of article [18] would have to prove that class of solutions to the mixed problem (8), (9), (18), which increase on time in accord with the a

priori exponential lower estimate (16) constructed by them, is not empty set. However, this proof was not carried out by the authors of paper [18].

The author of this article believes that if the authors of paper [18] took on proof of nonemptiness for class of solutions to the initial–boundary value problem (8), (9), (18), growing over time according to the a priori exponential estimate from below (16), then they would meet with insurmountable hardship. The reason of this hardship stems from the fact that positive constant value  $\mu$  of hyperbolic sine argument in the right part of inequality (16) is not arbitrary, but is subject to the restriction (17). This implies that not every rising in time solution to the mixed problem (8), (9), (18) will grow necessarily in accord with the a priori exponential lower estimate (16). In turn, this circumstance means that it is not possible to prove nonemptiness for class of solutions to the initial–boundary value problem (8), (9), (18), which increase on time according to the a priori exponential estimate from below (16).

Thus, contrary to assertion by the authors of article [18], scenario of instability evolution of steady–state 3D flows (4)–(6), (17) of an ideal incompressible fluid with respect to “isovortical” small 3D perturbations (8), (9), (18) may differ from what is supposedly predetermined by the a priori exponential lower estimate (16).

Given the results of paper [18], the purpose of further consideration is to design such a priori exponential estimate from below, so that, on the one hand, it testified to growth over time “isovortical” small 3D perturbations (8), (9) of exact stationary solutions (4)–(6) to the mixed problems (1) and the equation (3), and on the other hand, its increment would be free from the constraint (17) and the like.

With this aim, it is not difficult to see that, without loss of generality, the double inequality

$$-\alpha \xi_m^2 \leq \xi_j \xi_k \frac{\partial^2 P}{\partial x_j \partial x_k} \leq \alpha \xi_m^2 \quad (20)$$

is true for exact stationary solutions (4)–(6) to the initial–boundary value problem (1) and the relation (3) (here  $\alpha$  is some positive constant).

Then, using the expression (13) for functional  $E_1$ , virial equality (15), and the right part of double inequality (20), one can obtain the key relation — the basic differential inequality [20,22]:

$$M'' - 2\lambda M' + 2(\lambda^2 + \alpha)M \geq 0 \quad (21)$$

where  $\lambda$  is arbitrary positive constant value.

Indeed, virial equation (15) is multiplied at first by some constant  $\lambda$ , then, taking into account the expression (13) for integral  $E_1$ , one can derive the ratio

$$E'_\lambda = 2\lambda(E_\lambda - 2T_\lambda - T_1 - 2\Pi) \quad (22)$$

Here

$$E_\lambda \equiv \Pi_\lambda + T_\lambda, \quad 2\Pi_\lambda \equiv 2(\Pi + T_1) + \lambda^2 M$$

$$2T_\lambda \equiv 2T - \lambda M' + \lambda^2 M = \int_\tau \left[ \frac{\partial \xi}{\partial t} + (\mathbf{U} \cdot \nabla) \xi - \lambda \xi \right]^2 d\tau \geq 0$$

Now let  $\lambda > 0$ , and exact stationary solutions (4)–(6) to the mixed problem (1) and the equation (3) meet double inequality (20). In this case, it is not hard to ascertain, key differential inequality (21) can be extracted from the relation (22).

Specifically,

$$\begin{aligned} E'_\lambda &= 2\lambda(E_\lambda - 2T_\lambda - T_1 - 2\Pi) \\ &= 2\lambda(\Pi_\lambda - T_\lambda - T_1 - 2\Pi), \\ &\left( \Pi + T_1 + \lambda^2 M + T - \frac{\lambda}{2} M' \right)' \\ &= \lambda^3 M - 2 \times \lambda(T_\lambda + \Pi) \leq \lambda^3 M + \lambda \alpha M, \\ \lambda^2 M - \frac{\lambda}{2} M'' &\leq \lambda \times (\lambda^2 + \alpha) M, \\ M'' - 2\lambda M' + 2(\lambda^2 + \alpha)M &\geq 0 \end{aligned}$$

which required to be confirmed.

Unfortunately, the differential operator of basic relation (21) is not positive on semi–infinite interval  $[0, +\infty)$  [36]. This fact means that key differential inequality (21) can't be integrated over the half–interval by applying the Chaplygin method [37].

In this regard, it is proposed further such procedure of basic relation (21) studying, namely: 1) to divide semi–infinite interval  $[0, +\infty)$  into nonintersecting half–intervals countable set of key inequality (21) differential operator positivity so as to the left ends of these semi–intervals belonged to them also, whereas the right — no; 2) to integrate basic differential relation (21) by the Chaplygin method at each half–interval of the introduced above countable set; 3) by analyzing of conditions on left ends of the considered semi–intervals from countable set, to characterize the initial data for growing solutions to the mixed problem (8), (9); 4) using the results of key differential inequality (21) integration on countable set of half–intervals, to construct the a priori lower estimate for growing solutions to the initial–boundary value problem (8), (9) at semi–infinite interval  $[0, +\infty)$ , indicating that present solutions grow with time exponentially or faster.

In fact, the relation (21) can be formally integrated on the following half–intervals:

$$t \in \left[ \frac{2\pi n}{\sqrt{\lambda^2 + 2\alpha}}, \frac{\pi}{2\sqrt{\lambda^2 + 2\alpha}} + \frac{2\pi n}{\sqrt{\lambda^2 + 2\alpha}} \right) \quad (23)$$

$$n = 0, 1, 2, \dots$$

For this purpose, it is necessary to carry out a number of replacements for the functional  $M$  (14). Concretely,

$$1) M_1(t) \equiv \exp(-\lambda t) M(t)$$

$$M_1''(t) + (\lambda^2 + 2\alpha) M_1(t) \geq 0$$

$$2) M_2(t) \equiv \frac{M_1(t)}{\cos t \sqrt{\lambda^2 + 2\alpha}} \left[ M_2'(t) \cos t \sqrt{\lambda^2 + 2\alpha} \right]$$

$$-\sqrt{\lambda^2 + 2\alpha} M_2'(t) \times \sin t \sqrt{\lambda^2 + 2\alpha} \geq 0$$

$$3) M_3(t) \equiv M'_2(t) \cos^2 t \sqrt{\lambda^2 + 2\alpha} : M'_3(t) \geq 0$$

Integration of the last inequality and implementation of reverse replacements allow us to go out on the relation

$$M(t) \geq \left[ \begin{matrix} A_{1n} \sin t \sqrt{\lambda^2 + 2\alpha} \\ + A_{2n} \cos t \sqrt{\lambda^2 + 2\alpha} \end{matrix} \right] \times \exp \lambda t \quad (24)$$

where  $A_{1n}$  and  $A_{2n}$  are some constant values.

Relying on non-strict inequality (24), it is not difficult to express constants  $A_{1n}$  and  $A_{2n}$  ( $n = 0, 1, 2, \dots$ ) through values of the functional  $M$  and its derivative  $M'(t)$  in time points  $t_n \equiv 2\pi n / \sqrt{\lambda^2 + 2\alpha}$ . As a result, such final form can be given to the ratio (24):

$$M(t) \geq f(t),$$

$$f(t) \equiv \left\{ \begin{matrix} M(t_n) \cos t \sqrt{\lambda^2 + 2\alpha} \\ + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \left[ \begin{matrix} M'(t_n) \\ -\lambda M(t_n) \end{matrix} \right] \sin t \sqrt{\lambda^2 + 2\alpha} \end{matrix} \right\} \times \exp \lambda(t - t_n) \quad (25)$$

To substantiate the procedure of integration key differential inequality (21) on semi-intervals (23), which brought in result to the a priori exponential estimate from below (25), it is necessary to calculate ordinary first-order derivative of the function  $f$  on its argument  $t$ :

$$f'(t) = \left\{ \begin{matrix} \left[ \begin{matrix} M'(t_n) \cos t \sqrt{\lambda^2 + 2\alpha} \\ + \frac{\lambda}{\sqrt{\lambda^2 + 2\alpha}} \left[ \begin{matrix} M'(t_n) \\ -\lambda M(t_n) \end{matrix} \right] \sin t \sqrt{\lambda^2 + 2\alpha} \end{matrix} \right] \\ \times \left[ \begin{matrix} M'(t_n) \\ -\lambda M(t_n) \end{matrix} \right] \\ - \sqrt{\lambda^2 + 2\alpha} M(t_n) \end{matrix} \right\} \times \exp \lambda(t - t_n) \quad (26)$$

Considering ratios (25) and (26), it is possible to claim that the function  $f(t)$  will be positive and strictly increasing on half-intervals (23) in that and only in that case when inequalities

$$M(t_n) > 0, M'(t_n) \geq 2 \left( \lambda + \frac{\alpha}{\lambda} \right) M(t_n) \quad (27)$$

take place [38]. These inequalities serve just as required guarantees of rightfulness the described above procedure of basic differential ratio (21) integration on semi-intervals (23).

As half-intervals (23) do not intercross, values of the functional  $M$  and its derivative  $M'(t)$  can be set enough arbitrary manner on left ends of these semi-intervals. In particular, it is possible to take these values in the form

$$M(t_n) \equiv M(0) \exp \lambda t_n, M'(t_n) \equiv M'(0) \exp \lambda t_n$$

Then inequalities (27) will be true in that and only in that case, if

$$M(0) > 0, M'(0) \geq 2 \left( \lambda + \frac{\alpha}{\lambda} \right) M(0)$$

and the function  $f(t)$  will appear as

$$f(t) = \left[ \begin{matrix} M(0) \cos t \sqrt{\lambda^2 + 2\alpha} \\ + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \{ M'(0) - \lambda M(0) \} \\ \times \sin t \sqrt{\lambda^2 + 2\alpha} \end{matrix} \right] \exp \lambda t$$

Similar reasonings can be carried out when key differential ratio (21) should be integrated on all other temporary half-intervals. In view of this circumstance, the results of basic differential inequality (21) integration on the remained time semi-intervals are reported in the form of illustrating computation, without additional explaining comments ( $t_* \equiv \pi / 2\sqrt{\lambda^2 + 2\alpha}$ ):

$$a) t \in [t_* + t_n, 2t_* + t_n]; n = 0, 1, 2, \dots$$

$$1) M_1(t) = \exp(-\lambda t) M(t)$$

$$M''_1(t) + (\lambda^2 + 2\alpha) M_1(t) \geq 0$$

$$2) M_2(t) = \frac{M_1(t)}{\cos t \sqrt{\lambda^2 + 2\alpha}}$$

$$\left[ M'_2(t) \cos t \sqrt{\lambda^2 + 2\alpha} \right]' - \sqrt{\lambda^2 + 2\alpha} M'_2(t) \times \sin t \sqrt{\lambda^2 + 2\alpha} \geq 0$$

$$3) M_3(t) = M'_2(t) \cos^2 t \sqrt{\lambda^2 + 2\alpha} : M'_3(t) \leq 0$$

$$4) M(t) \geq \left[ A_{3n} \sin t \sqrt{\lambda^2 + 2\alpha} + A_{4n} \times \cos t \sqrt{\lambda^2 + 2\alpha} \right] \exp \lambda t; A_{3n}, A_{4n} - \text{const}$$

$$5) M(t) \geq f_1(t); f_1(t) \equiv \{ M(t_* + t_n) \times$$

$$\times \sin t \sqrt{\lambda^2 + 2\alpha} - \frac{1}{\sqrt{\lambda^2 + 2\alpha}} [ M'(t_* + t_n) - \lambda \times$$

$$\times M(t_* + t_n) ] \cos t \sqrt{\lambda^2 + 2\alpha} \} \exp \lambda(t - t_* - t_n)$$

$$f'_1(t) = \left[ M'(t_* + t_n) \sin t \sqrt{\lambda^2 + 2\alpha} - \left\{ \frac{\lambda}{\sqrt{\lambda^2 + 2\alpha}} \times \right. \right.$$

$$\times [ M'(t_* + t_n) - \lambda M(t_* + t_n) ] - \sqrt{\lambda^2 + 2\alpha} \times$$

$$\times M(t_* + t_n) \} \cos t \sqrt{\lambda^2 + 2\alpha} \left. \right] \exp \lambda(t - t_* - t_n)$$

$$6) M(t_* + t_n) > 0, M'(t_* + t_n) \geq 2 \left( \lambda + \frac{\alpha}{\lambda} \right) \times$$

$$\times M(t_* + t_n)$$

$$\begin{aligned}
7) M(t_* + t_n) &\equiv M(0) \exp \lambda(t_* + t_n), M'(t_* + t_n) \equiv \\
&\equiv M'(0) \exp \lambda(t_* + t_n); M(0) > 0, M'(0) \geq \\
&\geq 2\left(\lambda + \frac{\alpha}{\lambda}\right)M(0); f_1(t) = \left[ M(0) \sin t \sqrt{\lambda^2 + 2\alpha} - \right. \\
&\left. - \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \{M'(0) - \lambda M(0)\} \cos t \sqrt{\lambda^2 + 2\alpha} \right] \times \\
&\quad \times \exp \lambda t
\end{aligned}$$

$$b) t \in [2t_* + t_n, 3t_* + t_n]; n = 0, 1, 2, \dots$$

$$1) M_1(t) = \exp(-\lambda t) M(t)$$

$$M_1''(t) + (\lambda^2 + 2\alpha) M_1(t) \geq 0$$

$$\begin{aligned}
2) M_2(t) &= \frac{M_1(t)}{\cos t \sqrt{\lambda^2 + 2\alpha}} \\
&\left[ M_2'(t) \cos t \sqrt{\lambda^2 + 2\alpha} \right]' - \sqrt{\lambda^2 + 2\alpha} M_2'(t) \times \\
&\quad \times \sin t \sqrt{\lambda^2 + 2\alpha} \geq 0
\end{aligned}$$

$$3) M_3(t) = M_2'(t) \cos^2 t \sqrt{\lambda^2 + 2\alpha}; M_3'(t) \leq 0$$

$$\begin{aligned}
4) M(t) &\geq \left[ A_{5n} \sin t \sqrt{\lambda^2 + 2\alpha} + A_{6n} \times \right. \\
&\quad \left. \times \cos t \sqrt{\lambda^2 + 2\alpha} \right] \exp \lambda t; A_{5n}, A_{6n} - \text{const}
\end{aligned}$$

$$\begin{aligned}
5) M(t) &\geq f_2(t); f_2(t) \equiv -\{M(2t_* + t_n) \times \\
&\quad \times \cos t \sqrt{\lambda^2 + 2\alpha} + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} [M'(2t_* + t_n) - \lambda \times \\
&\quad \times M(2t_* + t_n)] \sin t \sqrt{\lambda^2 + 2\alpha}\} \exp \lambda(t - 2t_* - t_n) \\
f_2'(t) &= -\left[ M'(2t_* + t_n) \cos t \sqrt{\lambda^2 + 2\alpha} + \right. \\
&\quad \left. + \left\{ \frac{\lambda}{\sqrt{\lambda^2 + 2\alpha}} [M'(2t_* + t_n) - \lambda M(2t_* + t_n)] - \right. \right. \\
&\quad \left. \left. - \sqrt{\lambda^2 + 2\alpha} M(2t_* + t_n) \right\} \sin t \sqrt{\lambda^2 + 2\alpha} \right] \times \\
&\quad \times \exp \lambda(t - 2t_* - t_n)
\end{aligned}$$

$$\begin{aligned}
6) M(2t_* + t_n) &> 0, M'(2t_* + t_n) \geq 2\left(\lambda + \frac{\alpha}{\lambda}\right) \times \\
&\quad \times M(2t_* + t_n)
\end{aligned}$$

$$\begin{aligned}
7) M(2t_* + t_n) &\equiv M(0) \exp \lambda(2t_* + t_n) \\
M'(2t_* + t_n) &\equiv M'(0) \exp \lambda(2t_* + t_n); M(0) > 0 \\
M'(0) &\geq 2\left(\lambda + \frac{\alpha}{\lambda}\right) M(0); f_2(t) = -[M(0) \times
\end{aligned}$$

$$\begin{aligned}
&\times \cos t \sqrt{\lambda^2 + 2\alpha} + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \{M'(0) - \lambda M(0)\} \times \\
&\quad \times \sin t \sqrt{\lambda^2 + 2\alpha}] \exp \lambda t
\end{aligned}$$

$$c) t \in [3t_* + t_n, 4t_* + t_n]; n = 0, 1, 2, \dots$$

$$1) M_1(t) = \exp(-\lambda t) M(t)$$

$$M_1''(t) + (\lambda^2 + 2\alpha) M_1(t) \geq 0$$

$$\begin{aligned}
2) M_2(t) &= \frac{M_1(t)}{\cos t \sqrt{\lambda^2 + 2\alpha}} \\
&\left[ M_2'(t) \cos t \sqrt{\lambda^2 + 2\alpha} \right]' - \sqrt{\lambda^2 + 2\alpha} M_2'(t) \times \\
&\quad \times \sin t \sqrt{\lambda^2 + 2\alpha} \geq 0
\end{aligned}$$

$$3) M_3(t) = M_2'(t) \cos^2 t \sqrt{\lambda^2 + 2\alpha}; M_3'(t) \geq 0$$

$$\begin{aligned}
4) M(t) &\geq \left[ A_{7n} \sin t \sqrt{\lambda^2 + 2\alpha} + A_{8n} \times \right. \\
&\quad \left. \times \cos t \sqrt{\lambda^2 + 2\alpha} \right] \exp \lambda t; A_{7n}, A_{8n} - \text{const}
\end{aligned}$$

$$\begin{aligned}
5) M(t) &\geq f_3(t); f_3(t) \equiv \{-M(3t_* + t_n) \times \\
&\quad \times \sin t \sqrt{\lambda^2 + 2\alpha} + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} [M'(3t_* + t_n) - \lambda \times \\
&\quad \times M(3t_* + t_n)] \cos t \sqrt{\lambda^2 + 2\alpha}\} \exp \lambda(t - 3t_* - t_n)
\end{aligned}$$

$$\begin{aligned}
f_3'(t) &= \left[ -M'(3t_* + t_n) \sin t \sqrt{\lambda^2 + 2\alpha} + \right. \\
&\quad \left. + \left\{ \frac{\lambda}{\sqrt{\lambda^2 + 2\alpha}} [M'(3t_* + t_n) - \lambda M(3t_* + t_n)] - \right. \right. \\
&\quad \left. \left. - \sqrt{\lambda^2 + 2\alpha} M(3t_* + t_n) \right\} \cos t \sqrt{\lambda^2 + 2\alpha} \right] \times \\
&\quad \times \exp \lambda(t - 3t_* - t_n)
\end{aligned}$$

$$\begin{aligned}
6) M(3t_* + t_n) &> 0, M'(3t_* + t_n) \geq 2\left(\lambda + \frac{\alpha}{\lambda}\right) \times \\
&\quad \times M(3t_* + t_n)
\end{aligned}$$

$$\begin{aligned}
7) M(3t_* + t_n) &\equiv M(0) \exp \lambda(3t_* + t_n) \\
M'(3t_* + t_n) &\equiv M'(0) \exp \lambda(3t_* + t_n); M(0) > 0 \\
M'(0) &\geq 2\left(\lambda + \frac{\alpha}{\lambda}\right) M(0); f_3(t) = [-M(0) \times
\end{aligned}$$

$$\begin{aligned}
&\times \sin t \sqrt{\lambda^2 + 2\alpha} + \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \{M'(0) - \lambda M(0)\} \times \\
&\quad \times \cos t \sqrt{\lambda^2 + 2\alpha}] \exp \lambda t
\end{aligned}$$

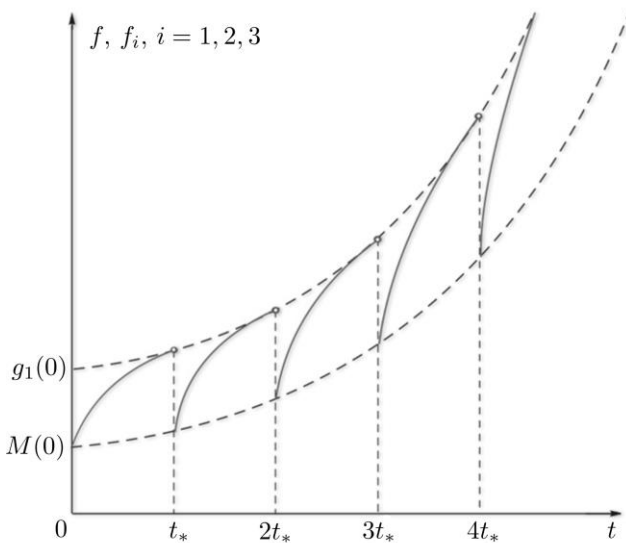
If to analyze final expressions for functions  $f(t)$ ,  $f_i(t)$  ( $i=1,2,3$ ), it is not hard to see that curves, lying across half-strip, which is directed at infinity exponentially in time, at that, their left ends are situated at bottom edge of this semi-strip

$$g(t) \equiv M(0)\exp \lambda t$$

and right ones adjoin to its upper bound

$$g_1(t) \equiv g_1(0)\exp \lambda t, g_1(0) \equiv \frac{M'(0) - \lambda M(0)}{\sqrt{\lambda^2 + 2\alpha}}$$

will be graphs of these functions on the corresponding temporal half-intervals (Figure 3).



**Figure 3.** Graphic representation the procedure of integration key differential ratio (21)

This analysis of geometrical properties functions  $f(t)$ ,  $f_i(t)$  ( $i=1,2,3$ ) enables us to make absolutely definite conclusion that the functional  $M$  (14) cannot grow on time more slowly, than exponential.

Summarizing the results of integration basic differential inequality (21) over temporary intervals  $[nt_*, (n+1)t_*]$  ( $n=0,1,2,\dots$ ), in the upshot, it is possible to put with complete confidence forward the following statement: when adding countable set of conditions in the form

$$\begin{aligned} M(nt_*) &> 0; n = 0, 1, 2, \dots; \\ M'(nt_*) &\geq 2\left(\lambda + \frac{\alpha}{\lambda}\right)M(nt_*); \\ M(nt_*) &\equiv M(0)\exp \lambda nt_*, \\ M'(nt_*) &\equiv M'(0)\exp \lambda nt_*; \\ M(0) &> 0, M'(0) &\geq 2\left(\lambda + \frac{\alpha}{\lambda}\right)M(0) \end{aligned} \tag{28}$$

to key differential ratio (21), the desirable prior exponential lower estimate of increase over time “isovortical” small 3D perturbations (8), (9) of exact stationary solutions (4)–(6), (20) to the mixed problem (1) and the equation (3) of the form

$$M(t) \geq C \exp \lambda t \tag{29}$$

will follow from it with need (here  $C$  is the known positive constant value).

Now, it is worth to discuss especially the question about reciprocal connection between the executed interval integration of basic differential inequality (21) and characteristic properties of solutions to the linearized initial-boundary value problem (8), (9).

To wit, this connection is that, for key differential ratio (21) by means of special initial data choice (see on couple of identical equalities in system of expressions (28)) on left ends of the studied temporal intervals, it turned out to specify uniform initial conditions (see on the last couple of inequalities from system of ratios (28)) for “isovortical” small 3D perturbations (8), (9) of exact stationary solutions (4)–(6), (20) to the mixed problems (1) and the equation (3), providing validity of requirements for positiveness and rigorous increase of functions  $f(t)$ ,  $f_i(t)$  ( $i=1,2,3$ ) (see on the first couple of inequalities in system of ratios (28)) at all considered temporary semi-intervals.

It should be noted that class of solutions to the linearized initial-boundary value problem (8), (9), which grow on time in consent with the designed prior exponential estimate from below (29), with supplementary conditions

$$M(0) > 0, M'(0) \geq 2\left(\lambda + \frac{\alpha}{\lambda}\right)M(0) \tag{30}$$

on initial data  $\xi_0(\mathbf{x})$  and  $(\partial \xi / \partial t)_0(\mathbf{x})$  is not empty.

Really, as the mixed problem (8), (9) is linear, it is solvable with respect to “isovortical” small 3D perturbations in the form of normal waves [9], [19]. Further, as the functional  $E_1$  (10) does not possess neither definiteness or constancy in sign, the initial-boundary value problem (8), (9) is solvable as well with regard to increasing over time “isovortical” small 3D perturbations in the form of normal waves [20,30]. At last, any growing in time solution to the mixed problem (8), (9), which meets to “isovortical” small 3D perturbation in the form of normal wave, will, owing to arbitrariness of a positive constant  $\lambda$ , satisfy to basic differential inequality (21), countable set of conditions (28), and the a priori exponential lower estimate (29) identically and automatically.

So, there are no obstacles whatever that increasing on time solutions, which correspond to “isovortical” small 3D perturbations in the form of normal waves, were available among solutions to the linearized initial-boundary value problem (8), (9), (30). Incidentally, this is confirmed by concrete analytical and numerical examples [20,23,30] constructed by the author of this article earlier.

Thus, according to the Lyapunov definition of unstable solution to system of differential equations [7], the a priori exponential estimate from below (29) demonstrates clearly that, at the minimum, one “isovortical” small 3D perturbation (8), (9) with initial data (30) of steady-state 3D flows (4)–(6), (20) of an inviscid incompressible fluid will grow over time not more slowly, than exponential. As this estimate is received without imposing of any additional restrictions on steady-state 3D flows (4)–(6),



(20), absolute instability of the last with respect to “isovortical” small 3D perturbations (8), (9), (30) follows from here also.

Besides, the first couple of inequalities from system of connections (28) allows us to interpret it as sufficient conditions for linear practical instability of steady-state 3D flows (4)–(6), (20) of an ideal incompressible fluid with regard to “isovortical” small 3D perturbations (8), (9), (30), and with respect to “isovortical” small 3D perturbations (8), (9), (30) in the form of normal waves — as necessary and sufficient ones (in view of the fact that positive constant value  $\lambda$  is arbitrary for the rest). It is important that these conditions for linear practical instabilities are constructive inherently, and so, they can be applied as mechanism for testing and monitoring during physical experiments, carrying-out of numerical calculations, and realization of technological processes [20,30].

It is also worth to pay separate attention to the fact that the integral  $M$  (14) represents just the required Lyapunov functional, increasing on time in consent with equations of the linearized mixed problem (8), (9), (30), in this paper. The characteristic feature of this growth serves great freedom which is peculiar to a positive constant  $\lambda$  in exponent index from right-hand side of the prior exponential lower estimate (29). It, along with other, gives us opportunity to perceive any solution to the initial-boundary value problem (8), (9), (30), increasing over time according to the constructed prior exponential estimate from below (29), as analog of incorrectness example in the Hadamard sense [39].

In summary, it is necessary to emphasize that shortcomings, which are inherent in the prior exponential lower estimate (16) constructed by the authors of article [18], are absent completely in the prior exponential estimate from below (29) designed in the present paper.

## 5. Conclusion

In this article, the problem on linear stability of steady-state 3D flows (4)–(6), (20) of an inviscid incompressible fluid, which entirely fills a volume with quiescent solid impenetrable walls, without mass forces is studied.

It is proved by the direct Lyapunov method that these flows are absolutely unstable with regard to “isovortical” small 3D perturbations (8), (9), (30), whereas states of equilibrium (rest) are, on the contrary, absolutely stable. The constructive conditions for linear practical instability are given. The a priori exponential lower estimate (29), testifying to growth in time of the considered “isovortical” small 3D perturbations, is constructed. The conditional character of a priori exponential estimate from below (16), which is designed in paper [18], is revealed.

It is worth noticing that, from the mathematical point of view, the present article results are, in the majority, the prior as existence theorems of solutions to the studied mixed problems for systems of differential equations with partial derivatives were not proved.

At last, concerning interrelation of this paper results with results received by other authors earlier, it is logical to concentrate attention on a number of principal circumstances. Specifically,

1) as there are no data on exact stationary solutions (4)–(6) to the initial-boundary value problem (1) and the ratio (3) in key differential inequality (21), it is necessary to expect that the inequality will in this or that look arise during consideration other mathematical models of hydrodynamic type [20,21,23-29,31,32] too;

2) depending on, whether derivation of basic differential inequality (21) and to it similar is accompanied by imposing on that or other studied steady-state flows of these or those extra restrictions, the conclusion about properties of instability (absolute or conditional) for these flows can be made: if yes, then these restrictions will represent sufficient conditions of instability for steady-state flows under consideration; if no, then the studied steady-state flows will be absolutely unstable [20-32];

3) the fact of existence key differential inequalities of the form (21) leads to immediate conversion of the known sufficient conditions for linear stability of these or those steady-state flows under consideration; in other words, it is possible to find exclusively necessary and sufficient conditions of linear stability and anything else in the presence of basic differential inequalities of the type (21) [20,21,23,25-27,29,31,32];

4) key differential inequalities of the form (21) are impossible to construct, if analogs of energy integrals for those or other linearized mixed problems in terms of the Lagrangian displacements field [19] are definite or constant in sign [20-32].

Relying on the listed above circumstances, there are every reason to conclude that the described in this article method for designing of the Lyapunov functionals, for which is characteristic to increase on time along solutions to the considered linearized initial-boundary value problems, will be, undoubtedly, good help in the course of studying still unresolved linear problems of the mathematical theory hydrodynamic stability.

## References

- [1] Arnold, V.I., “Variational principle for three-dimensional steady-state flows of an ideal fluid (Steady flow possessing extremal kinetic energy compared to equivortex flow for stability analysis),” *PMM — J. Appl. Math. Mech.*, 29 (5), 1002-1008, 1965.
- [2] Hsiao, K.-L., “Heat and mass mixed convection for MHD visco-elastic fluid past a stretching sheet with ohmic dissipation,” *Commun. Nonlinear Sc. Numer. Simul.*, 15 (7), 1803-1812, 2010.
- [3] Hsiao, K.-L., “Corrigendum to “Heat and mass mixed convection for MHD visco-elastic fluid past a stretching sheet with ohmic dissipation” [Commun. Nonlinear Sci. Numer. Simul., 15 (7) 1803-1812, 2010],” *Commun. Nonlinear Sc. Numer. Simul.*, 28 (1), 232, 2015.
- [4] Hsiao, K.-L., “Nanofluid flow with multimedia physical features for conjugate mixed convection and radiation,” *Comput. Fluids*, 104, 1-8, 2014.
- [5] Hsiao, K.-L., “MHD mixed convection for viscoelastic fluid past a porous wedge,” *Int. J. Nonlinear Mech.*, 46 (1), 1-8, 2011.
- [6] Lyapunov, A.M., *The general problem of the stability of motion*, Taylor & Francis, London, 1992.
- [7] Demidovich, B.P., *Lectures on the mathematical stability theory*, Nauka, Moscow, 1967 (in Russian).
- [8] Chetaev, N.G., *Stability of motion*, Nauka, Moscow, 1990 (in Russian).
- [9] Chandrasekhar, S., *Hydrodynamic and hydromagnetic stability*, Clarendon Press, Oxford, 1961.
- [10] Moiseyev, N.N. and Rumyantsev, V.V., *Dynamic stability of bodies containing fluid*, Springer-Verlag, New York, 1968.

- [11] Dikii, L.A., *Hydrodynamical stability and dynamics of atmosphere*, Gidrometeoizdat, Leningrad, 1976 (in Russian).
- [12] Joseph, D.D., *Stability of fluid motions*, Springer-Verlag, New York, 1976.
- [13] Drazin, P.G. and Reid, W.H., *Hydrodynamic stability*, Cambridge University Press, Cambridge, 1981.
- [14] Kozyrev, O.R. and Stepanyants, Yu.A., "Method of integral relations in the linear theory of hydrodynamical stability," *VINITI*, 25, 3-89, 1991 (in Russian).
- [15] Drazin, P.G., *Introduction to hydrodynamic stability*, Cambridge University Press, Cambridge, 2002.
- [16] Vladimirov, V.A., "Application of conservation laws to derivation of conditions of stability for stationary flows of an ideal fluid," *J. Appl. Mech. Techn. Phys.*, 28 (3), 351-358, 1987.
- [17] Holm, D.D., Marsden, J.E., Ratiu, T., Weinstein, A., "Nonlinear stability of fluid and plasma equilibria," *Phys. Reports*, 123 (1 & 2), 1-116, 1985.
- [18] Vladimirov, V.A. and Ilin, K.I., "Virial functionals in fluid dynamics," *Moscow Math. J.*, 3 (2), 691-709, 2003.
- [19] Chandrasekhar, S., *Ellipsoidal figures of equilibrium*, Yale University Press, New Haven, 1969.
- [20] Gubarev, Yu.G., *The direct Lyapunov method. The stability of quiescent states and steady-state flows of fluids and gases*, Palmarium Academic Publishing, Saarbrücken, 2012 (in Russian).
- [21] Gubarev, Yu.G., "Stability of steady-state shear jet flows of an ideal fluid with a free boundary in an azimuthal magnetic field against small long-wave perturbations," *J. Appl. Mech. Techn. Phys.*, 45 (2), 239-248, 2004.
- [22] Gubarev, Y.G., "On instability of hydrodynamical flows," in *Zababakhin Scientific Talks. 2005: Intern. Conf. High Energy Density Physics*, American Institute of Physics, 335-340.
- [23] Gubarev, Yu.G., "On stability of steady-state plane-parallel shearing flows in a homogeneous in density ideal incompressible fluid," *Nonlinear Anal. Hybrid Syst.*, 1 (1), 103-118, 2007.
- [24] Gubarev, Yu.G., "On stability of steady-state three-dimensional flows of an ideal compressible medium," in *Youth Conf. "Stability and Turbulence of Flows of Homogeneous and Heterogeneous Fluids"*, Parallel, 107-110 (in Russian).
- [25] Gubarev, Yu.G., *Progress in nonlinear analysis research*, Nova Science Publishers, New York, 2009, 137-181.
- [26] Gubarev, Yu.G., "On the stability of jetlike magnetohydrodynamic flows," *J. Appl. Ind. Math.*, 4 (3), 318-331, 2010.
- [27] Gubarev, Yu.G., "Linear stability criterion for steady screw magnetohydrodynamic flows of ideal fluid," *T & A*, 16 (3), 407-418, 2009.
- [28] Gubarev, Yu.G., "Sufficient conditions for linear long-wave instability of steady-state axisymmetric flows of an ideal liquid with a free boundary in an azimuthal magnetic field," *Tech. Phys.*, 56 (3), 345-350, 2011.
- [29] Gavrilieva, A.A. and Gubarev, Yu.G., "Stability of steady-state plane-parallel shear flows of an ideal stratified fluid in the gravity field," *Vestnik of the NEFU named after M.K. Ammosov*, 9 (3), 15-21, 2012 (in Russian).
- [30] Gubarev, Yu.G., "The problem of adequate mathematical modeling for liquids fluidity," *AJFD*, 3 (3), 67-74, 2013.
- [31] Gavrilieva, A.A., Gubarev, Yu.G., Lebedev, M.P., "Rapid approach to resolving the adequacy problem of mathematical modeling of physical phenomena by the example of solving one problem of hydrodynamic instability," *IJTMP*, 3 (4), 123-129, 2013.
- [32] Gubarev, Yu.G. and Latkin, I.E., "On stability of spherically symmetric dynamic equilibrium states of self-gravitating Vlasov-Poisson gas," in *All-Russian Conf. with Participation of Foreign Scientists "Modern Problems of Rarefied Gas Dynamics"*, Institute for Thermophysics SB RAS, 82-84 (in Russian).
- [33] Karacharov, K.A. and Pilyutik, A.G., *Introduction in technical theory of motion stability*, Fizmatgiz, Moscow, 1962 (in Russian).
- [34] La Salle, J. and Lefschetz, S., *Stability by Liapunov's direct method with applications*, Academic Press, New York, 1961.
- [35] McConnell, A.J., *Application of tensor analysis*, Dover, New York, 1957.
- [36] Bekkenbach, E.F. and Bellman, R., *Inequalities*, Springer-Verlag, Berlin, 1961.
- [37] Chaplygin, S.A., *New method of approximate integration of differential equations*, GITTL, Moscow, 1950 (in Russian).
- [38] Il'in, V.A., Sadovnochi, V.A., Sendov, Bl.Kh., *Mathematical analysis*, Nauka, Moscow, 1979 (in Russian).
- [39] Godunov, S.K., *Equations of mathematical physics*, Nauka, Moscow, 1979 (in Russian).