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A Fixed Point Result of Expanding Mappings in Complete Cone Metric Spaces

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Abstract In this paper, we prove a fixed point theorem for expanding onto self-mappings in complete cone metric spaces. Our results improve and extend some comparable results in the literature.

Keywords: *cone metric space, fixed point, expanding mapping*

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1. Introduction

In 2007, Huang and Zhang [5] introduced cone metric spaces replacing the real numbers by an ordered Banach space, and they have proved some fixed point theorems for self-mapping satisfying different types of contractive conditions in cone metric spaces. Later on, many authors have generalized and extended Huang and Zhang [5] fixed point theorems (see, e.g., [1,2,3,7,8]). In 1984, the concept of expanding mappings was introduced by Wang et. al. [9]. In 1992, Daffer and Kaneko [4] defined expanding mappings for pair of mappings in complete metric spaces and proved some fixed point theorems. In 2012, X. Huang, Ch. Zhu and Xi Wen [6] proved some fixed point theorems for expanding mappings cone metric spaces and they have also extended the results of Daffer and Kaneko [4]. The main aim of this paper is we proved a fixed point theorem for expanding mappings in cone metric spaces, our result extends and improves the results of [6].

The following definitions and properties are due to Huang and Zhang [5].

Definition 1.1. Let B be a real Banach space and θ is the zero element of B, P a subset of B. The set P is called a cone if and only if:

(i) P is closed, non–empty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;

(iii) $P \cap (-P) = \theta$.

For a cone P in a Banach space B, define partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{Int}\, P$, where Int P denotes the interior of the set P. This cone P is called an order cone.

Let B be a Banach space and $P \subset B$ be an order cone .The order cone P is called normal if there exists $K > 0$ such that for all $x, y \in B$,

$$\theta \leq x \leq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.2. Let X be a nonempty set of B. Suppose that the map d: $X \times X \to B$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and

$d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3. Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c >> \theta$, there is N such that for all $n, m > N$, $d(x_n, x_m) << c$;

(ii) a convergent sequence if for any $c >> \theta$, there is an N such that for all $n > N$, $d(x_n, x) << c$, for some fixed x in X. We write $x_n \to x$ (as $n \to \infty$).

The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent [5].

Definition 1.4. **[5]** Let (X, d) be a cone metric space and T: $X \to X$, then T is called a expanding mapping, if for every $x, y \in X$ there exists a number $k > 1$ such that $d(Tx, Ty) \geq k\, d(x, y)$.

2. Main Result

In this section, we prove a fixed point theorem for expanding mappings in complete cone metric spaces.

We prove a Lemma which is useful in the main theorem.

Lemma 2.1. Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. If there exists a number $\lambda \in (0,1)$ such that

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \quad n=1,2,\ldots \qquad (1)$$

then $\{x_n\}$ is a Cauchy sequence in X.

Proof. By the induction and the condition (1), we have

$$d(x_{n+1},x_n) \le \lambda d(x_n,x_{n-1}) \le \lambda^2 d(x_{n-1},x_{n-2}) \le \ldots \le \lambda^n d(x_1,x_0).$$

For n > m

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n-1}) \le d(x_{n-1},x_{n-2}) \\ &\le \ldots \le d(x_{m+1},x_m). \\ &\le (\lambda^{n-1}+\lambda^{n-2}+\ldots+\lambda^m)d(x_1,x_0). \\ &\le \lambda^m/1+\lambda d(x_1,x_0). \end{aligned}$$

Let $\theta << c$ be given. Choose $r > 0$ such that $c + N_r(\theta) \subseteq P$, where $N_r(\theta) = \{x \in E : \| x \| < r\}$. Also choose a natural number N_1 such that $\lambda^m/1+\lambda d(x_1,x_0) \in N_r(\theta)$, for all $m \ge N_1$. Thus

$$d(x_n,x_m) \le \lambda^m/1+\lambda d(x_1,x_0) << c, \text{ for all } m \ge N_1.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X.

The following theorem improved and extended the Theorem 2.1. of [6].

Theorem 2.2. Let (X, d) be a complete cone metric space and T: $X \to X$ be a surjection. Suppose that there exists $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \ge 0$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4 > 1$ such that

$$d(Tx,Ty) \ge \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty) + \alpha_4 d(y,Tx) \quad \ldots \quad (2)$$

for all $x, y \in X$, $x \ne y$. Then T has a fixed point in X.

Proof. By our assumption, it is clear that T is injective. Let F be the inverse mapping of T.

Let $x_0 \in X$, then $x_1 = F(x_0), x_2 = F(x_1) = F^2(x_0), \ldots$, $x_{n+1} = F(x_n) = F^{n+1}(x_0), \ \ldots$

We assume that $x_{n-1} \ne x_n$ for all n = 1, 2, 3… otherwise $x_{n_0-1} = x_{n_0}$, for some n_0, then x_0 is a fixed point of T.

From the condition (2) it follows that

$$\begin{aligned} d(x_{n-1},x_n) &= d\left(TT^{-1}x_{n-1},TT^{-1}x_n\right), \\ &\ge \alpha_1 d\left(T^{-1}x_{n-1},T^{-1}x_n\right)+\alpha_2 d\left(T^{-1}x_{n-1},TT^{-1}x_{n-1}\right) \\ &\quad +\alpha_3 d\left(T^{-1}x_n,TT^{-1}x_n\right)+\alpha_4 d\left(T^{-1}x_n,TT^{-1}x_{n-1}\right), \\ &= \alpha_1 d(Fx_{n-1},Fx_n)+\alpha_2 d(Fx_{n-1},x_{n-1}) \\ &\quad +\alpha_3 d(Fx_n,x_n)+\alpha_4 d(Fx_n,x_{n-1}) \\ &= \alpha_1 d(x_n,x_{n+1})+\alpha_2 d(x_n,x_{n-1}) \\ &\quad +\alpha_3 d(x_{n+1},x_n)+\alpha_4 d(x_{n+1},x_{n-1}) \\ &= \alpha_1 d(x_n,x_{n+1})+\alpha_2 d(x_n,x_{n-1}) \\ &\quad +\alpha_3 d(x_{n+1},x_n)+\alpha_4 d(x_{n+1},x_n)+\alpha_4 d(x_n,x_{n-1}) \\ &= (\alpha_1+\alpha_3+\alpha_4)d(x_n,x_{n+1}) \\ &\quad +(\alpha_2+\alpha_4)d(x_n,x_{n-1}) \end{aligned}$$

$$\Rightarrow [1-(\alpha_2+\alpha_4)]d(x_n,x_{n-1}) \ge (\alpha_1+\alpha_3+\alpha_4)d(x_n,x_{n+1}).$$

If $\alpha_1+\alpha_3+\alpha_4 = 0$, then

Since $\alpha_1+\alpha_2+\alpha_3+2\alpha_4 > 1 \Rightarrow \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_4 > 1$ $\Rightarrow \alpha_2+\alpha_4 > 1$.

Then the above inequality implies that a negative number is ≥ 0, which is not possible.

So, $\alpha_1+\alpha_3+\alpha_4 \ne 0$ and $1-\alpha_2-\alpha_4 > 0$.

Therefore,

$$\begin{aligned} d(x_{n+1},x_n) &\le (\alpha_1+\alpha_3+\alpha_4)/(1-\alpha_2-\alpha_4)d(x_{n-1},x_n) \\ &\le hd(x_{n-1},x_n), \end{aligned}$$

where, $h = (\alpha_1+\alpha_3+\alpha_4)/(1-\alpha_2-\alpha_4) < 1$.

By the Lemma 2.1, we get that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{x_n\}$ converges to a point $z \in X$. Let $z = Tp$, $p \in X$, we have

$$\begin{aligned} d(x_n,z) &= d(Tx_{n+1},Tp) \\ &\ge \alpha_1 d(x_{n+1},p)+\alpha_2 d(x_{n+1},x_{n-1}) \\ &\quad +\alpha_3 d(p,Tp)+\alpha_4 d(p,x_n). \end{aligned}$$

Letting $n \to \infty$, we get that

$$\begin{aligned} \theta &\ge \alpha_1 d(z,p)+\alpha_2 d(z,z)+\alpha_3 d(p,z)+\alpha_4 d(p,z), \\ &= (\alpha_1+\alpha_3+\alpha_4)d(p,z). \\ \Rightarrow \theta &\ge (\alpha_1+\alpha_3+\alpha_4)d(p,z). \end{aligned}$$

That is, $(\alpha_1+\alpha_3+\alpha_4)d(p, z) \le \theta$.

$\Rightarrow d(p, z) = \theta$. That is, $p = z$.

Therefore, $p = z = Tp$.

Therefore, z is a fixed point of T.

Remark 2.3. If we choose $\alpha_4 = 0$ in Theorem 2.1, then we get that Theorem 2.1. of [6].

Remark 2.4. If we choose $\alpha_1 = k$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$ in Theorem 2.1, then we get that Corollary 2.1. of [6].

References

[1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341(2008) 416-420.

[2] M. Arshad, A. Azam, P. Vetro, Some common fixed point results on cone metric spaces, Fixed Point Theory Appl. (2009)11, Article ID 493965.

[3] A. Azam, M. Arshad, I. Beg, Banach contraction principle on cone rectangular metric spaces,Appl. Anal. Discrete Math., 3(2),(2009), 236-241.

[4] P. Z. Daffer and H. Kaneko, On expansive mappings, Math. Japon 37 (1992), 733-735.

[5] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2) (2007)1468-1476.

[6] X. Huang, Ch. Zhu and Xi Wen, Fixed point theorems for expanding mappings in cone metric spaces, Math Reports 14(64),2(2012), 141-148.

[7] D. Ilic, V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341(2008) 876-886.

[8] Sh. Rezapour and R. Hamlbarani, Some notes on the paper cone metric spaces and fixed point theorem of contractive mappings, J. Math. Anal. Appl. 345(2008), 719-724.

[9] S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iseki, Some fixed point theorems for expansion mappings, Math. Japon 29(1984), 631-636.