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On Generalized Trigonometric Functions

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Abstract A new trigonometric functions called generalized trigonometric functions are perversely defined by a system of first order nonlinear ordinary differential equations with initial conditions. This system is related to the Hamilton system. In this paper, we define these functions using the equation $|x|^m + |y|^m = 1$, for $m > 0$. We study the graphs, the trigonometric identities and some of common properties of these functions. We find the first derivatives which have different forms when m is even and when m is odd.

Keywords: *trigonometric functions, generalized trigonometric functions, trigonometric identities*

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1. Introduction

Ordinary trigonometry studies triangles in the Euclidean plane $\mathbb{R}^2$. There are some ways to defining the ordinary trigonometric functions on real numbers such as right-angled triangle definition, unit-circle definition, series definition, definitions via differential equations, and definition using functional equations. Trigonometric functions are one of the important group of the elementary functions. Using them, we can solve geometric problems, complex analytic problems and problems involving Fourier series. Also they are important because they are periodic. All the six trigonometric functions can defined through the sine and cosine functions.

In many papers, (see [1,2,3,4]), a new trigonometric functions are defined using a system of first order nonlinear ordinary differential equations with initial conditions. This system is related to the Hamilton system. The new functions are called generalized trigonometric functions and denoted by $\sin_m x, \cos_m x, \tan_m x, \cot_m x, \sec_m x, \csc_m x$ for $m > 0$. It was proved that if $x = \cos_m \theta$ and $y = \sin_m \theta$, then $|x|^m + |y|^m = 1$. In this paper, we define these functions directly using the equation $|x|^m + |y|^m = 1$, for $m > 0$. We study the graphs and the trigonometric identities of these functions. Then we study the first derivative for special cases when m is natural number. Since trigonometric functions are used in Fourier series, Fourier transform, and signal processing, we look to improve the efficiency of signal processing and reduce the noice effects by using the generalized trigonometric functions. Moreover, the generalized trigonometric functions can be used to obtain analytic solutions to the equation of a nonlinear spring-max system.

Now, consider the equation $|x|^m + |y|^m = 1$. The graph of this equation in the Cartesian plan is symmetric about the axes. For a special case, if $m = 1$, the graph of $|x| + |y| = 1$ is a unit square centered at $(0,0)$ with vertices at $(1,0), (0,1), (-1,0), (0,-1)$. For $m = 2$ the graph is the unit circle. For any $m > 0$, let S_m be the graph of the equation $|x|^m + |y|^m = 1$. We call S_m *a unit semi-square*. For an angle θ placed in the center of S_m with initial ray on the positive $x-axis$ and terminal ray intersects the graph of S_m n a point $P(x,y)$, we say that θ is placed in the *standard position* of S_m. Suppose that the angle $\theta = \frac{\pi}{4}$ is placed in the standard position of S_m and $P_m(x,y)$ s the point of intersection of the terminal ray and S_m. Then along the terminal ray, we have the following:

1. $P_m \to (0,0)$ as $m \to 0$, and
2. $P_m \to (1,1)$ as $m \to \infty$.

In Figure 1, we graph the equation $|x|^m + |y|^m = 1$ for several values of $m > 0$ showing the point P_m.

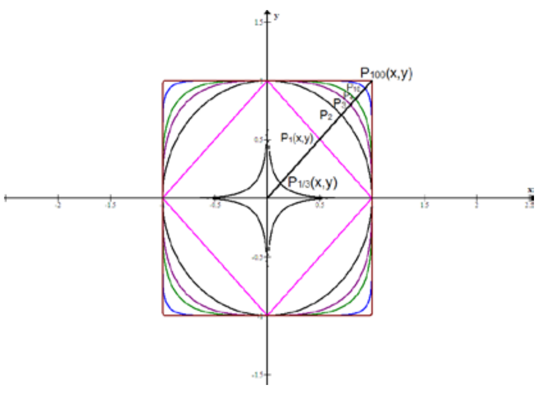

Figure 1. Graph of $|x|^m + |y|^m = 1$ for $m = \frac{1}{3}, 1, 2, 3, 4, 10, 100$

Definition 1.1 *In the Cartesian plane, if θ is an angle placed in the center of the plane with initial ray at the positive $x-axis$, then we say that:*

1. $\theta \in Q_1$ if the terminal ray lies between the positive $x-axis$ and the positive $y-axis$. $\theta \in Q_1^*$ if $\theta \in Q_1$ or $\theta = 2n\pi, n \in \mathbb{Z}$.
2. $\theta \in Q_2$ if the terminal ray lies between the negative $x-axis$ and the positive $y-axis$. $\theta \in Q_2^*$ if $\theta \in Q_2$ or $\theta = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$.
3. $\theta \in Q_3$ if the terminal ray lies between the negative $x-axis$ and the negative $y-axis$. $\theta \in Q_3^*$ if $\theta \in Q_3$ or $\theta = \pi + 2n\pi, n \in \mathbb{Z}$.
4. $\theta \in Q_4$ if the terminal ray lies between the positive $x-axis$ and the negative $y-axis$. $\theta \in Q_4^*$ if $\theta \in Q_4$ or $\theta = \frac{3\pi}{2} + 2n\pi, n \in \mathbb{Z}$.

2. Generalized Trigonometric Functions; Definitions and Graphs

Definition 2.1 *For a given $m > 0$, and for a unit semi-square S_m, let θ be an angle placed in the standard position. Suppose that the terminal ray intersects S_m in a point $P(x, y)$ (as seenin Figure 2). We define the six generalized trigonometric functions of θ as follows:*

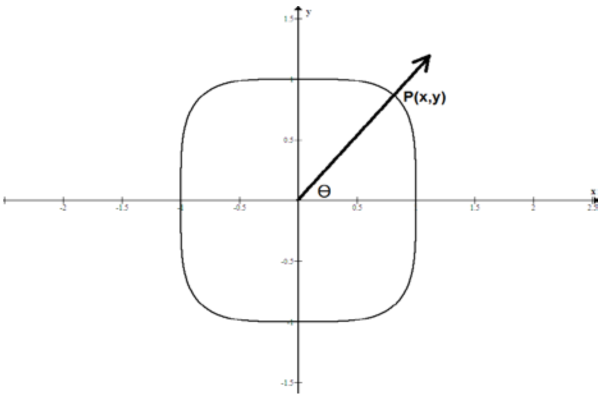

Figure 2. θ in standard position on S_m

1. g-sine of $\theta : \sin_m \theta = y$.
2. g-cosine of $\theta : \cos_m \theta = x$.
3. g-tangent of $\theta : \tan_m \theta = \frac{y}{x}$ *provided* $x \neq 0$.
4. g-cosecant of $\theta : \csc_m \theta = \frac{1}{y}$ *provided* $y \neq 0$.
5. g-secant of $\theta : \sec_m \theta = \frac{1}{x}$ *provided* $x \neq 0$.
6. g-cotangent of $\theta : \cot_m \theta = \frac{x}{y}$ *provided* $y \neq 0$.

The following table gives some values of g-trigonometric functions for some special angles:

Table 1. Values of $\sin_m \theta, \cos_m \theta$ **, and** $\tan_m \theta$ **for selected** θ

θ	**0**	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin_m \theta$	0	1	0	-1	0
$\cos_m \theta$	1	0	-1	0	1
$\tan_m \theta$	0	*Undefined*	0	*Undefined*	0

Theorem 2.2 *For all $\theta \in \mathbb{R}$, we have the following:*

1. $|\cos_m \theta|^m + |\sin_m \theta|^m = 1$.
2. $1 + |\tan_m \theta|^m = |\sec_m \theta|^m$.
3. $1 + |\cot_m \theta|^m = |\csc_m \theta|^m$.

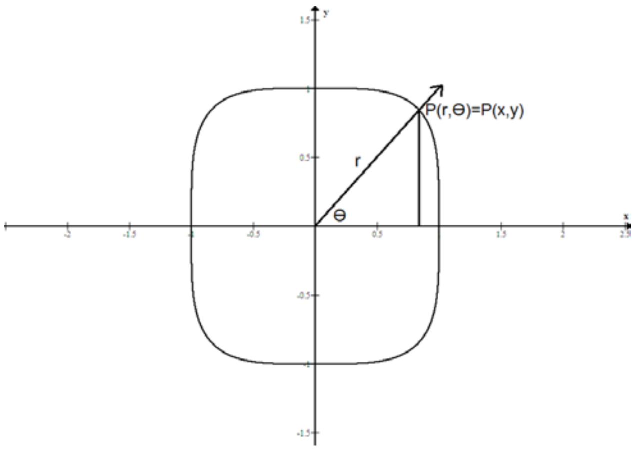

Figure 3. Polar versus Cartesian coordinate

The polar equation of the unit semi-square S_m is

$$r = \frac{1}{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}.$$

Using this equation,we get relations between usual trigonometric and g- trigonometric functions in the following theorem:

Theorem 2.3 *For any $\theta \in \mathbb{R}$, we have the following:*

1. $\sin_m \theta = \frac{\sin\theta}{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}$.
2. $\cos_m \theta = \frac{\cos\theta}{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}$.
3. $\tan_m \theta = \tan\theta$, *provided* $\cos\theta \neq 0$.
4. $\csc_m \theta = \frac{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}{\sin\theta}$, *provided* $\sin\theta \neq 0$.
5. $\sec_m \theta = \frac{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}{\cos\theta}$, *provided* $\cos\theta \neq 0$.
6. $\cot_m \theta = \cot\theta$, *provided* $\sin\theta \neq 0$.

Corollary 2.4 *Let $y = f(\theta)$ be a g-trigonometric functions.For any $k \in \mathbb{Z}$ and for any $\theta \in \mathbb{R}$, we have that $f(\theta + 2k\pi) = f(\theta)$. That is, g-trigonometric functions are periodic functions.*

Theorem 2.5 *Let $\theta \in \mathbb{R}$. Then:*

1. $\sin_m(\theta+\frac{\pi}{2})=\cos_m\theta.$
2. $\cos_m(\theta+\frac{\pi}{2})=-\sin_m\theta.$
3. $\sin_m(\theta\mp\pi)=-\sin_m\theta.$
4. $\cos_m(\theta\mp\pi)=\pm\cos_m\theta.$

Using the relations between g-trigonometric functions and usual trigonometric functions and using the Graph-4.4.2 grapher program, we give graphs of g-trigonometric functions for $m=\frac{1}{3}, 1$ and 4. We neglect the graphs of $\tan_m\theta$ and $\cot_m\theta$ since they are exactly the graphs of $\tan\theta$ and $\cot\theta$ respectively.

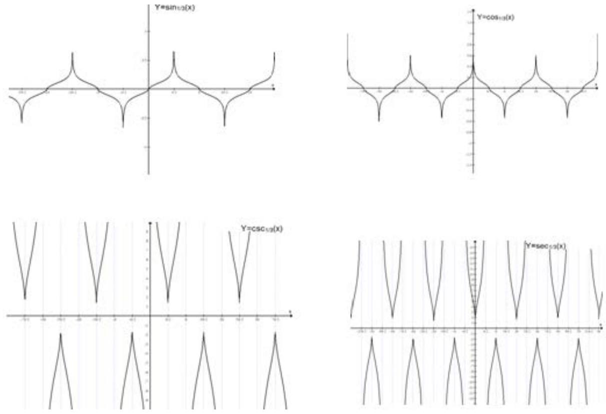

Figure 4. The graph of g-trigonometric functions when $m=1$

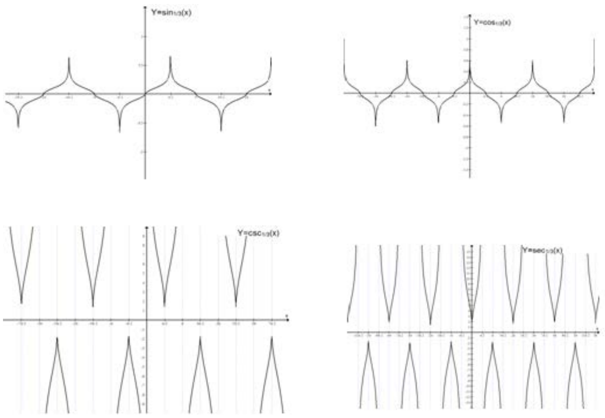

Figure 5. The graph of g-trigonometric functions when $m=\frac{1}{3}$

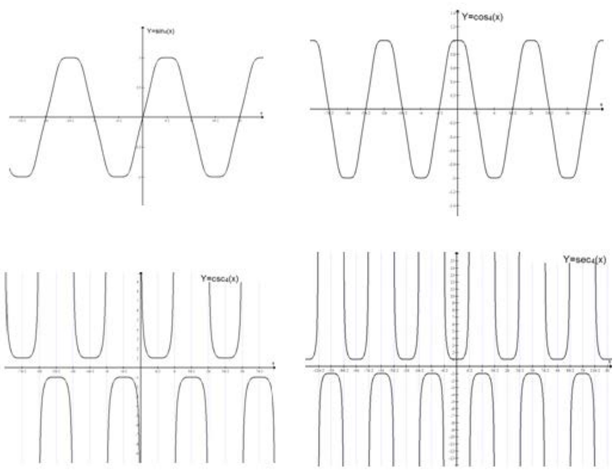

Figure 6. The graph of g-trigonometric functions when $m=4$

3. Identities and Some Common Properties

Evidently, and for $m>0$, the g-trigonometric functions have the following direct common properties:

1. $|\sin_m x|\le 1$ *and* $|\cos_m x|\le 1.$
2. All g-trigonometric functions are periodic. Moreover, the functions $\sin_m x, \cos_m x, \sec_m x$ and $\csc_m x$ have period 2π, while the functions $\tan_m x$ and $\cot_m x$ have period π.
3. The g-trigonometric functions $y=\cos_m x$ and $y=\sec_m x$ are even functions, while the other g-trigonometric functions are odd functions.

Theorem 3.1 *For any* $\theta_1,\theta_2\in\mathbb{R},$ *and* $m>0,$

$$\cos_m(\theta_1-\theta_2)=\frac{\cos_m\theta_1\cos_m\theta_2+\sin_m\theta_1\sin_m\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos_m\theta_1\cos_m\theta_2+\sin_m\theta_1\sin_m\theta_2|^m\\+|\sin_m\theta_1\cos_m\theta_2-\cos_m\theta_1\sin_m\theta_2|^m\end{array}\right]}}$$

Proof. Consider S_m the graph of $|x|^m+|y|^m=1$ in the first quadrant, as shown Figure 7. Draw the two vectors $\overleftarrow{oa}$ and $\overleftarrow{oc}$ as terminal rays of the two angles θ_1 and θ_2 respectively. So, we have the following:

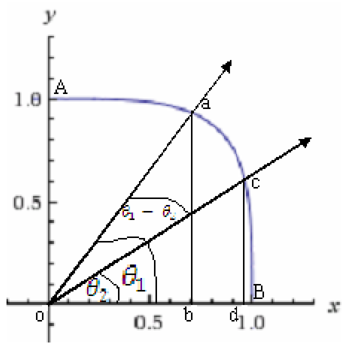

Figure 7. The graph of S_m in the first quadrant

$$ob=\cos_m\theta_1, od=\cos_m\theta_2, ab=\sin_m\theta_1, cd=\sin_m\theta_2.$$

Hence

$$cos_m(\theta_1-\theta_2)=\frac{\cos(\theta_1-\theta_2)}{\sqrt[m]{|\cos(\theta_1-\theta_2)|^m+|\sin(\theta_1-\theta_2)|^m}}$$

$$=\frac{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2|^m\\+|\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2|^m\end{array}\right]}}$$

$$=\frac{ob.od+ab.cd}{\sqrt[m]{\left[|ob.od+ab.cd|^m+|ab.od-ob.cd|^m\right]}}\cdot\frac{\sqrt[m]{|oa.oc|^m}}{oa.oc}$$

$$=\frac{ob.od+ab.cd}{\sqrt[m]{|ob.od+ab.cd|^m+|ab.od-ob.cd|^m}}$$

$$=\frac{\cos_m\theta_1\cos_m\theta_2+\sin_m\theta_1\sin_m\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos_m\theta_1\cos_m\theta_2+\sin_m\theta_1\sin_m\theta_2|^m\\+|\sin_m\theta_1\cos_m\theta_2-\cos_m\theta_1\sin_m\theta_2|^m\end{array}\right]}}.$$

Corollary 3.2 *For any* $\theta_1, \theta_2 \in \mathbb{R}$,

1. $$\cos_m(\theta_1+\theta_2) = \frac{\cos_m\theta_1\cos_m\theta_2 - \sin_m\theta_1\sin_m\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos_m\theta_1\cos_m\theta_2 - \sin_m\theta_1\sin_m\theta_2|^m \\ +|\sin_m\theta_1\cos_m\theta_2 + \cos_m\theta_1\sin_m\theta_2|^m\end{array}\right]}}.$$

2. $$\cos_m 2\theta = \frac{\cos_m^2\theta - \sin_m^2\theta}{\sqrt[m]{|\cos_m^2\theta - \sin_m^2\theta|^m + |2\sin_m\theta\cos_m\theta|^m}}.$$

The proof of the following theorem can be done directly or similar to the proof ofTheorem 3.1. Either way, it is easy and we will omit it.

Theorem 3.3 *For any* $\theta_1, \theta_2 \in \mathbb{R}$,

1. $$\sin_m(\theta_1-\theta_2) = \frac{\sin_m\theta_1\cos_m\theta_2 - \cos_m\theta_1\sin_m\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos_m\theta_1\cos_m\theta_2 + \sin_m\theta_1\sin_m\theta_2|^m \\ +|\sin_m\theta_1\cos_m\theta_2 - \cos_m\theta_1\sin_m\theta_2|^m\end{array}\right]}}.$$

2. $$\sin_m(\theta_1+\theta_2) = \frac{\sin_m\theta_1\cos_m\theta_2 + \cos_m\theta_1\sin_m\theta_2}{\sqrt[m]{\left[\begin{array}{l}|\cos_m\theta_1\cos_m\theta_2 - \sin_m\theta_1\sin_m\theta_2|^m \\ +|\sin_m\theta_1\cos_m\theta_2 + \cos_m\theta_1\sin_m\theta_2|^m\end{array}\right]}}.$$

3. $$\sin_m(2\theta) = \frac{2\sin_m(\theta)\cos_m\theta}{\sqrt[m]{\left[\begin{array}{l}|\cos_m^2\theta - \sin_m^2\theta|^m \\ +|2\sin_m\theta\cos_m\theta|^m\end{array}\right]}}.$$

4. $$\tan_m(\theta_1-\theta_2) = \frac{\tan_m\theta_1 - \tan_m\theta_2}{1+\tan_m\theta_1\tan_m\theta_2}.$$

5. $$\tan_m(\theta_1+\theta_2) = \frac{\tan_m\theta_1 + \tan_m\theta_2}{1-\tan_m\theta_1\tan_m\theta_2}.$$

6. $$\tan_m(2\theta) = \frac{2\tan_m(\theta)}{1-\tan_m^2(\theta)}.$$

7. $\cos_m^2(\theta) + \sin_m^2(\theta) = r^2$.

Theorem 3.4 *For any* $m > 0$, *we have that* $\lim_{\theta\to 0}\frac{\sin_m\theta}{\theta} = 1.$

4. Derivatives of g-Trigonometric Functions

In this section and unless otherwise statement, we consider m as a natural number. Let m be an even natural number and S_m be a unit semi-square. Then the polar equation of S_m has the form $r = (\cos^m\theta + \sin^m\theta)^{-\frac{1}{m}}$. So, the first derivative of r with respect to θ is

$$\frac{dr}{d\theta} = r(\cos_m^{m-1}\theta\sin_m\theta - \sin_m^{m-1}\theta\cos_m\theta)$$

Theorem 4.1 *If* m *is even, then*

1. $\frac{d}{d\theta}(\cos_m\theta) = -\sin_m^{m-1}\theta(\sin_m^2\theta + \cos_m^2\theta).$
2. $\frac{d}{d\theta}(\sin_m\theta) = \cos_m^{m-1}\theta(\sin_m^2\theta + \cos_m^2\theta).$
3. $\frac{d}{d\theta}(\tan_m\theta) = \sec^2\theta.$
4. $\frac{d}{d\theta}(\sec_m\theta) = \sin_m^{m-1}\theta(\tan_m^2\theta + 1).$
5. $\frac{d}{d\theta}(\csc_m\theta) = -\cos_m^{m-1}\theta(\cot_m^2\theta + 1).$
6. $\frac{d}{d\theta}(\cot_m\theta) = -\csc^2\theta.$

Mathematically, if m is even, then $|x|^m = x^m$ for all $x \in \mathbb{R}$. But if m is odd, the value of $|x|^m$ depends on the sign of x. So, there is a quite difference between the forms of derivatives of the g-trigonometric functions when m is even and when m is odd. Moreover, we will see that in the case when m is odd, the derivatives of the g-trigonometric functions have different forms when $m = 1$ and when $m > 1$. If m is odd, the polar equation of S_m has the following piecewise definition function with four cases depending on θ:

$$r = \begin{cases} \frac{1}{\sqrt[m]{\cos^m\theta + \sin^m\theta}}, & \theta \in Q_1^*; \\ \frac{1}{\sqrt[m]{-\cos^m\theta + \sin^m\theta}}, & \theta \in Q_2^*; \\ \frac{1}{\sqrt[m]{-\cos^m\theta - \sin^m\theta}}, & \theta \in Q_3^*; \\ \frac{1}{\sqrt[m]{\cos^m\theta - \sin^m\theta}}, & \theta \in Q_4^*. \end{cases}$$

In order to simplify the derivatives of the generalized trigonometric functions when m is odd, define the following four functions:

$$\delta(\theta) = \begin{cases} 1, \theta \in Q_1 \cup Q_2, \\ -1, \theta \in Q_3 \cup Q_4. \end{cases} \quad \delta^*(\theta) = \begin{cases} 1, \theta \in Q_1^* \cup Q_2^*, \\ -1, \theta \in Q_3^* \cup Q_4^*. \end{cases}$$

$$\gamma(\theta) = \begin{cases} 1, \theta \in Q_1 \cup Q_4, \\ -1, \theta \in Q_2 \cup Q_3. \end{cases} \quad \gamma^*(\theta) = \begin{cases} 1, \theta \in Q_1^* \cup Q_4^*, \\ -1, \theta \in Q_2^* \cup Q_3^*. \end{cases}$$

So, we have that for $n \in \mathbb{Z}$, if $\theta \neq (2n+1)\frac{\pi}{2}$, then $\delta(\theta) = \frac{|\sin\theta|}{\sin\theta}$, and if $\theta \neq n\pi$, then $\gamma(\theta) = \frac{|\cos\theta|}{\cos\theta}$.

Theorem 4.2 *Let* m *be an odd natural number, and let* $r = \frac{1}{\sqrt[m]{|\cos\theta|^m + |\sin\theta|^m}}$ *be the polar equation of* S_m.

Then for $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}$,

$$\frac{dr}{d\theta} = r\left(\gamma(\theta)\sin_m\theta\cos_m^{m-1}\theta - \delta(\theta)\sin_m^{m-1}\theta\cos_m\theta\right)$$

Proof. Derive directly, we get that

$$\frac{dr}{d\theta}=\frac{\left[-\frac{1}{m}(|\cos\theta|^m+|\sin\theta|^m)^{\frac{-m+1}{m}}\begin{pmatrix}m|\cos\theta|^{m-1}\frac{|\cos\theta|}{\cos\theta}(-\sin\theta)\\+m|\sin\theta|^{m-1}\frac{|\sin\theta|}{\sin\theta}(\cos\theta)\end{pmatrix}\right]}{(|\cos\theta|^m+|\sin\theta|^m)^{\frac{2}{m}}}.$$

As m is odd, $m-1$ is even and we have that

$$\frac{dr}{d\theta}=\left(|\cos\theta|^m+|\sin\theta|^m\right)^{\frac{-1}{m}-1}$$
$$\times\left(\cos^{m-1}\theta\sin\theta\frac{|\cos\theta|}{\cos\theta}-\sin^{m-1}\theta\cos\theta\frac{|\sin\theta|}{\sin\theta}\right)$$
$$=r.r^m\left(\cos^{m-1}\theta\sin\theta\frac{|\cos\theta|}{\cos\theta}-\sin^{m-1}\theta\cos\theta\frac{|\sin\theta|}{\sin\theta}\right)$$
$$=r(\cos_m^{m-1}\theta\sin_m\theta\frac{|\cos\theta|}{\cos\theta}-\sin_m^{m-1}\theta\cos_m\theta\frac{|\sin\theta|}{\sin\theta})$$

Since $\theta\neq n\frac{\pi}{2}, n\in\mathbb{Z}$,

$$\frac{dr}{d\theta}=r\begin{pmatrix}\gamma(\theta)\cos_m^{m-1}\theta\sin_m\theta\\-\delta(\theta)\sin_m^{m-1}\theta\cos_m\theta\end{pmatrix}.$$

Theorem 4.3 *If $m=1$, then $\frac{dr}{d\theta}$ does not exist for all $\theta=n\frac{\pi}{2}, n\in\mathbb{Z}$.*

Proof. For $m=1$, the right hand and the left hand derivatives of r with respect to θ at $\theta=n\pi$ are

$$\left(\frac{dr}{d\theta}\right)^+=-1 \text{and} \left(\frac{dr}{d\theta}\right)^-=1.$$

Hence, $\frac{dr}{d\theta}$ does not exist for all $\theta=n\pi$, $n\in\mathbb{Z}$.

Similarly, $\left(\frac{dr}{d\theta}\right)^+\neq\left(\frac{dr}{d\theta}\right)^-$ for all $\theta=(2n+1)\frac{\pi}{2}$.

Theorem 4.4 *If m is odd, and $m\neq1$, then $\frac{dr}{d\theta}=0$ for all $\theta=n\frac{\pi}{2}, n\in\mathbb{Z}$.*

Proof. If $m>1$, then $\forall\theta=n\frac{\pi}{2}, n\in\mathbb{Z}$, both $\cos_m^{m-1}\theta\sin_m\theta=0$ and $\sin_m^{m-1}\theta\cos_m\theta=0$. This implies that

$$\left(\frac{dr}{d\theta}\right)^+\Bigg|_{\theta=n\frac{\pi}{2}}=\left(\frac{dr}{d\theta}\right)^-\Bigg|_{\theta=n\frac{\pi}{2}}=0.$$

Remark 4.5 *For all $m>0$ and for $\theta\neq n\frac{\pi}{2}, n\in\mathbb{Z}$, we have $\delta(\theta)=\delta^*(\theta)$ and $\gamma(\theta)=\gamma^*(\theta)$, so we get that*

$$\frac{dr}{d\theta}=r\begin{pmatrix}\gamma^*(\theta)\cos_m^{m-1}\theta\sin_m\theta\\-\delta^*(\theta)\sin_m^{m-1}\theta\cos_m\theta\end{pmatrix}.$$

Moreover, if $m\neq1$ and at $\theta=n\frac{\pi}{2}, n\in\mathbb{Z}$, we have that, $r=1$, $\delta^*(\theta)$ and $\gamma^*(\theta)$ have values 1 or -1, and $\cos_m^{m-1}\theta\sin_m\theta=0$, $\sin_m^{m-1}\theta\cos_m\theta=0$. Hence

$$\frac{dr}{d\theta}=r\begin{pmatrix}\gamma^*(\theta)\cos_m^{m-1}\theta\sin_m\theta\\-\delta^*(\theta)\sin_m^{m-1}\theta\cos_m\theta\end{pmatrix}$$
$$=1(\gamma^*(\theta)\cdot0-\delta^*(\theta)\cdot0)=0.$$

Using this remark, we have the following summary theorem.

Theorem 4.6 *Let m be an odd natural number, and let $r=(|\cos\theta|^m+|\sin\theta|^m)^{-\frac{1}{m}}$ be the polar equation of the unit semi-square S_m. Then*

a. For $m=1$, and for $\theta\neq n\frac{\pi}{2}, n\in\mathbb{Z}$,

$$\frac{dr}{d\theta}=r(\gamma(\theta)\sin_1\theta-\delta(\theta)\cos_1\theta).$$

b. For $m\neq1$, and for all θ,

$$\frac{dr}{d\theta}=r\left(\gamma^*(\theta)\cos_m^{m-1}\theta\sin_m\theta-\delta^*(\theta)\sin_m^{m-1}\theta\cos_m\theta\right).$$

Theorem 4.7

a. For $m=1$, and for $\theta\neq n\frac{\pi}{2}, n\in\mathbb{Z}$,

1. $\frac{d}{d\theta}(\cos_1\theta)=-\delta(\theta)(\cos_1^2\theta+\sin_1^2\theta)$.
2. $\frac{d}{d\theta}(\sin_1\theta)=\gamma(\theta)(\cos_1^2\theta+\sin_1^2\theta)$.
3. $\frac{d}{d\theta}(\sec_1\theta)=\delta(\theta)(1+\tan_1^2\theta)$.
4. $\frac{d}{d\theta}\csc_1\theta=-\gamma(\theta)(1+\cot_1^2\theta)$.

b. For $m\neq1$, and for all θ,

1. $\frac{d}{d\theta}(\cos_m\theta)=-\sin_m^{m-2}\theta|\sin_m\theta|\left(\sin_m^2\theta+\cos_m^2\theta\right)$.
2. $\frac{d}{d\theta}\sin_m\theta=\cos_m^{m-2}\theta|\cos_m\theta|\left(\cos_m^2\theta+\sin_m^2\theta\right)$.
3. $\frac{d}{d\theta}\sec_m\theta=\sin_m^{m-2}\theta|\sin_m\theta|(1+\tan_m^2\theta)$.
4. $\frac{d}{d\theta}\csc_m\theta=-\cos_m^{m-2}\theta|\cos_m\theta|(1+\cot_m^2\theta)$.

Proof. a) (1) For $m=1$, nd for $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}$, we have

$$\frac{d}{d\theta}\left(\cos_1 \theta\right)=\frac{d}{d\theta}\left(r\cos\theta\right)=-r\sin\theta+\cos\theta\frac{dr}{d\theta}$$
$$=-r\sin\theta+r\cos\theta\left(\gamma(\theta)\sin_1\theta-\delta(\theta)\cos_1\theta\right)$$
$$=-\sin_1\theta+\left|r\cos\theta\right|\sin_1\theta-\delta\left(\theta\right)\cos_1\theta r\cos\theta$$
$$=-\sin_1\theta(1-|\cos_1\theta|)-\delta(\theta)\cos_1^2\theta$$
$$=-\sin_1\theta\,|\sin_1\theta|-\delta(\theta)\cos_1^2\theta$$
$$=-\sin_1\theta r\left|\sin\theta\right|-\delta\left(\theta\right)\cos_1^2\theta$$
$$=-r\sin_1\theta\delta\left(\theta\right)\sin\theta-\delta\left(\theta\right)\cos_1^2\theta$$
$$=-\delta(\theta)(\sin_1^2\theta+\cos_1^2\theta)$$

b)(1) For $m \neq 1$, we have two cases:

Case 1: $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}.$

In this case, $\delta^*(\theta)=\frac{|\sin\theta|}{\sin\theta}$ and $\gamma^*(\theta)=\frac{|\cos\theta|}{\cos\theta}$. So,

$$\frac{d}{d\theta}(\cos_m\theta)=\frac{d}{d\theta}(r\cos\theta)=-r\sin\theta+\cos\theta\frac{dr}{d\theta}$$
$$=-\sin_m\theta+\cos\theta(r\begin{pmatrix}\gamma^*\left(\theta\right)\cos_m^{m-1}\theta\sin_m \\ -\delta^*(\theta)\sin_m^{m-1}\theta\cos_m\theta\end{pmatrix}).$$

Since m is odd, $m-1$ is even, so

$$\frac{d}{d\theta}\left(\cos_m\theta\right)$$
$$=-\sin_m+r\cos\theta\begin{pmatrix}|\cos_m\theta|^{m-1}\sin_m\theta\frac{|\cos\theta|}{\cos\theta} \\ -|\sin_m\theta|^{m-1}\cos_m\theta\frac{|\sin\theta|}{\sin\theta}\end{pmatrix}$$
$$=-\sin_m\theta+r\begin{pmatrix}|\cos_m\theta|^{m-1}\sin\theta\,|\cos_m\theta| \\ -|\sin_m\theta|^{m-1}\cos^2\theta\frac{|\sin_m\theta|}{\sin\theta}\end{pmatrix}$$
$$=-\sin_m\theta+r\begin{pmatrix}|\cos_m\theta|^m\sin\theta \\ -|\sin_m\theta|^m\frac{1}{\sin\theta}(1-\sin^2\theta)\end{pmatrix}$$
$$=-\sin_m\theta+r\begin{pmatrix}\sin\theta(|\cos_m\theta|^m+|\sin_m\theta|^m) \\ -|\sin_m\theta|^m\frac{1}{\sin\theta}\end{pmatrix}$$
$$=-\sin_m\theta+r\left(\sin\theta-|\sin_m\theta|^m\frac{1}{\sin\theta}\right)$$
$$=-r(\sin_m\theta)^{m-1}\frac{|\sin_m\theta|}{\sin\theta}$$
$$=-r^2\sin_m^{m-2}\theta\,|\sin_m\theta|$$
$$=-\sin_m^{m-2}\theta\,|\sin_m\theta|\,(\sin_m^2\theta+\cos_m^2\theta).$$

Case 2: $\theta=n\frac{\pi}{2}, n\in\mathbb{Z}$. In this case, $\frac{dr}{d\theta}=0$. So,

$$\frac{d}{d\theta}(\cos_m\theta)=\frac{d}{d\theta}(r\cos\theta)$$
$$=-r\sin\theta+\cos\theta\frac{dr}{d\theta}=-\sin_m\theta.$$

Now if $\theta=n\pi$, then $\frac{d}{d\theta}(\cos_m\theta)=0$. In this case,

$$(-\sin_m^{m-2}\theta\,|\sin_m\theta|\,(\sin_m^2\theta+\cos_m^2\theta))=0.$$

And if $\theta=(2n+1)\frac{\pi}{2}, \frac{d}{d\theta}(\cos_m\theta)=1$. In this case,

$$(-\sin_m^{m-2}\theta\,|\sin_m\theta|\,(\sin_m^2\theta+\cos_m^2\theta))=1.$$

In both cases, we have that

$$\frac{d}{d\theta}(\cos_m\theta)=-\sin_m^{m-2}\theta\,|\sin_m\theta|\,(\sin_m^2\theta+\cos_m^2\theta).$$

Remark 4.8 *Since* $\tan_m\theta=tan\theta$, *and* $\cot_m\theta=cot\theta$, *we have that*

$$\frac{d}{d\theta}(\tan_m\theta)=\sec^2\theta and\frac{d}{d\theta}(\cot_m\theta)=-\csc^2\theta.$$

References

[1] Burgoyre F.D.,"Generalization trigonometric functions.", Mathematics of Computation, vol.18, pp 3-14-316,1964.

[2] Edmunds D. and Lang J.,"Generalizing trigonometric functions from different points of view," http://www.math.osu.edu/mri/preprints/2009, Tech. Rep., 2009.

[3] Lang J. and Edmunds D.,"Eigenvalues, Embeddings and Generalised Trigonometric Functions", Springer, 2011.

[4] Shelupsky D.,"A generalization of trigonometric functions.", The American Mathematical Monthly, Vol.66 no. 10, pp 879-884,1959.

[5] Wei D.,Elgindi Y. L..,and Elgindi M. B., "Some Generalized Trigonometric Sine Functions and Their Applications ", Applied Mathematical Sciences, Vol. 6, no. 122, 6053-6068, 2012.