

Fermat Collocation Method for Solving a Class of the First Order Nonlinear Differential Equations

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Abstract In this study, we present a reliable numerical approximation of the some first order nonlinear ordinary differential equations with the mixed condition by the using a new Fermat collocation method. The solution is obtained in the form of a truncated Fermat series with easily determined components. Also, the method can be used to solve Riccati equation. The numerical results show the effectuality of the method for this type of equations. Comparing the methodology with some known techniques shows that the existing approximation is relatively easy and highly accurate.

Keywords: Nonlinear ordinary differential equations, Riccati equation, Fermat polynomials, collocation points

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1. Introduction

Nonlinear ordinary differential equations are frequently used to model a wide class of problems in many areas of scientific fields; chemical reactions, spring-mass systems bending of beams, resistor-capacitor-inductance circuits, pendulums, the motion of a rotating mass around another body and so forth [1,2]. These equations here also demonstrated their usefulness in ecology, economics, biology, astrophysics and engineering. Thus, methods of solution for these equations are of great importance to engineers and scientists [3,4].

In this paper, for our aim we consider the first order nonlinear ordinary differential equation of the form

$$\begin{aligned} &P(x)y(x) + Q(x)y'(x) + R(x)y^2(x) \\ &+ S(x)y(x)y'(x) + T(x)(y'(x))^2 \\ &= g(x), a \leq x \leq b \end{aligned} \quad (1)$$

under the mixed conditions

$$\alpha y(a) + \beta y(b) = \lambda \quad (2)$$

and look for the approximate solution in the form

$$\begin{aligned} y(x) &= \sum_{n=0}^N y_n F_n(x), \\ F_n(x) &= 3xF_{n-1}(x) - 2F_{n-1}(x), \quad (3) \quad [5,6] \\ &a \leq x \leq b \end{aligned}$$

which is a Fermat polynomial of degree N , where $y_n(n = 0, 1, \dots, N)$ are the coefficients to be determined. Here $P(x)$, $Q(x)$, $R(x)$, $S(x)$, $T(x)$ and $g(x)$ are the functions defined on $a \leq x \leq b$; the real coefficients α, β and λ are appropriate

constants. Note that, if $S(x)=T(x)=0$ in Eq. (1), it is a Riccati Equation [7,8,9].

2. Fundamental Matrix Relations

Our aim is to find the matrix form of each term in the nonlinear equation given by Eq. (1). Firstly, we consider the solution $y(x)$ defined by a truncated series (3) and then we can convert to the matrix form

$$y(x) = \mathbf{F}(x)\mathbf{Y} \quad (4)$$

where

$$\begin{aligned} \mathbf{F}(x) &= [F_0(x) \quad F_1(x) \quad \dots \quad F_N(x)] \\ \mathbf{Y} &= [y_0 \quad y_1 \quad \dots \quad y_N]^T \end{aligned}$$

If we differentiate expression (4) with respect to x , we obtain

$$y'(x) = \mathbf{F}'(x)\mathbf{Y} = \mathbf{F}(x)\mathbf{D}\mathbf{Y} \quad (5)$$

where

$$\mathbf{D} = \begin{bmatrix} 0 & 3 & 0 & 6 & 0 & 12 & 0 & 24 & 0 & \dots \\ 0 & 0 & 6 & 0 & 12 & 0 & 24 & 0 & 48 & \dots \\ 0 & 0 & 0 & 9 & 0 & 18 & 0 & 36 & 0 & \dots \\ 0 & 0 & 0 & 0 & 12 & 0 & 24 & 0 & 48 & \dots \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & 30 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 36 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

$(N+1) \times (N+1)$

On the other hand, the matrix form of expression $y^2(x)$ or shortly is obtained as

$$y^2(x) = \begin{bmatrix} 1 & 3x & 9x^2 - 2 & \dots & F_N(x) \end{bmatrix} \begin{bmatrix} \mathbf{F}(x) & 0 & \dots & 0 \\ 0 & \mathbf{F}(x) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbf{F}(x) \end{bmatrix} \begin{bmatrix} y_0 \mathbf{Y} \\ y_1 \mathbf{Y} \\ \vdots \\ y_N \mathbf{Y} \end{bmatrix}$$

or briefly

$$y^2(x) = \mathbf{F}(x) \mathbf{F}^*(x) \mathbf{Y}^* \tag{6}$$

where

$$\mathbf{F}^*(x) = \begin{bmatrix} \mathbf{F}(x) & 0 & \dots & 0 \\ 0 & \mathbf{F}(x) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbf{F}(x) \end{bmatrix}, \tag{10}$$

$$\mathbf{Y}^* = [y_0 \mathbf{Y} \quad y_1 \mathbf{Y} \quad \dots \quad y_N \mathbf{Y}]^T,$$

By using the expression (4), (5) and (6) we obtain

$$y(x)y'(x) = \mathbf{F}(x) \mathbf{F}^*(x) \mathbf{D}^* \mathbf{Y}^* \tag{7}$$

Following a similar way to (6), we have

$$(y'(x))^2 = \mathbf{F}(x) \mathbf{D} \mathbf{F}^*(x) \mathbf{D}^* \mathbf{Y}^* \tag{8}$$

where

$$\mathbf{D}^* = \begin{bmatrix} \mathbf{D} & 0 & \dots & 0 \\ 0 & \mathbf{D} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbf{D} \end{bmatrix}$$

3. Matrix Relations Based on Collocation Points

Let us use the collocation points defined by

$$x_i = a + \frac{b-a}{N} i, \quad i = 0, 1, \dots, N \tag{9}$$

in order to

$$a = x_0 < x_1 < \dots < x_n = b.$$

By putting the collocation points (9) into Eq. (1), we get the equation

$$\begin{aligned} &P(x_i)y(x_i) + Q(x_i)y'(x_i) \\ &+ R(x_i)y^2(x_i) + S(x_i)y(x_i)y'(x_i) \\ &+ T(x_i)(y'(x_i))^2 = g(x_i) \end{aligned} \tag{10}$$

$$i = 0, 1, \dots, N;$$

$$a \leq x_i \leq b$$

By using the relations (4), (5), (6), (7) and (8); the system (10) can be written in the matrix form

$$\begin{aligned} &(\mathbf{P}\mathbf{F} + \mathbf{Q}\mathbf{F}\mathbf{D})\mathbf{Y} \\ &+ (\mathbf{R}\mathbf{F}\mathbf{F}^* + \mathbf{S}\mathbf{F}\mathbf{F}^*\mathbf{D}^* + \mathbf{T}\mathbf{F}\mathbf{D}\mathbf{F}^*\mathbf{D}^*)\mathbf{Y}^* = \mathbf{G} \end{aligned}$$

$$\mathbf{W}\mathbf{Y} + \mathbf{V}\mathbf{Y}^* = \mathbf{G} \tag{11}$$

where

$$\mathbf{P} = \begin{bmatrix} P(x_0) & 0 & \dots & 0 \\ 0 & P(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & P(x_N) \end{bmatrix};$$

$$\mathbf{Q} = \begin{bmatrix} Q(x_0) & 0 & \dots & 0 \\ 0 & Q(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Q(x_N) \end{bmatrix};$$

$$\mathbf{R} = \begin{bmatrix} R(x_0) & 0 & \dots & 0 \\ 0 & R(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & R(x_N) \end{bmatrix};$$

$$\mathbf{S} = \begin{bmatrix} S(x_0) & 0 & \dots & 0 \\ 0 & S(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & S(x_N) \end{bmatrix};$$

$$\mathbf{T} = \begin{bmatrix} T(x_0) & 0 & \dots & 0 \\ 0 & T(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & T(x_N) \end{bmatrix}; \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix};$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}(x_0) \\ \mathbf{F}(x_1) \\ \vdots \\ \mathbf{F}(x_N) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3x_0 & 9x_0^2 - 2 & \dots & 3x_0 F_{N-1}(x_0) \\ & & & & -2F_{N-2}(x_0) \\ 1 & 3x_1 & 9x_1^2 - 2 & \dots & 3x_1 F_{N-1}(x_1) \\ & & & & -2F_{N-2}(x_1) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 3x_N & 9x_N^2 - 2 & \dots & 3x_N F_{N-1}(x_N) \\ & & & & -2F_{N-2}(x_N) \end{bmatrix};$$

$$\mathbf{F}^* = \begin{bmatrix} \mathbf{F}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{F}(x_1) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbf{F}(x_N) \end{bmatrix}.$$

4. Method of Solution

The fundamental matrix equation (11) corresponding to Eq. (1), can be written as

$$\mathbf{W}\mathbf{Y} + \mathbf{V}\mathbf{Y}^* = \mathbf{G}$$

or

$$[\mathbf{W}; \mathbf{V} : \mathbf{G}] \tag{12}$$

where

$$\begin{aligned}\mathbf{W} &= [w_{pq}] = \mathbf{PF} + \mathbf{QFD}, \\ \mathbf{V} &= [v_{pq}] = \mathbf{RFF}^* + \mathbf{SFF}^* \mathbf{D}^* + \mathbf{TFDF}^* \mathbf{D}^*, \\ p, q &= 0, 1, \dots, N\end{aligned}$$

We can find the corresponding matrix equation for the condition (2), using the relation (4), as follows:

$$\{\alpha \mathbf{F}(a) + \beta \mathbf{F}(b)\} \mathbf{Y} = [\lambda] \quad (13)$$

so that

$$\begin{aligned}\mathbf{F}(a) &= \begin{bmatrix} 1 & 3a & 9a^2 - 2 & \dots & 3aF_{N-1}(a) - 2F_{N-2}(a) \end{bmatrix} \\ \mathbf{F}(b) &= \begin{bmatrix} 1 & 3b & 9b^2 - 2 & \dots & 3bF_{N-1}(b) - 2F_{N-2}(b) \end{bmatrix}.\end{aligned}$$

$$\overline{[\mathbf{W}; \mathbf{V}; \mathbf{G}]} = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & ; & v_{00} & v_{01} & v_{02} & \dots & v_{0N} & : & g(x_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & ; & v_{10} & v_{11} & v_{12} & \dots & v_{1N} & : & g(x_1) \\ \vdots & \vdots & \vdots & \dots & \vdots & ; & \vdots & \vdots & \vdots & \dots & \vdots & : & \vdots \\ w_{N-1,0} & w_{N-1,1} & w_{N-1,2} & \dots & w_{N-1,N} & ; & v_{N-1,0} & v_{N-1,1} & v_{N-1,2} & \dots & v_{N-1,N} & : & g(x_{N-1}) \\ z_0 & z_1 & z_2 & \dots & z_N & ; & 0 & 0 & 0 & \dots & 0 & : & \lambda \end{bmatrix}$$

or the corresponding matrix equation

$$\overline{\mathbf{W}}\mathbf{Y} + \overline{\mathbf{V}}\mathbf{Y} = \overline{\mathbf{G}} \quad (15)$$

where

$$\begin{aligned}\overline{\mathbf{W}} &= \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} \\ w_{10} & w_{11} & \dots & w_{1N} \\ \vdots & \vdots & & \vdots \\ w_{N-1,0} & w_{N-1,1} & \dots & w_{N-1,N} \\ z_0 & z_1 & \dots & z_N \end{bmatrix}, \\ \overline{\mathbf{V}} &= \begin{bmatrix} v_{00} & v_{01} & \dots & v_{0N} \\ v_{10} & v_{11} & \dots & v_{1N} \\ \vdots & \vdots & & \vdots \\ v_{N-1,0} & v_{N-1,1} & \dots & v_{N-1,N} \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \overline{\mathbf{G}} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_{N-1}) \\ \lambda \end{bmatrix}\end{aligned}$$

The unknown coefficients set $\{y_0, \dots, y_n\}$ can be determined from the nonlinear system (15). As a result, we can obtain approximate solution in the truncated series form (3).

5. Accuracy of Solution

We can check the accuracy of the solution by following procedure [11,12,13]: The truncated Fermat series in (3) have to be approximately satisfying Eq. (1); that is, for each $x=x_i \in [a, b]$, $i=1, 2, \dots$

$$E(x_i) = \begin{vmatrix} P(x_i)y(x_i) + Q(x_i)y'(x_i) \\ +R(x_i)y^2(x_i) + S(x_i)y(x_i)y'(x_i) \\ +T(x_i)(y'(x_i))^2 - g(x_i) \end{vmatrix} \cong 0$$

and $E(x_i) \leq 10^{-k_i}$ (k_i is any positive integer).

We can write the corresponding matrix form (13) for the mixed condition (2) in the augmented matrix form as

$$[\mathbf{Z}; \mathbf{0}; \lambda] \quad (14)$$

where

$$\begin{aligned}\mathbf{Z} &= [z_0 \ z_1 \ \dots \ z_N] = \alpha \mathbf{F}(a) + \beta \mathbf{F}(b) \\ \mathbf{Z} &= [z_{ij}] \\ &= \begin{bmatrix} 1 & 3a & 9a^2 - 2 & \dots & 3aF_{N-1}(a) - 2F_{N-2}(a) \\ + [1 & 3b & 9b^2 - 2 & \dots & 3bF_{N-1}(b) - 2F_{N-2}(b) \end{bmatrix} \\ \mathbf{0} &= [0 \ 0 \ \dots \ 0]_{1 \times (N+1)}.\end{aligned}$$

To obtain the approximate solution of Eq. (1) with the mixed condition (2) in the terms of Fermat polynomials, by replacing the row matrix (14) by the last row of the matrix (11), we obtain the required augmented matrix:

If $\max(10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k} .

6. Numerical Examples

In this section, two numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 6.1. [14] Let us first consider the first-order nonlinear differential equation

$$\begin{aligned}(y'(x))^2 - xy(x)y'(x) + y'(x) \\ - x^2 y(x) + (y(x))^2 = 2 - x^3\end{aligned} \quad (16)$$

with condition

$$y(-1) + 2y(1) = 1, -1 \leq x \leq 1$$

and the approximate solution $y(x)$ by the truncated Taylor polynomial

$$y(x) = \sum_{n=0}^2 y_n F_n(x), \quad -1 \leq x \leq 1$$

where

$$\begin{aligned}P(x) &= -x^2, \quad Q(x) = 1, \quad R(x) = 1, \\ S(x) &= -x, \quad T(x) = 1, \quad g(x) = 2 - x^3.\end{aligned}$$

For $N=2$ the collocation points become

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1.$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as

$$(\mathbf{PF} + \mathbf{QFD})\mathbf{Y} + (\mathbf{RFF}^* + \mathbf{SFF}^*\mathbf{D}^* + \mathbf{TFDF}^*\mathbf{D}^*)\mathbf{Y}^* = \mathbf{G}$$

and

$$\{\mathbf{F}(-1) + 2\mathbf{F}(1)\}\mathbf{Y} = [1]$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -3 & 7 \\ 1 & 0 & -2 \\ 1 & 3 & 7 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F}^* = \begin{bmatrix} 1 & -3 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 \end{bmatrix},$$

$$\mathbf{Y}^* = [y_0y_0 \quad y_0y_1 \quad y_0y_2 \quad y_1y_0 \quad y_1y_1 \quad y_1y_2 \quad y_2y_0 \quad y_2y_1 \quad y_2y_2]^T$$

The augmented matrix for this fundamental matrix equation is calculated

$$[\overline{\mathbf{W};\mathbf{V};\mathbf{G}}] = \begin{bmatrix} -1 & 6 & -25 & ; & 1 & 0 & -11 & -3 & 0 & 6 & 7 & 12 & -139 & : & 3 \\ 0 & 3 & 0 & ; & 1 & -3 & 7 & 0 & 9 & 0 & 0 & -2 & -6 & : & 2 \\ 3 & 3 & 21 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 1 \end{bmatrix}$$

From the obtained system, the coefficients y_0, y_1 and y_2 are found as $y_0=0, y_1=1/3$ and $y_2=0$.

Hence we have the Fermat polynomial solution

$$y(x) = x.$$

Example 6.2. [14] Consider the following nonlinear differential equation given by

$$y'(x) - x^2y(x) + x(y(x))^2 = -x^2 + x + 1 \quad (17)$$

with the initial condition

$$y(0) = -1, \quad -1 \leq x \leq 1.$$

So that

$$P(x) = -x^2, \quad Q(x) = 1, \quad R(x) = x,$$

$$S(x) = 0, \quad T(x) = 0, \quad g(x) = -x^2 + x + 1.$$

For $N=2$ the collocation points become

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1.$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as

$$(\mathbf{PF} + \mathbf{QFD})\mathbf{Y} + (\mathbf{RFF}^* + \mathbf{SFF}^*\mathbf{D}^* + \mathbf{TFDF}^*\mathbf{D}^*)\mathbf{Y}^* = \mathbf{G}$$

and

$$\mathbf{F}(0)\mathbf{Y} = [-1]$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -3 & 7 \\ 1 & 0 & -2 \\ 1 & 3 & 7 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated

$$[\overline{\mathbf{W};\mathbf{V};\mathbf{G}}] = \begin{bmatrix} -1 & 6 & -25 & ; & -1 & 3 & -7 & 3 & 0 & -6 & -7 & -21 & -49 & : & -1 \\ 0 & 3 & 0 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 1 \\ 1 & 0 & -2 & ; & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & -1 \end{bmatrix}$$

From the obtained system, the coefficients y_0, y_1 and y_2 are found as $y_0=0, y_1=1/3$ and $y_2=0$.

Hence we have the Fermat polynomial solution

$$y(x) = x - 1$$

Example 6.3. Consider the following nonlinear differential equation given by

$$x(y'(x))^2 - xy^2(x) + y(x) = e^x \quad (18)$$

with the initial condition

$$y(0) = 1, \quad 0 \leq x \leq 1.$$

So that

$$P(x) = 1, \quad Q(x) = 0, \quad R(x) = -x,$$

$$S(x) = 0, \quad T(x) = x, \quad g(x) = e^x.$$

The solutions obtained for $N=2,5,7$ are compared with the exact solution is e^x , which are given in Figure 1. We compare the numerical solution and absolute errors for $N=2,5,7$ in Table 1.

Table 1. Comparison of the numerical errors of Example 6.3

x_i	Exact solution	$N = 2$ Numerical solution	$N = 5$ Numerical solution	$N = 7$ Numerical solution	$N = 2$ Error	$N = 5$ Error	$N = 7$ Error
0	1	1	1	1	0	0	0
0.1	1.105170918	1.105012	1.105172	1.105171	1.594E-4	5.471E-7	1.778E-8
0.2	1.221402758	1.220046	1.221405	1.221403	1.357E-3	2.257E-6	9.627E-8
0.3	1.349858807	1.345104	1.349863	1.349859	4.755E-3	4.576E-6	2.721E-7
0.4	1.491824697	1.480184	1.491829	1.491825	1.164E-2	4.609E-6	5.714E-7
0.5	1.648721271	1.625287	1.648715	1.648722	2.343E-2	5.747E-6	9.765E-7
0.6	1.822118801	1.780414	1.822075	1.822121	4.171E-2	4.403E-5	1.338E-6
0.7	2.013752707	1.945564	2.013609	2.013754	6.819E-2	1.429E-4	1.196E-6
0.8	2.225540928	2.120736	2.225183	2.225541	1.05E-1	3.576E-4	5.613E-7
0.9	2.459603111	2.305932	2.458828	2.459597	1.54E-1	7.749E-4	6.256E-6
1	2.718281828	2.501115	2.716757	2.718262	2.18E-1	1.525E-3	2.029E-5

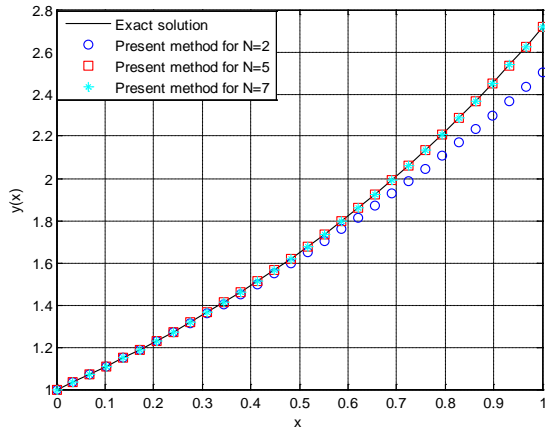


Figure 1. Numerical and exact solution of Example 6.3 for $N=2,5,7$

Example 6.4. Let us first consider the first-order nonlinear differential equation

$$x(y'(x))^2 + xy^2(x) + y(x) = x + \sin x \quad (19)$$

with condition

$$y(0) = 0, 0 \leq x \leq \pi.$$

Taking $N=3,5,7$ we obtain the approximate solution of this nonlinear differential equation.

The values of this solution are compared with the exact solution is $x + \sin x$ and given errors in Table 2.

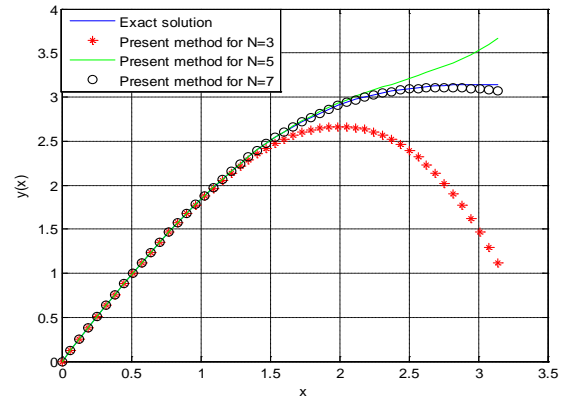


Figure 2. Numerical and exact solution of Example 6.4 for $N=3,5,7$

Table 2. Comparison of the absolute errors of Example 6.4

x_i	Exact solution	$N=3 y(x_i)$		$N=5 y(x_i)$		$N=7 y(x_i)$	
		Absolute errors		Absolute errors		Absolute errors	
0	0	0	0	0	0	0	0
$\frac{\pi}{10}$	0.6232	0.6232	2.5509E-05	0.6232	4.6066E-08	0.6232	1.8559E-08
$\frac{2\pi}{10}$	1.2161	1.2153	8.0897E-04	1.2161	7.5326E-06	1.2161	1.6691E-07
$\frac{3\pi}{10}$	1.7515	1.7454	6.0693E-03	1.7516	1.2921E-04	1.7515	1.8993E-06
$\frac{4\pi}{10}$	2.2077	2.1825	2.5157E-02	2.2087	9.6029E-04	2.2077	2.1518E-05
$\frac{\pi}{2}$	2.5708	2.4956	7.5176E-02	2.5753	4.5249E-03	2.5706	1.5647E-04
$\frac{6\pi}{10}$	2.8360	2.6537	1.8234E-1	2.8520	1.5975E-02	2.8352	7.9892E-04
$\frac{7\pi}{10}$	3.0081	2.6257	3.8245E-1	3.0543	4.6183E-02	3.0050	3.1655E-03
$\frac{8\pi}{10}$	3.1011	2.3806	7.2041E-1	3.2163	1.1527E-1	3.0907	1.0399E-02
$\frac{9\pi}{10}$	3.1365	1.8876	0.12489E-1	3.3935	2.5702E-1	3.1069	2.9592E-02
π	3.1416	1.1154	0.20262E-1	3.6657	5.2409E-1	3.0664	7.5161E-02

7. Conclusion

In this study, a new Fermat approximation method for the solution of a class of first order nonlinear differential equations has been presented. The principal advantage of this method is the capability to succeed in the solution up to all term of Fermat expansion. It is seen from Example 6.3 that Fermat collocation method gives well results for the different values N . To obtained numerical results show that accuracy improves when N is increased as shown from Example 6.4. Examples, tables and figures indicate that the present method is convenient, reliable and effective. Also it is important to note that Fermat coefficients of the solution are found very simply by using the computer programs [15].

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