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W. Blaschke's Theory Application in Digital Image Processing

O. Samarina*, V. Slavsky

Yugra State University, Khanty-Mansiysk, Russia
*Corresponding author: samarina_ov@mail.ru

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Abstract Texture classification is one of the basic images processing tasks. In this paper we present a geometrical approach to the images analysis and processing. We introduce topological invariants of the RGB-image based on W. Blaschke's web geometry. The approach to images processing presented in the given work, can be used at the solving of images classification problems, their recognition, and also at construction of statistical methods of group of images processing.

Keywords: *invariants, multichannel image, three-webs*

1. Introduction

This paper will be discussed methods of texture classification, based on topological images invariants. In the central topics of this paper will be researched such invariants as connection form, element of surface, curvature and RGB-images canonical form. The algorithm of numerical calculation for invariants will be also described by the authors. Additionally there will be introduced the degree of regularity. This characteristic will be analysed for three types of images: picture of Earth surface from space, photo and simulated image.

2. Basic Constructions

Let's consider a three-channel RGB-image. Suppose that RGB-image is a set of three non-negative functions $\mathrm{u_i}(\mathrm{x,y})$, $i = 1,2,3$ in a two-dimensional domain D. Families of lines for this functions can be written as follows:

$$L_1 = \{(x,y) : u_1(x,y) = const\}$$

$$L_2 = \{(x,y) : u_2(x,y) = const\}$$

$$L_3 = \{(x,y) : u_3(x,y) = const\}$$

Call these three families of lines an image's topographical grid (or three-webs) [1]. The three-webs function is any function $W(u_1, u_2, u_3)$ nonidentically equal to a constant such that in domain D:

$$W(u_1(x,y), u_2(x,y), u_3(x,y)) \equiv 0$$

W. Blaschke's suggested to consider "topological" differential geometry.

He studied differential-geometrical (the local!) properties of various objects invariant to topological transformations. Thus use of the classical differential geometry forces to be limited transformations set by functions, differentiated sufficient times or analytical.

3. Pfaffian Form

Let Σ be the families of curves Ξ_j:

$$u_j(x,y) = u_j = const, \ j = 1,2,3$$

Multiplying du_j by $g_j(x,y) \neq 0$ we get the Pfaffian form $\sigma_1 = g_1 du_1$, $\sigma_2 = g_2 du_2$, $\sigma_3 = g_3 du_3$.

Curves Ξ_j are defined by differential equation $\sigma_j = 0$. Multiplies g_j are any functions distinct from zero.

Pfaffian forms can be normalized in such a way as the normalization condition (for some g_j)

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \quad (1)$$

hold. Pfaffian form is possible to represent as

$$\sigma_j^* = \frac{\sigma_j}{g}$$

where $g(x,y) \neq 0$.

Now we find g_j. For this purpose define U:D $\rightarrow$ R^3, G:D $\rightarrow$ R^3:

$$U(x,y) = \{u_1(x,y), u_2(x,y), u_3(x,y)\},$$

$$G(x,y) = \{g_1(x,y), g_2(x,y), g_3(x,y)\}.$$

Then normalization condition is equivalent to system of equalities

$$\left\langle \frac{\partial U}{\partial x}, G \right\rangle = 0, \left\langle \frac{\partial U}{\partial y}, G \right\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is scalar product of vectors. Make use vector product and put

$$G = \left[\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right].$$

From here we will receive vector components G:

$$G = \begin{pmatrix} u_3^{(0,1)}(x,y)u_2^{(1,0)}(x,y) - u_2^{(0,1)}(x,y)u_3^{(1,0)}(x,y) \\ u_1^{(0,1)}(x,y)u_3^{(1,0)}(x,y) - u_3^{(0,1)}(x,y)u_1^{(1,0)}(x,y) \\ u_2^{(0,1)}(x,y)u_1^{(1,0)}(x,y) - u_1^{(0,1)}(x,y)u_2^{(1,0)}(x,y) \end{pmatrix}$$

Clearly

$$\sigma_1 = du_1 \begin{vmatrix} u_2^{(1,0)} & u_2^{(0,1)} \\ u_3^{(1,0)} & u_3^{(0,1)} \end{vmatrix},$$

$$\sigma_2 = du_2 \begin{vmatrix} u_3^{(1,0)} & u_3^{(0,1)} \\ u_1^{(1,0)} & u_1^{(0,1)} \end{vmatrix},$$

$$\sigma_3 = du_3 \begin{vmatrix} u_1^{(1,0)} & u_1^{(0,1)} \\ u_2^{(1,0)} & u_2^{(0,1)} \end{vmatrix}.$$

From (1) for form σ_j follows

$$\sigma_2 \wedge \sigma_3 = \sigma_3 \wedge \sigma_1 = \sigma_1 \wedge \sigma_2 .$$

Equation $\sigma_1 \wedge \sigma_2 = \Omega$ is called element of surface

$$\Omega = \begin{vmatrix} u_1^{(1,0)} & u_1^{(0,1)} \\ u_2^{(1,0)} & u_2^{(0,1)} \end{vmatrix} \cdot \begin{vmatrix} u_2^{(1,0)} & u_2^{(0,1)} \\ u_3^{(1,0)} & u_3^{(0,1)} \end{vmatrix} \cdot \begin{vmatrix} u_3^{(1,0)} & u_3^{(0,1)} \\ u_1^{(1,0)} & u_1^{(0,1)} \end{vmatrix} dx \wedge dy$$

However Ω depends from normalization condition

$$\Omega^* = \frac{\Omega}{g^2} .$$

External differentials of Pfaffian form σ_j for web Σ is differ only by scalar multiplier h_j from element of surface Ω : $d\sigma_j = h_j \Omega$. It follows from (1) that $h_1 + h_2 + h_3 = 0$. Values h_j are called Christoffel symbols for three-web. To find expression for h_j let's enter some designations

$$\nabla f = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\}, \nabla^\perp f = \left\{ -\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right\}.$$

$$d^2 f(a,b) = \frac{\partial^2 f}{\partial x^2} a_1 b_1 + \frac{\partial^2 f}{\partial x \partial y}(a_1 b_2 + a_2 b_1) + \frac{\partial^2 f}{\partial y^2} a_2 b_2$$

$$\left\langle \nabla^\perp f, \nabla g \right\rangle = \begin{vmatrix} f^{(1,0)} & f^{(0,1)} \\ g^{(1,0)} & g^{(0,1)} \end{vmatrix}.$$

In this designations

$$h_1 = \frac{d^2 u_2 \left(\nabla^\perp u_1, \nabla^\perp u_3 \right) - d^2 u_3 \left(\nabla^\perp u_1, \nabla^\perp u_2 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_2, \nabla u_3 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle},$$

$$h_2 = \frac{d^2 u_3 \left(\nabla^\perp u_2, \nabla^\perp u_1 \right) - d^2 u_1 \left(\nabla^\perp u_2, \nabla^\perp u_3 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_2, \nabla u_3 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle},$$

$$h_3 = \frac{d^2 u_1 \left(\nabla^\perp u_3, \nabla^\perp u_2 \right) - d^2 u_2 \left(\nabla^\perp u_3, \nabla^\perp u_1 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_2, \nabla u_3 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle}.$$

Construct the Pfaffian form

$\gamma = h_3 \sigma_2 - h_2 \sigma_3 = h_1 \sigma_3 - h_3 \sigma_1 = h_2 \sigma_1 - h_1 \sigma_2$.

This form is called connection form Σ for web. Normalized γ can be transformed so [1]:

$$\gamma^* = \gamma - d \ln g \tag{2}$$

This way, connection form changes only on full differential. Hence, in force (2) integral

$$\int_D d\gamma = \int_{\partial D} \gamma \tag{3}$$

is renormalization invariant. In the same way the norm element of surface $d\gamma$ is invariant $d\gamma = \kappa \Omega$.

The function κ is called the web curvature. It follows from last equation that

$$\kappa^* = g^2 \kappa$$

This means that the web curvature κ is a relative invariant of weight two (e.g. [3]). The integrated Formula (3) reminds the Gauss formula

$$\gamma = h_2 \sigma_1 - h_1 \sigma_2$$
$$= \frac{d^2 u_3 \left(\nabla^\perp u_2, \nabla^\perp u_1 \right) - d^2 u_1 \left(\nabla^\perp u_2, \nabla^\perp u_3 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle} - \frac{d^2 u_2 \left(\nabla^\perp u_1, \nabla^\perp u_3 \right) - d^2 u_3 \left(\nabla^\perp u_1, \nabla^\perp u_2 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_2, \nabla u_3 \right\rangle}$$

It follows that

$$d\gamma = d \left[\frac{d^2 u_3 \left(\nabla^\perp u_2, \nabla^\perp u_1 \right) - d^2 u_1 \left(\nabla^\perp u_2, \nabla^\perp u_3 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle} \right] \wedge du_1 - d \left[\frac{d^2 u_2 \left(\nabla^\perp u_1, \nabla^\perp u_3 \right) - d^2 u_3 \left(\nabla^\perp u_1, \nabla^\perp u_2 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_2, \nabla u_3 \right\rangle} \right] \wedge du_2 =$$
$$= \left\langle \nabla^\perp A, \nabla u_1 \right\rangle dx \wedge dy - \left\langle \nabla^\perp B, \nabla u_2 \right\rangle dx \wedge dy.$$

where functions A и B are

$$A = \frac{d^2 u_3 \left(\nabla^\perp u_2, \nabla^\perp u_1 \right) - d^2 u_1 \left(\nabla^\perp u_2, \nabla^\perp u_3 \right)}{\left\langle \nabla^\perp u_1, \nabla u_2 \right\rangle \cdot \left\langle \nabla^\perp u_3, \nabla u_1 \right\rangle},$$

$$B=\frac{d^2u_2\left(\nabla^\perp u_1,\nabla^\perp u_3\right)-d^2u_3\left(\nabla^\perp u_1,\nabla^\perp u_2\right)}{\left\langle\nabla^\perp u_1,\nabla u_2\right\rangle\cdot\left\langle\nabla^\perp u_2,\nabla u_3\right\rangle}.$$

For three-web curvature we have:

$$\kappa=\frac{\left\langle\nabla^\perp A,\nabla u_1\right\rangle-\left\langle\nabla^\perp B,\nabla u_2\right\rangle}{\left\langle\nabla^\perp u_1,\nabla u_2\right\rangle\cdot\left\langle\nabla^\perp u_2,\nabla u_3\right\rangle\cdot\left\langle\nabla^\perp u_3,\nabla u_1\right\rangle}.$$

4. Connection Form, Element of Surface and Curvature

We may assume that at the point P_0 satisfy a condition

$$u_1(P_0)=u_2(P_0)=u_3(P_0)=0$$

Decompose the web function in a series $\{u_1,u_2,u_3\}$:

$$W=W_1u_1+W_2u_2+W_3u_3+\frac{1}{2}W_1u_1^2+W_{12}u_1u_2+...$$

Here W_i, W_{ij} are private derivatives of the first and second order with respect to u_i.

In the paper [1] connection form Σ, norm element of surface $d\gamma$ and curvature κ are defined as three-web invariants:

$$\gamma=-\sum_{i=1,2,3}\frac{\partial}{\partial u_i}\ln(W_i)du^i+d\ln(W_1W_2W_3),$$

$$d\gamma=\frac{1}{2}\sum_{r<s}\frac{\partial^2}{\partial u_r\partial u_s}\ln(\frac{W_r}{W_s})\left[du^r,du^s\right],$$

$$\kappa=A_{23}+A_{31}+A_{12},$$

where A_{rs}:

$$A_{rs}=\frac{1}{W_rW_s}\frac{\partial^2}{\partial u_r\partial u_s}\ln\frac{W_r}{W_s}=$$

$$=\frac{W_{rrs}}{W_r^2W_s}-\frac{W_{rss}}{W_rW_s^2}+\frac{W_{rs}}{W_rW_s}\left(\frac{W_{ss}}{W_s^2}-\frac{W_{rr}}{W_r^2}\right).$$

Curvature κ for three-channel image can be defined in another way. Let's assume (using local diffeomorphism) that first two functions are coordinate functions $u_1(x,y)=x$, $u_2(x,y)=y$ and the third function $u_3(x,y)$ is three times differentiable function. Using Taylor decomposition third order with centre in the arbitrary point of area we take:

$$u_3(x,y)=a+p_1x+p_2y+\frac{1}{2}\left(b_{11}x^2+b_{12}xy+b_{22}y^2\right)+...$$
$$+\frac{1}{6}\left(b_{111}x^3+b_{112}x^2y+b_{122}xy^2+b_{222}y^3\right)+o\left(\sqrt{(x^2+y^2)^3}\right).$$

From the discrete grid (numeral images are defined on the discrete grid of points) it's easy to calculate coefficients of Taylor decomposition [2]. For this functions the RGB-images curvature take the form

$$\kappa=\frac{p_1p_2(b_{112}p_2-b_{122}p_1)+b_{12}(b_{22}p_1^2-b_{11}p_2^2}{p_1^3p_2^3}.$$

Thus, the connection form Σ, element of surface $d\gamma$ and curvature κ can be used as RGB-images invariants to wide group of transformations at the decision of various tasks of digital image processing.

5. RGB-Images Canonical Form

In [1] it is proved that web-function W by some transformations can take the canonical form in a vicinity of any point:

$$W_0=u_1+u_2+u_3+a\left(u_2^2u_3+u_3^2u_1+u_1^2u_2\right)+...$$
$$+\left(b_1u_1+b_2u_2+b_3u_3\right)u_1u_2u_3+...,$$

where constants b_1,b_2,b_3 satisfy to condition

$$b_1+b_2+b_3=0$$

The first type of transformations has the form [1]:

$$I:W^*(u_1,u_2,u_3)=W\left[u_1\left(u_1^*\right),u_2\left(u_2^*\right),u_3\left(u_3^*\right)\right]$$

That is parameter replacement in each family by means of unequivocal functions $u_i=u_i\left(u_i^*\right)$ (this transformations correspond to channels calibration). The second and third transformations have the following form

$$II:\hat{W}(u_1,u_2,u_3)=H(u_1,u_2,u_3)W(u_1,u_2,u_3),$$

$$III:\tilde{W}(u_1,u_2,u_3)=F\left[W(u_1,u_2,u_3)\right],$$

where $H\neq0,\ F(0)=0,\ F'(0)\neq0.$ These transformations allow to normalize definitely channels u_i , $i=\overline{1,3}$ and web-function in a vicinity of the chosen point W .

Let's consider algorithm of reduction to canonical representation for RGB-images web-function. Assume that condition $u_1(x_0,y_0)=u_{21}(x_0,y_0)=u_3(x_0,y_0)=0$ the point $P_0=(x_0,y_0)$ is satisfied. Decompose web-function W in sedate a number on u_i:

$$W=W_1u_1+W_2u_2+W_3u_3+\frac{1}{2}W_{11}u_1^2+W_{12}u_1u_2+...$$

We will apply to function W transformation (II)

$$\hat{W}=H(u_1,u_2,u_3)W,$$

where

$$H(u_1,u_2,u_3)=1+a_1u_1+a_2u_2+a_3u_3+a_{11}u_1^2+a_{12}u_1u_2+...$$

Now take a_1,a_2,a_3 such that in decomposition $\hat{W}$ remove elements $\hat{W}_{12}u_1u_2$, $\hat{W}_{13}u_1u_3$, $\hat{W}_{23}u_2u_3$

$$a_1=-\frac{-W_{23}W_2+W_2W_{13}+W_3W_{12}}{2W_3W_2},$$

$$a_2 = -\frac{W_{23}W_1 - W_2W_{13} + W_3W_{12}}{2W_3W_1},$$

$$a_3 = -\frac{W_{23}W_1 + W_2W_{13} - W_3W_{12}}{2W_2W_1}.$$

Thus we delete from decomposition all mixed derivative functions $\hat{W}$ of the second order. Further at calculation we will consider that members $W_{12}u_1u_2$, $W_{13}u_1u_3$ and $W_{23}u_2u_3$ are absent also function H looks like.

$$H(u_1,u_2,u_3) = 1 + a_{11}u_1^2 + a_{12}u_1u_2 + ...$$

Let's enter following replacements

$$u_1^* = \frac{1}{24}W_{1111}u_1^4 + \frac{1}{6}W_{111}u_1^3 + \frac{1}{2}W_{11}u_1^2 + W_1u_1,$$

$$u_2^* = \frac{1}{24}W_{2222}u_2^4 + \frac{1}{6}W_{222}u_2^3 + \frac{1}{2}W_{22}u_2^2 + W_2u_2,$$

$$u_3^* = \frac{1}{24}W_{3333}u_3^4 + \frac{1}{6}W_{333}u_3^3 + \frac{1}{2}W_{33}u_3^2 + W_3u_3.$$

Choose a representation of a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} such what $\hat{W}_{112}$, $\hat{W}_{113}$, $\hat{W}_{122}$, $\hat{W}_{123}$, $\hat{W}_{133}$, $\hat{W}_{223}$, $\hat{W}_{233}$ will depend on a. For this purpose consider equality

$$W_0\left[u_1^*, u_2^*, u_3^*\right] = \hat{W} = W \cdot H \qquad (4)$$

We get system of equations

$$2\left(-aW_1W_1^2 + a_{12}W_1 + a_{11}W_2 + W_{112}\right) = 0$$

$$2\left(-aW_1W_3^2 + a_{13}W_3 + a_{33}W_1 + W_{133}\right) = 0$$

$$2\left(-aW_3W_2^2 + a_{23}W_2 + a_{22}W_3 + W_{223}\right) = 0$$

$$2\left(a_{11}W_3 + a_{13}W_1 + W_{113}\right) = 0$$

$$2\left(a_{12}W_2 + a_{22}W_1 + W_{122}\right) = 0$$

$$2\left(a_{23}W_3 + a_{33}W_2 + W_{233}\right) = 0$$

$$a_{12}W_3 + a_{13}W_2 + a_{23}W_1 + W_{123} = 0$$

Solving this system of equation concerning a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} and a we receive following expressions:

$$a_{11} = -\frac{W_{112}W_3^2 - W_1W_{123}W_3 + 2W_2W_{113}W_3 - W_{233}W_1^2 + W_2W_1W_{133}}{3W_3^2W_2},$$

$$a_{12} = -\frac{-W_{113}W_2^2 - W_1W_{123}W_2 + W_3W_{112}W_2 - W_{223}W_1^2 - W_3W_1W_{122}}{3W_1W_2W_3},$$

$$a_{13} = -\frac{-W_{112}W_3^2 - W_1W_{123}W_2 + W_2W_{113}W_3 - W_{223}W_1^2 - W_2W_1W_{133}}{3W_1W_2W_3},$$

$$a_{22} = -\frac{W_{113}W_2^2 - W_1W_{123}W_2 + W_3W_{112}W_2 - W_{223}W_1^2 - 2W_3W_1W_{122}}{3W_3W_1^2},$$

$$a_{23} = -\frac{W_{122}W_3^2 - W_1W_{223}W_3 + W_2W_{123}W_3 + W_{133}W_2^2 - W_2W_{233}W_1}{3W_3W_2W_1},$$

$$a_{33} = -\frac{W_{122}W_3^2 - W_1W_{223}W_3 - W_2W_{123}W_3 - W_{133}W_2^2 - 2W_2W_1W_{233}}{3W_1W_2^2},$$

$$a = -\frac{W_1W_{122}W_3^2 - W_2W_{112}W_3^2 - W_{223}W_1^2W_3}{3W_1^2W_2^2W_3^2} + .$$
$$+\frac{-W_{113}W_2^2W_3 - W_2W_{233}W_1^2 + W_1W_{133}W_2^2}{3W_1^2W_2^2W_3^2}$$

Further we received a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} and a in expression (4).

Now take a_{111}, a_{112}, a_{113}, a_{121}, a_{122}, a_{123}, a_{133}, a_{222}, a_{223}, a_{333} and b_1, b_2 such that $\hat{W}_{1112}$, ..., $\hat{W}_{2333}$ in equality (4) on the left and on the right have coincided. Adding the condition $b_1 + b_2 + b_3 = 0$ to system of 12 equations and b_1, b_2, b_3 to 10 factors, we will receive system of 13 equations on 13 unknown variables:

$$6\left(a_{111}\hat{W}_3 + a_{113}\hat{W}_1 + a_{13}\hat{W}_{11} + \hat{W}_{1113}\right) = 0$$

$$6\left(a_{122}\hat{W}_2 + a_{12}\hat{W}_{22} + a_{222}\hat{W}_1 + \hat{W}_{1222}\right) = 0$$

$$6\left(a_{233}\hat{W}_3 + a_{23}\hat{W}_{33} + a_{333}\hat{W}_2 + \hat{W}_{2333}\right) = 0$$

$$6\left(a\hat{W}_1\hat{W}_{11}\hat{W}_2 + a_{111}\hat{W}_2 + a_{112}\hat{W}_1 + a_{12}\hat{W}_{11} + \hat{W}_{1112}\right) = 0$$

$$6\left(-a\hat{W}_2\hat{W}_{22}\hat{W}_3 + a_{222}\hat{W}_3 + a_{223}\hat{W}_2 + a_{23}\hat{W}_{22} + \hat{W}_{2223}\right) = 0$$

$$6\left(-a\hat{W}_1\hat{W}_{33}\hat{W}_3 + a_{133}\hat{W}_3 + a_{13}\hat{W}_{33} + a_{333}\hat{W}_1 + \hat{W}_{1333}\right) = 0$$

$$2\left(-b_3\hat{W}_2\hat{W}_1\hat{W}_3^2 + a_{123}\hat{W}_3 + a_{12}\hat{W}_{33} + a_{133}\hat{W}_2 + a_{233}\hat{W}_1 + \hat{W}_{1233}\right) = 0$$

$$2\left(-b_2\hat{W}_3\hat{W}_1\hat{W}_2^2 + a_{123}\hat{W}_2 + a_{13}\hat{W}_{22} + a_{122}\hat{W}_3 + a_{223}\hat{W}_1 + \hat{W}_{1223}\right) = 0$$

$$2\left(-b_1\hat{W}_3\hat{W}_2\hat{W}_1^2 + a_{123}\hat{W}_1 + a_{23}\hat{W}_{11} + a_{112}\hat{W}_3 + a_{113}\hat{W}_2 + \hat{W}_{1123}\right) = 0$$

$$-2a\hat{W}_{33}\hat{W}_2^2 + 4a_{323}\hat{W}_2 + 4a_{223}\hat{W}_3 + 4a_{22}\hat{W}_{33} + 4a_{33}\hat{W}_{22} + 4\hat{W}_{2233} = 0$$

$$-2a\hat{W}_{11}\hat{W}_3^2 + 4a_{113}\hat{W}_3 + 4a_{133}\hat{W}_1 + 4a_{11}\hat{W}_{33} + 4a_{33}\hat{W}_{11} + 4\hat{W}_{1133} = 0$$

$$-2a\hat{W}_{22}\hat{W}_1^2 + 4a_{122}\hat{W}_1 + 4a_{112}\hat{W}_2 + 4a_{11}\hat{W}_{22} + 4a_{22}\hat{W}_{11} + 4\hat{W}_{1122} = 0$$

As a result we will receive required expressions for a_{111}, ..., a_{333} and b_1, b_2, b_3.

Using this expressions we can take the canonical representation for $\hat{W}$:

$$W\left(u_1^*,u_2^*,u_3^*\right)=u_1+u_2+u_3+$$
$$+a\left(u_2^2u_3+u_3^2u_1+u_1^2u_2\right)$$
$$+\left(b_1u_1+b_2u_2+b_3u_3\right)u_1u_2u_3+...$$

Canonical form of three-web function is invariant characteristic of the numerical RGB-image (e.g. [1,2]). It can be used at the decision of various tasks in images analysis, such as pattern recognition, analysis of biomedical images and other tasks.

6. Numerical Calculation of RGB-images Invariants

Let's define the algorithm for a numerical finding of three-webs function for three-channel discrete digital image. Decompose $W=W\left(u_1,u_2,u_3\right)$ to the third order in u_i :

$$W=W_1u_1+W_2u_2+W_3u_3+W_{11}u_1^2+W_{22}u_2^2+$$
$$+W_{33}u_3^2+W_{12}u_1u_2+W_{13}u_1u_3+W_{23}u_2u_3+W_{111}u_1^3+$$
$$+W_{222}u_2^3+W_{333}u_3^3+W_{112}u_1^2u_2+W_{113}u_1^2u_3+W_{123}u_1u_2u_3+$$
$$+W_{122}u_1u_2^2+W_{133}u_1u_3^2+W_{233}u_2u_3^2+W_{223}u_2^2u_3.$$

At all points $W\left(u_1,u_2,u_3\right)$ satisfies to condition

$$W\left(u_1\left(x,y\right),u_2\left(x,y\right),u_3\left(x,y\right)\right)\equiv 0$$

Thus values of three-web function in rectangular grid in the size of 5×5 is equal to zero.

Let's delete central and angular points from rectangular grid (black points in Figure 1). Then receive twenty points for definition of three-web function. Set the system from 20 equations on 19 unknown variables:

$$A\cdot H=W$$

Let H be a row-vector of decompositions coefficients $W=W(u_1,u_2,u_3)$ to the third order, and W be a zero vector.

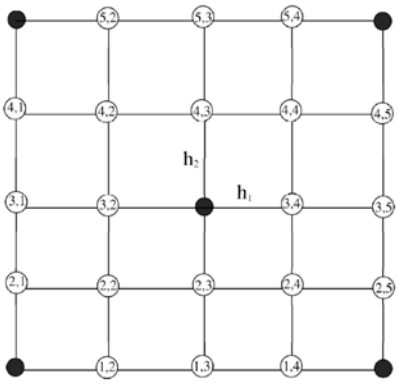

Figure 1. Rectangular grid

For a finding of non-trivial numerical decision of the system we will add to the considered list of the equations a following condition for functions W :

$$W_1+W_2+W_3=1.$$

Define the system of 21 equations in 19 unknown variables. Solving this system by the least squares method we have

$$H=P\cdot W,$$

where $P=A^{-1}$ be a pseudo-inverse matrix.

The factors of three-web function decomposition may be used for calculate without effort three-channel images connection form, norm element of surface and curvature.

7. Numerical Calculation of the Parameter a

In this section we introduce approach to numerical calculation of the parameter a which is present in canonical representation of RGB - images web-function:

$$a=\frac{W_1W_{122}W_3^2-W_2W_{112}W_3^2-W_{223}W_1^2W_3}{3W_1^2W_2^2W_3^2}$$
$$+\frac{-W_{113}W_2^2W_3-W_2W_{223}W_1^2-W_1W_{133}W_2^2}{3W_1^2W_2^2W_3^2}$$

For experiment we will consider three thematic types of images: a picture of Earth surface, photo and simulated image. For each kind of images we will spend a separate series of tests.

To investigate dependence of a on a point choice, we will calculate a for nine points of the image located on a grid in the size 3×3 (e.g. Figure 2).

For estimation of the received resulting importance we will notice that for continuously differentiated function on a plane from its regularity in the vicinity centre (i.e. the gradient is distinct from zero) follows that the contour lines passing through the vicinity centre breaks its two coherent components. By analogy we will make definition of a regularity discrete function defined on pixels.

We will construct for each pixel of the image a matrix in the size 3×3. The central element of a matrix we will accept for zero. Boundary values will be accepted for "+", if it is more than central element and boundary values will be accepted for "–" if it is less than central element.

If as a result the set of boundary pixels is divided into two coherent pieces we will name the image regular on the given pixel. If a component of connectivity is two or one we will name the image extended regular. If a component is more than two the image is irregular.

We will define degree of regularity for digital image as probability of that the image is extended regular in casually chosen pixel. The given property of the digital image is important for correct morphological images processing.

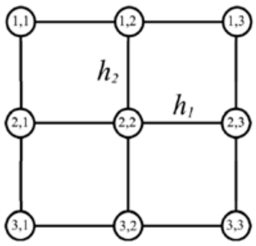

Figure 2. Rectangular grid 3×3

Let's estimate probability of coherent areas formation for any stochastic function.

We will display set of boundary pixels in the form of circular area for visual representation (e.g. Figure 3). Number of all possible arrangements of signs equally $Pm = 2^8 = 256$.

By the direct calculations gets the Table 1.

Table 1. Number of connectivity components

n_1	n_2	n_4	n_6	n_8
2	56	140	56	2

Where n_i is number of arrangements to which corresponds i=1,2,4,6,8 component of connectivity of set of boundary pixels. Hence, for the random digital image regularity degree is equal:

$$\frac{n_1 + n_2}{n_1 + n_2 + n_4 + n_6 + n_8} = \frac{29}{128} = 0.226563$$

Let's estimate degree of images regularity for various types of the images.

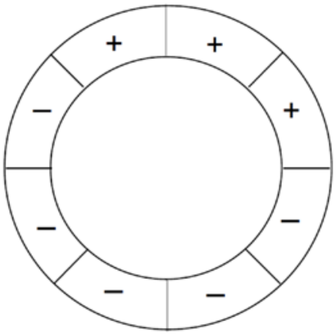

Figure 3. Set of boundary pixels

7.1. Picture of Earth's Surface

This image is an eight-channel picture of earth's surface received from Sputnik Landsat 7 (e.g. Figure 4). We will choose for processing three first channel with ranges of the spectral permission 0.45-0.52, 0.52-0.6 and 0.63-0.69, corresponding to blue, green and red colour zones.

Figure 4. Picture of earth's surface

For this type of images 10 000 tests have been spent, in 61% positive results have been received. That is the set of pixels was divided into one or two coherent areas. Degree of regularity for this image is.

$$\frac{n_1 + n_2}{n_1 + n_2 + n_4 + n_6 + n_8} = \frac{1857 + 4179}{10000} = 0.6036$$

7.2. Photo

The given type of the image is presented of a format jpg (e.g. Figure 5). Pixels values contain in a range [0,255] for three colour channels.

Figure 5. Photo

Degree of a regularity of the given image has made.

$$\frac{n_1 + n_2}{n_1 + n_2 + n_4 + n_6 + n_8} = \frac{1560 + 3652}{10000} = 0.5212$$

7.3. Simulated Image

Simulated image has been generated with use of mathematical package Matlab (e.g. Figure 6).

Degree of regularity for the image is equal.

$$\frac{n_1 + n_2}{n_1 + n_2 + n_4 + n_6 + n_8} = \frac{1058 + 6005}{10000} = 0.7063$$

Using Laplace integration theorem it is easy to convince that the received values can't be random.

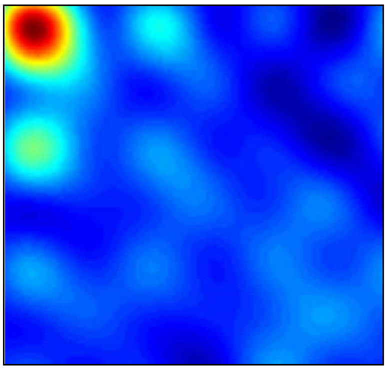

Figure 6. Simulated image

8. Conclusion

Automatic processing of the visual information is one of the major directions in the field of artificial intellect and now the greatest attention is being paid to it all over the world. The important part in the theoretical base of processing systems, analysis and identification of images is performed by the images invariants construction device.

Images invariants are the effective characteristics which can be used in the various applied problems. It can be used at biomedical images analysis, geological researches, problems of images classification and their recognition (e.g. [2]).

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Statement of Competing Interests

The authors have no competing interests.

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