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Discretizing the Information Based Asset Price Dynamics

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Abstract The dynamics of the asset process and variance process are driven by continuous time processes in the Information Based Asset Pricing Framework as proposed by Brody, Hughson and Macrina, also known as the BHM Model. To make use of numerical simulation, the continuous time processes can be discretized to discrete time processes. Here, two discretization schemes will be looked at: Euler scheme and Milstein scheme. The main objective of this study is to apply the two discretization schemes to the Information Based Asset Picing Framework. The two schemes will first be applied to the Black-Scholes and the Heston models and then extended to the BHM model. Studies have shown that the Euler scheme approach to discretization can be inefficient which makes the use of the Milstein scheme approach to discretization more accurate due to the expansion of the coefficients involved in the stochastic differential equation.

Keywords: *discretization, Euler scheme, Milstein scheme, Black-Scholes model, Hes-ton model, BHM model*

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1. Introduction

The classical model of asset prices which was published in 1973 is known as the Black-Scholes model. One of the key assumptions made under the model is that the volatility of asset returns is constant. This implies that the dynamics of the prices of risky assets can be modelled by geometric Brownian motions (GBMs). This results in a distribution which is lognormal.

Empirical studies demonstrate that the volatility of asset returns is not constant with Rosenberg [1] coming to the same conclusion through studies of implied volatility. Much work has followed with Scott [2] also providing empirical evidence showing that volatility changes with time and that the changes are unpredictable. The studies also show that volatility has a tendency to revert to a long-run average.

The Heston model is an extension of the Black-Scholes model which makes the assumption of stochastic volatility and that the volatility and the underlying asset price are correlated. In so doing, the Heston model is able to capture various properties of the financial information which the Black-Scholes model doesn't. However, the reliability of the model is questionable because the assumptions on the volatility and the underlying asset price dynamics display an ad-hoc nature.

Another approach suggested later by Brody, Hughson and Macrina called the BHM approach or model obtains the asset price dynamics using a more realistic approach towards the structure of the market unlike the Heston model where the dynamics of the volatility and price are pre-specified. The approach specifies a model for the structure of the information available in the market since asset prices are determined by expectations on the future cash flows given the market information available.

The later approach doesn't assume any dynamic model for the asset prices, it's observed that the asset price dynamics derived with the assumed information structure naturally have stochastic volatility giving a different view of the volatility nature. Accordingly, the model illustrates that the volatility of volatility is stochastic. In the BHM approach, the stochastic process governing the dynamics of an asset are deduced as compared to being imposed at the beginning of the process in an arbitrary way.

The aim of this study is to make the BHM Model suitable for numerical simulation through discretization of its stochastic differential equations (SDEs). Discretization refers to this process of approximating a continuous time SDE to a discrete time SDE. Both in literature and in practice, a lot of focus has been directed to discrete time approximations of SDEs which has led to the development of schemes that accomplish this.

The Euler scheme approach to discretization was first studied by Maruyama [3]. In order to simulate a realization of the Euler approximation, the independent random variables involved need to be generated. Gatheral [4] shows that a Euler discretization of the variance process may result in a negative variance.

There are several approaches that can be taken to fix the problem of the variance becoming negative with a non zero probability which makes the computation of the square root of the variance impossible. One approach is by the reflection scheme which involves taking the absolute value of the variance, that is if V_t denotes the variance, then, its absolute value, $|V_t|$ will be used.

Another approach that can be used to counter the problem of the variance becoming negative is the full truncation scheme which involves taking the positive value of the variance which is denoted by V_t^+, that is $V_t^+ = max(0, V_t)$. This study adopts the second approach, that is the full truncation scheme.

According to Platen [5], the Euler scheme approach can be inefficient and often shows poor stability properties. This means that other stochastic numerical methods need to be considered in discretization of the BHM model. The Milstein scheme increases the accuracy of the discretization by considering expansion of the coefficients involved in the SDE. In addition, the scheme significantly alleviates the problem of the variance becoming negative which makes it preferred to the Euler scheme.

The study starts by looking at the existing work in discretization as applied to the Black-Scholes and Heston Model as shown in Gatheral [4]. A similar approach is then extended to the BHM Model. Both the Euler scheme and Milstein scheme will be looked at in the discretization process.

An assumption is made that the asset price, S_t is driven by the SDE:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \tag{1.1}$$

where W_t is a Brownian motion.

S_t is simulated over the interval $[0, T]$, which is assumed to be discretized as $0 = t_1 < t_2 < ... < t_m = T$. The time intervals are equally spaced with width Δt.

Integrating equation 1.1 over the interval $[t, t+\Delta t]$ leads to:

$$\int_t^{t+\Delta t} dS_s = \int_t^{t+\Delta t} \mu(S_s, s)ds + \int_t^{t+\Delta t} \sigma(S_s, s)dW_s \tag{1.2}$$

$$\int_t^{t+\Delta t} dS_s = S_{t+\Delta t} - S_t$$

$$S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} \mu(S_s, s)ds + \int_t^{t+\Delta t} \sigma(S_s, s)dW_s \tag{1.3}$$

The value at time t for S_t in equation 1.3 is known, the problem is to find the value for $S_{t+\Delta t}$.

2. Euler Scheme Discretization

Euler discretization is equivalent to getting an approximate integral value using the left-point rule since the value at time t is known. From integral 1.3:

$$\int_t^{t+\Delta t} \mu(S_s, s)ds \approx \mu\left(S_t, t\right)\int_t^{t+\Delta t} du$$
$$= \mu\left(S_t, t\right)\Delta t.$$

Similarly:

$$\int_t^{t+\Delta t} \sigma(S_s, s)dW_s \approx \sigma\left(S_t, t\right)\int_t^{t+\Delta t} dW_s$$
$$= \sigma\left(S_t, t\right)\left(W_{t+\Delta t} - W_t\right)$$
$$= \sigma\left(S_t, t\right)\sqrt{\Delta t}Z$$

where $\left(W_{t+\Delta t} - W_t\right)$ and $\sqrt{\Delta t}Z$ have the same distribution.

Thus,

$$S_{t+\Delta t} = S_t + \mu\left(S_t, t\right)\Delta t + \sigma\left(S_t, t\right)\sqrt{\Delta t}Z \tag{2.1}$$

2.1. The Black-Scholes Model

Under this model, the asset price dynamics under the risk neutral measure, $\mathbb{Q}$, are:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

where B_t is a Brownian motion, S_t denotes the underlying asset price, σ is the volatility of the underlying asset and r is the risk free rate of interest.

Integrating equation 2.2 over the interval $[t, t+\Delta t]$ gives:

$$\int_t^{t+\Delta t} dS_s = \int_t^{t+\Delta t} rS_s ds + \int_t^{t+\Delta t} \sigma S_s dB_s$$

$$S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} rS_s ds + \int_t^{t+\Delta t} \sigma S_s dB_s$$

$$\int_t^{t+\Delta t} rS_s ds \approx rS_t \Delta t$$

$$\int_t^{t+\Delta t} \sigma S_s dB_s \approx \sigma \int_t^{t+\Delta t} S_s dB_s$$
$$= \sigma S_t\left(B_{t+\Delta t} - B_t\right) = \sigma S_t \sqrt{dt}Z.$$

Thus, the Euler discretization for the Black-Scholes model is given as:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sigma S_t\sqrt{\Delta t}Z. \tag{2.3}$$

2.2. The Heston Model

Under a risk-neutral measure, $\mathbb{Q}$, the Heston [6] model assumes that an underlying asset price, S_t has a stochastic variance, V_t, that follows a Cox, Ingersoll and Ross [7] process with long-run mean θ and rate or reversion κ while σ is the volatility of the volatility. This process is represented by the following dynamical system:

$$dS_t = (r-q)S_t dt + \sqrt{V_t}S_t dW_t \tag{2.4}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_t \tag{2.5}$$

where r is a constant risk-free interest rate and q is a constant dividend. All the parameters κ, θ and σ are positive constant. The terms W_t and Z_t are Wiener processes that must be correlated with each other, that is

$$(dW_t dZ_t) = \rho dt \tag{2.6}$$

In equation 2.6, the term ρ is the correlation coefficient between the return of the underlying asset and the changes in the variance. The correlation, which is often negative, will ensure that the volatility for example will rise if the underlying asset value falls dramatically. In addition the variance is also mean-reverting, which is also evident in the market. The mean-reverting process is the term $\kappa(\theta - v)$.

2.2.1. Discretization of the Asset Process

Integrating equation 2.4 over the interval $[t, t+\Delta t]$:

$$\int_t^{t+\Delta t} dS_r = \int_t^{t+\Delta t} (r-q) S_r dr + \int_t^{t+\Delta t} \sqrt{V_r} S_r dW_r$$

Euler scheme approximates the integrals as follows:

$$\int_t^{t+\Delta t} dS_r \approx S_{t+\Delta t} - S_t$$

$$\int_t^{t+\Delta t} (r-q) S_r dr \approx (r-q) S_t \Delta t$$

$$\int_t^{t+\Delta t} \sqrt{V_r} S_r dW_r \approx \sqrt{V_t} S_t \sqrt{\Delta t} Z_p$$

Thus:

$$S_{t+\Delta t} = S_t + (r-q) S_t \Delta t + \sqrt{V_t} S_t \sqrt{\Delta t} Z_p \quad (2.7)$$

To counter the occurrence of a negative variance, the full truncation scheme is applied which involves substituting V_t with V_t^+ which results in:

$$S_{t+\Delta t} = S_t + (r-q) S_t \Delta t + \sqrt{V_t^+} S_t \sqrt{\Delta t} Z_p. \quad (2.8)$$

2.2.2. Discretization of the Variance Process

Integrating equation 2.5 over the interval $[t, t+\Delta t]$, gives:

$$\int_t^{t+\Delta t} dV_s = \int_t^{t+\Delta t} \kappa(\theta - V_s) ds + \int_t^{t+\Delta t} \sigma \sqrt{V_s} dZ_s$$

$$\int_t^{t+\Delta t} dV_s = V_{t+\Delta t} - V_t$$

$$\int_t^{t+\Delta t} \kappa(\theta - V_s) ds = \kappa(\theta - V_t) \Delta t$$

$$\int_t^{t+\Delta t} \sigma \sqrt{V_s} dZ_s = \sigma \sqrt{V_s} (Z_{v(t+\Delta t)} - Z_{vt}) = \sigma \sqrt{V_t} \sqrt{\Delta t} Z_v.$$

Thus:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t) \Delta t + \sigma \sqrt{V_t} \sqrt{\Delta t} Z_v \quad (2.9)$$

where $Z_v \sim N[0,1]$.

To counter the occurrence of a negative variance, the full truncation scheme is applied which involves substituting V_t with V_t^+ which results in:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t^+) \Delta t + \sigma \sqrt{V_t^+} \sqrt{\Delta t} Z_v \quad (2.10)$$

2.2.3. Discretized Heston Model Dynamics

With initial values S_t for the asset price and V_t for the variance, equations 2.13 and 2.14 can be used to obtain S_{t+dt} and V_{t+dt}.

Z_p and Z_v have a correlation ρ, two independent standard normal variables B_p and B_v are generated such that:

$$Z_v = B_v$$

$$Z_p = \rho B_v + \sqrt{1-\rho^2} B_p.$$

2.3. The BHM Model

According to Brody, Hughson and Macrina, [8], the dynamics of the price process are given as:

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t$$

In this study, an assumption will be made that r_t is a constant which implies that $r_t = r$. Thus;

$$dS_t = r S_t dt + \Gamma_{tT} dW_t \quad (2.11)$$

where Γ_{tT} denotes the absolute volatility process:

$$\Gamma_{tT} = P_{tT} \frac{\sigma T}{T-t} V_t$$

P_{tT} denotes the discount factor and r_t denotes the short rate.

The approach by Macrina [9] is used to obtain the dynamics for the volatility in the BHM model which are given as:

$$dV_t = -v_t^2 V_t^2 dt + v_t \kappa_t dW_t \quad (2.12)$$

where $\kappa_t = \mathbb{E}[(X_T - \mathbb{E}[X_T])^3]$ and

$$v_t = \sigma_t + \frac{1}{T-t} \int_0^t \sigma_s ds.$$

2.3.1. Discretization of the Asset Process

Integrating equation 2.11 over the interval $[t, t+\Delta t]$, results in:

$$\int_t^{t+\Delta t} dS_s = \int_t^{t+\Delta t} r S_s ds + \int_t^{t+\Delta t} \Gamma_{sT} dW_s$$

$$\int_t^{t+\Delta t} dS_s = S_{t+\Delta t} - S_t$$

$$\int_t^{t+\Delta t} r S_s ds = r S_t \Delta t$$

$$\int_t^{t+\Delta t} \Gamma_{sT} dW_s = \Gamma_{tT} (W_{t+\Delta t} - W_t) = \Gamma_{tT} \sqrt{\Delta t} Z.$$

Thus, the Euler discretization is given as:

$$S_{t+\Delta t} = S_t + r S_t \Delta t + \Gamma_{tT} \sqrt{\Delta t} Z \quad (2.13)$$

2.3.2. Discretization of the Variance Process

Integrating equation 2.12 over the interval $[t, t+\Delta t]$, gives:

$$\int_t^{t+\Delta t} dV_s = -\int_t^{t+\Delta t} v_s^2 V_s^2 ds + \int_t^{t+\Delta t} v_s \kappa_s dW_s$$

$$\int_t^{t+\Delta t} dV_s = V_{t+\Delta t} - V_t$$

$$-\int_t^{t+\Delta t} v_s^2 V_s^2 ds = -v_t^2 V_t^2 \Delta t$$

$$\int_t^{t+\Delta t} v_s \kappa_s dW_s = v_t \kappa_t \sqrt{W_{t+\Delta t} - W_t} = v_t \kappa_t \sqrt{\Delta t} Z$$

Thus, the Euler discretization is given as:

$$V_{t+\Delta t} = V_t - v_t^2 V_t^2 \Delta t + v_t \kappa_t \sqrt{\Delta t} Z \qquad (2.14)$$

2.3.3. Discretized BHM Model Dynamics

With initial values S_t for the asset price and V_t for the variance, equations 2.13 and 2.14 can be used to obtain S_{t+dt} and V_{t+dt}.

3. Milstein Scheme Discretization

Glasserman [10] shows that the Milstein scheme refines the Euler scheme by the introduction of an additional term.

This scheme is applicable where the coefficients $\mu(S_t)$ and $\sigma(S_t)$ depend only on S_t. This implies that an assumption will be made such that:

$$dS_t = \mu_t dt + \sigma_t dW_t \qquad (3.1)$$

Integrating equation 3.1 over the interval $[t, t+\Delta t]$, results in:

$$\int_t^{t+\Delta t} dS_s = \int_t^{t+\Delta t} \mu_s ds + \int_t^{t+\Delta t} \sigma_s dW_s$$

$$\int_t^{t+\Delta t} dS_s = S_{t+\Delta t} - S_t$$

$$S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} \mu_s ds + \int_t^{t+\Delta t} \sigma_s dW_s \qquad (3.2)$$

Through the application of Ito's lemma, the accuracy of the Milstein scheme in discretization is enhanced by considering the expansions of the coefficients $\mu_t = \mu(S_t)$ and $\sigma_t = \sigma(S_t)$ as follows:

Using Ito's lemma:

$$d\mu_t = \frac{\partial \mu_t}{\partial t} dt + \frac{\partial \mu_t}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 \mu_t}{\partial S_t^2}\left(dS_t\right)^2 \qquad (3.3)$$

$$\frac{\partial \mu_t}{\partial t} = 0, \ \frac{\partial \mu_t}{\partial S_t} = \mu_t', \ \frac{\partial^2 \mu_t}{\partial S_t^2} = \frac{1}{2}\mu_t^n$$

$$d\mu_t = \mu_t' dS_t + \frac{1}{2}\mu_t^n \left(dS_t\right)^2. \qquad (3.4)$$

Getting the square of equation 3.1 and substituting the result in equation 3.4, leads to:

$$(dS_t)^2 = \sigma_t^2 dt$$

$$d\mu_t = \mu_t'\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\mu_t''\sigma_t^2 dt$$

$$= \mu_t'\mu_t dt + \mu_t'\sigma_t dW_t + \frac{1}{2}\mu_t''\sigma_t^2 dt.$$

Thus:

$$d\mu_t = \left(\mu_t'\mu_t + \frac{1}{2}\mu_t''\sigma_t^2\right)dt + \mu_t'\sigma_t dW_t \qquad (3.5)$$

Using a similar approach, from Ito's lemma:

$$d\sigma_t = \frac{\partial \sigma_t}{\partial t} dt + \frac{\partial \sigma_t}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 \sigma_t}{\partial S_t^2}\left(dS_t\right)^2$$

$$\frac{\partial \sigma_t}{\partial t} = 0, \ \frac{\partial \sigma_t}{\partial S_t} = \sigma_t', \ \frac{\partial^2 \sigma_t}{\partial S_t^2} = \frac{1}{2}\sigma_t''$$

$$d\sigma_t = \sigma_t' dS_t + \frac{1}{2}\sigma_t''\left(dS_t\right)^2$$

$$= \sigma_t'\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\sigma_t''\sigma_t^2 dt$$

$$= \sigma_t'\mu_t dt + \sigma_t'\sigma_t dW_t + \frac{1}{2}\sigma_t''\sigma_t^2 dt.$$

Thus:

$$d\sigma_t = \left(\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2\right)dt + \sigma_t'\sigma_t dW_t. \qquad (3.6)$$

Integrating equation 3.5 over the interval [*t*, *s*] where $(t < s < t+\Delta t)$, results in:

$$\int_t^s d\mu_r = \int_t^s \left(\mu_r'\mu_r + \frac{1}{2}\mu_r''\sigma_r^2\right)dr + \int_t^s \mu_r'\sigma_r dW_r$$

$$\int_t^s d\mu_r = \mu_s - \mu_t$$

$$\mu_s = \mu_t + \int_t^s \left(\mu_r'\mu_r + \frac{1}{2}\mu_r''\sigma_r^2\right)dr + \int_t^s \mu_r'\sigma_r dW_r \qquad (3.7)$$

Integrating equation 3.6 over the interval [*t*, *s*] where $(t < s < t+\Delta t)$, leads to:

$$\int_t^s d\sigma_r = \int_t^s \left(\sigma_r'\mu_r + \frac{1}{2}\sigma_r''\sigma_r^2\right)dr + \int_t^s \sigma_t'\sigma_r dW_r$$

$$\int_t^s d\sigma_r = \sigma_s - \sigma_t$$

$$\sigma_s = \sigma_t + \int_t^s \left(\sigma_r'\mu_r + \frac{1}{2}\sigma_r''\sigma_r^2\right)dr + \int_t^s \sigma_t'\sigma_r dW_r \qquad (3.8)$$

Substituting for equation 3.7 and 3.8 in equation 3.2 gives the following integral:

$$S_{t+\Delta t}$$

$$= S_t + \int_t^{t+\Delta t}\left(\mu_t + \int_t^s \left(\mu_r'\mu_r + \frac{1}{2}\mu_r''\sigma_r^2\right)dr + \int_t^s \mu_r'\sigma_r dW_r\right)ds$$

$$+ \int_t^{t+\Delta t}\left(\sigma_t + \int_t^s \left(\sigma_r'\mu_r + \frac{1}{2}\sigma_r''\sigma_r^2\right)dr + \int_t^s \sigma_r'\sigma_r dW_r\right)dW_s.$$

Ignoring the terms *drds*, $drdW_s$ and $dsdW_r$ since they are higher than order one results in:

$$S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} \mu_t ds$$

$$+ \int_t^{t+\Delta t}\left(\sigma_t + \int_t^s \left(\sigma_r'\mu_r + \frac{1}{2}\sigma_r''\sigma_r^2\right)dr + \int_t^s \sigma_r'\sigma_r dW_r\right)dW_s$$

$$S_{t+\Delta t} = S_t + \mu_t \int_t^{t+\Delta t} ds + \sigma_t \int_t^{t+\Delta t} dW_s + \int_t^{t+\Delta t}\int_t^s \sigma_r'\sigma_r dW_r dW_s. \qquad (3.9)$$

Applying Euler scheme discretization to the last term in equation 3.9 leads to:

$$\int_t^{t+\Delta t}\int_t^s \sigma_r'\sigma_r dW_r dW_s \approx \sigma_t'\sigma_t \int_t^{t+\Delta t}\int_t^s dW_r dW_s$$
$$= \sigma_t'\sigma_t \int_t^{t+\Delta t}\left(W_s - W_t\right)dW_s$$
$$= \sigma_t'\sigma_t \int_t^{t+\Delta t}\left(W_s dW_s - W_t dW_s\right)$$
$$= \sigma_t'\sigma_t \left(\int_t^{t+\Delta t} W_s dW_s - \int_t^{t+\Delta t} W_t dW_s\right)$$
$$= \sigma_t'\sigma_t \left(\int_t^{t+\Delta t} W_s dW_s - W_t\left(W_{t+\Delta t} - W_t\right)\right).$$

Thus

$$\int_t^{t+\Delta t}\int_t^s \sigma_r'\sigma_r dW_r dW_s = \sigma_t'\sigma_t \left(\int_t^{t+\Delta t} W_s dW_s - W_t W_{t+\Delta t} + W_t^2\right). \quad (3.10)$$

Let $Y_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$, using Ito's lemma,

$$dY_t = \frac{\partial Y_t}{\partial t}dt + \frac{\partial Y_t}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial W_t^2}\left(dW_t\right)^2$$

$$\frac{\partial Y_t}{\partial t} = -\frac{1}{2}, \qquad \frac{\partial Y_t}{\partial W_t} = W_t, \qquad \frac{\partial^2 Y_t}{\partial W_t^2} = 1$$

$$dY_t = -\frac{1}{2}dt + W_t dW_t + \frac{1}{2}\left(dW_t\right)^2$$

$$dY_t = -\frac{1}{2}dt + W_t dW_t + \frac{1}{2}dt$$

$$dY_t = W_t dW_t \quad (3.11)$$

This implies that:

$$\int_t^{t+\Delta t} W_s dW_s = \int_t^{t+\Delta t} dY_s = Y_{t+\Delta t} - Y_t$$
$$= \left(\frac{1}{2}W_{t+\Delta t}^2 - \frac{1}{2}(t+\Delta t)\right) - \left(\frac{1}{2}W_t^2 - \frac{1}{2}t\right)$$
$$= \left(\frac{1}{2}W_{t+\Delta t}^2 - \frac{1}{2}t - \frac{1}{2}\Delta t\right) - \left(\frac{1}{2}W_t^2 - \frac{1}{2}t\right).$$

$$\int_t^{t+\Delta t} W_s dW_s = \frac{1}{2}W_{t+\Delta t}^2 - \frac{1}{2}W_t^2 - \frac{1}{2}\Delta t. \quad (3.12)$$

Substituting the result of equation 3.12 to equation 3.10 results in:

$$\int_t^{t+\Delta t}\int_t^s \sigma_r'\sigma_r dW_r dW_s$$
$$\approx \sigma_t'\sigma_t \left(\frac{1}{2}W_{t+\Delta t}^2 - \frac{1}{2}W_t^2 - \frac{1}{2}\Delta t - W_t W_{t+\Delta t} + W_t^2\right)$$
$$= \frac{1}{2}\sigma_t'\sigma_t \left(W_{t+\Delta t}^2 - W_t^2 - \Delta t - 2W_t W_{t+\Delta t} + 2W_t^2\right) \quad (3.13)$$
$$= \frac{1}{2}\sigma_t'\sigma_t \left(\left(W_{t+\Delta t} - W_t\right)^2 - \Delta t\right)$$
$$= \frac{1}{2}\sigma_t'\sigma_t \left(\Delta t Z^2 - \Delta t\right).$$

From equation 3.10 and 3.13, the Milstein discretization is given by:

$$S_{t+\Delta t} = S_t + \mu_t \int_t^{t+\Delta t} ds + \sigma_t \int_t^{t+\Delta t} dW_s + \frac{1}{2}\sigma_t'\sigma_t\left(\Delta t Z^2 - \Delta t\right)$$

$$\mu_t \int_t^{t+\Delta t} ds \approx \mu_t \Delta_t$$

$$\sigma_t \int_t^{t+\Delta t} dW_s \approx \sigma_t (W_{t+\Delta t} - W_t) = \sigma_t \sqrt{\Delta t} Z.$$

Thus:

$$S_{t+\Delta t} = S_t + \mu_t \Delta_t + \sigma_t \sqrt{\Delta t} Z + \frac{1}{2}\sigma_t'\sigma_t \Delta t\left(Z^2 - 1\right). \quad (3.14)$$

3.1. The Black-Scholes Model

Using the assumption made by the Milstein scheme in equation 3.1, under the Black-Scholes model, $\mu_t = rS_t$ and $\sigma_t = \sigma S_t$, such that

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (3.15)$$

Substituting for μ_t and σ_t in equation 3.14leads to:

$$S_{t+\Delta t} = S_t + rS_t \Delta_t + \sigma S_t \sqrt{\Delta t} Z + \frac{1}{2}\sigma_t'\sigma S_t \Delta t\left(Z^2 - 1\right)$$

$$\sigma_t' = \frac{\partial \sigma_t}{\partial S_t} = \frac{\partial(\sigma S_t)}{\partial S_t} = \sigma \frac{\partial S_t}{\partial S_t} = \sigma$$

Thus

$$S_{t+\Delta t} = S_t + rS_t \Delta_t + \sigma S_t \sqrt{\Delta t} Z + \frac{1}{2}\sigma^2 S_t \Delta t\left(Z^2 - 1\right) \quad (3.16)$$

This scheme adds a correction term of $\frac{1}{2}\sigma^2 S_t \Delta t\left(Z^2 - 1\right)$ to the Euler Scheme result in equation 2.3.

3.2. The Heston Model

3.2.1. Discretization of the Asset Process

For the Heston model asset price process, a substitution is done in equation 3.14, such that $\mu_t = (r-q)S_t$ and $\sigma_t = \sqrt{V_t} S_t$ resulting in:

$$S_{t+\Delta t} = S_t + (r-q)S_t \Delta_t + \sqrt{V_t} S_t \sqrt{\Delta t} Z_p + \frac{1}{2}\sigma_t' \sqrt{V_t} S_t \Delta t\left(Z_p^2 - 1\right)$$

$$\sigma_t' = \frac{\partial \sigma_t}{\partial S_t} = \frac{\partial(\sqrt{V_t} S_t)}{\partial S_t} = \sqrt{V_t}\frac{\partial S_t}{\partial S_t} = \sqrt{V_t}$$

Thus

$$S_{t+\Delta t} = S_t + (r-q)S_t \Delta_t + \sqrt{V_t} S_t \sqrt{\Delta t} Z_p + \frac{1}{2}V_t S_t \Delta t\left(Z_p^2 - 1\right). \quad (3.17)$$

A correction term of $\frac{1}{2}V_t S_t \Delta t\left(Z_p^2 - 1\right)$ is added to the Euler discretization given in equation 2.7.

Applying the full truncation scheme, equation 3.17 becomes:

$$S_{t+\Delta t} = S_t + (r-q)S_t\Delta_t + \sqrt{V_t^+}S_t\sqrt{\Delta t}Z_p + \frac{1}{2}V_t^+ S_t \Delta t\left(Z_p^2 - 1\right). \quad (3.18)$$

3.2.2. Discretization of the Variance Process

For the Heston model variance process, a substitution is done in equation 3.14 such that $\mu_t = \kappa(\theta - V_t)$ and $\sigma_t = \sigma\sqrt{V_t}$ which leads to:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta_t + \sigma\sqrt{V_t}\sqrt{\Delta t}Z_v + \frac{1}{2}\sigma_t'\sigma\sqrt{V_t}\Delta t\left(Z_v^2 - 1\right)$$

$$\sigma_t' = \frac{\partial \sigma_t}{\partial V_t} = \frac{\partial(\sigma\sqrt{V_t})}{\partial V_t} = \sigma\frac{\partial\sqrt{V_t}}{\partial V_t} = \frac{1}{2\sqrt{V_t}}\sigma.$$

Thus:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta_t + \sigma\sqrt{V_t}\sqrt{\Delta t}Z_v + \frac{1}{4}\sigma^2\Delta t\left(Z_v^2 - 1\right). \quad (3.19)$$

This approach produces less negative values for the variance as compared to Euler discretization. To counter the occurrence of a negative variance, the full truncation scheme is applied which involves substituting V_t with V_t^+ in equation 3.19 resulting in:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t^+)\Delta_t + \sigma\sqrt{V_t^+}\sqrt{\Delta t}Z_v + \frac{1}{4}\sigma^2\Delta t\left(Z_v^2 - 1\right). \quad (3.20)$$

The result in Equation 3.19 is the same result obtained in Gatheral [4] for the Milstein scheme discretization of the Heston Model variance process.

Comparing the result obtained in equation 3.20 and that obtained under the Euler scheme in equation 2.10, there's an additional term of $\frac{1}{4}\sigma^2\Delta t\left(Z_v^2 - 1\right)$.

3.2.3. Discretized Heston Model Dynamics

With initial values S_t for the asset price and V_t for the variance, equation 3.18 and equation 3.20 can be used to obtain S_{t+dt} and V_{t+dt}.

Z_p and Z_v have a correlation ρ, two independent standard normal variables B_p and B_v are generated such that:

$$Z_v = B_v$$

$$Z_p = \rho B_v + \sqrt{1-\rho^2}B_p.$$

3.3. The BHM Model

3.3.1. Discretization of the Asset Process

Using the assumption made by the Milstein scheme in equation 3.1, under the BHM model, $\mu_t = rS_t$ and $\sigma_t = \Gamma_{tT}$, such that

$$dS_t = rS_t dt + \Gamma_{tT}dW_t \quad (3.21)$$

Substituting for μ_t and σ_t in equation 3.14 leads to:

$$S_{t+\Delta t} = S_t + rS_t\Delta_t + \Gamma_{tT}\sqrt{\Delta t}Z + \frac{1}{2}\sigma_t'\Gamma_{tT}\Delta t\left(Z^2 - 1\right)$$

$$\sigma_t' = \frac{\partial \Gamma_{tT}}{\partial S_t} = \frac{\partial\left(P_{tT}\frac{\sigma T}{T-t}V_t\right)}{\partial S_t} = 0.$$

Thus:

$$S_{t+\Delta t} = S_t + rS_t\Delta_t + \Gamma_{tT}\sqrt{\Delta t}Z. \quad (3.22)$$

The result obtained in equation 3.22 is the same as the result obtained under the Euler scheme in equation 2.13. This means that for the BHM model, the asset process discretization remains the same under both the Euler scheme and Milstein scheme.

3.3.2. Discretization of the Variance Process

For the BHM model variance process, a substitution in equation 3.14 is made such that, $\mu_t = -v_t^2 V_t^2$ and $\sigma_t = v_t\kappa_t$ leading to:

$$V_{t+\Delta t} = V_t - v_t^2 V_t^2\Delta_t + v_t\kappa_t\sqrt{\Delta t}Z_v + \frac{1}{2}\sigma_t' v_t\kappa_t\Delta t\left(Z_v^2 - 1\right)$$

$$\sigma_t' = \frac{\partial \sigma_t}{\partial V_t} = \frac{\partial(v_t\kappa_t)}{\partial V_t} = 0.$$

Thus:

$$V_{t+\Delta t} = V_t - v_t^2 V_t^2\Delta_t + v_t\kappa_t\sqrt{\Delta t}Z_v \quad (3.23)$$

The result obtained in equation 3.23 is the same as the result obtained under the Euler scheme in equation 2.14. This means that for the BHM model, the variance process discretization remains the same under both the Euler scheme and Milstein scheme.

3.3.3. Discretized BHM Model Dynamics

The dynamics under the BHM model Milstein discretization scheme will take the same form as the BHM model under the Euler discretization scheme.

4. Conclusion

A brief discussion of the asset pricing models is given in the introduction starting from the Black-Scholes model which is the basic model from which the stochastic volatility models are derived from. The Heston model which addresses the biggest weakness of the Black-Scholes model which is the asumption of constant volatility is also looked at. The BHM model is then presented which addresses the weakness of the other two models of imposing the asset dynamics.

The Euler discretization is applied to the Black-Scholes model and the Heston Model. This approach is then extended to the BHM Model which is the main objective of this study. One of the disadvantages of the Euler Scheme is that the variance can become negative. To counter this problem, the full truncation scheme approach is applied.

Another discretization scheme, Milstein scheme is shown to be more accurate since it considers the expansion of the coefficients involved in the SDE. In addition, the Milstein scheme significantly alleviates the problem of the variance becoming negative which makes it preferred to the Euler approach to discretization.

From this study, it is observed that the asset process and variance process in the information based asset pricing framework take a similar form under both the Euler Scheme and Milstein Scheme.

Further works will be to extend this approach to the multi-asset framework for the information based asset pricing framework and to consider the case where r_t wich denotes the interest rate is not assumed to be constant.

References

[1] Rosenberg N. (1972) "Factors affecting the diffusion of technology," Explorations in Economic History 10(1), 3-33.

[2] Scott, L. (1987), "Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application," Journal of Financial and Quantitative Analysis, 22, 419-438.

[3] Maruyama, G., Continuous Markov processes and stochastic equations. Rend. Circ. Mat. Palermo 4 (1955), 48-49. 12. LOEVE, M., "Probability Theory," 3rd.

[4] Gatheral, Jim (2006). "The Volatility Surface. A Practitioner's Guide". New York, NY: John Wiley and Sons.

[5] Platen (1999). An introduction to numerical methods for Stochastic Differential Equations. Cambridge University Press, 197-246.

[6] Heston Steven L (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options". In The Review of Financial Studies 6(2) 327-343.

[7] Cox C, Ingersoll E and Ross A (1985). "A Theory of the Term Structure of Interest Rates". In: Journal of the Econometric Society 53 (2), 385-407.

[8] Brody D, Hughston L and Macrina A (2008). "Information-based asset pricing". In: International Journal of Theoretical and Applied finance 11, 107-142.

[9] Macrina, A. (2006). "An Information-Based Framework for Asset Pricing: X-Factor Theory and its Applications", PhD Thesis, King's College London.

[10] Glasserman, P. (2003). Monte Carlo Methods in Financial Engineering. New York, NY: Springer.