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The Moments for Solution of the Cox-Ingersoll-Ross Interest Rate Model

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Abstract In finance, the Cox-Ingersoll-Ross model (or CIR model) explains the evolution of interest rates. This model has no general explicit solution. In this paper, the moments of solution for the CIR model are obtained explicitly.

Keywords: *Cox-Ingersoll-Ross model, interest rate model, the moments for solution of CIR model, square-root diffusion*

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1. Introduction

A Stochastic differential equation (SDE) is a differential equation in which one or more of the terms has a random components. In general, SDE has the following form

$$dX(t)=\mu(X(t),t)dt+\sigma(X(t),t)dW(t), \qquad (1)$$

where $\mu(X(t),t)$ is drift term, and $\sigma(X(t),t)$ is diffusion therm. SDEs are used in finance (for e.g. in the stock-price models, option pricing, and interest rate models), biology (for e.g. in the epidemic models, predator-prey models, and population models), physics (for e.g. in the ion transport, nuclear reactor kinetics, chemical reaction, and cotton fiber breakage), and stochastic control to model various phenomena. In finance, SDEs are used to model stock and asset prices, and interest rates. The dynamics of interest rates, have an important role in the decisions related to an investment, and the transactions based on lending rate. Interest rates have properties such as positivity, boundedness, and return to equilibrium which make them behave differently from stock and asset prices. The short-term interest rate is one of the key financial variables in any economy. SDEs have been applied to model the evolution of interest rates for a short-time period. The first stochastic model for interest rates introduced by Oldrich Vasicek in 1977. This model satisfies the following stochastic differential equation form

$$dr(t)=\kappa(\theta-r(t))dt+\sigma dW(t). \qquad (2)$$

In this model $r(t)\sim N\left(e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right),\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa t}\right)\right)$.

Therefore, the main disadvantage of this model is that it is theoretically possible for the interest rate to become negative. The CIR model was introduced by John Carrington Cox, Jonathan Edwards Ingersoll and Stephen Alan Ross in 1985 and satisfies the following stochastic differential equation form (for further see [3,6,7] and references therein)

$$dr(t)=\kappa(\theta-r(t))dt+\sigma\sqrt{r(t)}dW(t). \qquad (3)$$

where κ, θ, and σ are positive constants. κ is the speed of mean reversion. θ is mean level or long term rate constant and σ regulates the volatility. In this model, the stochastic term $\sigma\sqrt{r(t)}dW(t)$ has a standard deviation proportion to the square root of the current rate. This implies that as the rate increase, its standard deviation increase and as it falls and approach zero, the stochastic term also approach zero. In this model, the probability density of interest rate at time *t*, conditional on its value at the current time *s*, is given by [2]

$$f(r(t);r(s),s)=ce^{-u-v}\left(\frac{v}{u}\right)^{q/2}I_q\left(2\sqrt{uv}\right), \qquad (4)$$

where

$$\begin{cases} c=\dfrac{2\kappa}{\sigma^2\left(1-e^{-\kappa(t-s)}\right)}, \\ u=cr(s)e^{-\kappa(t-s)}, \\ v=cr(t), \\ q=\dfrac{2\kappa\theta}{\sigma^2}-1, \end{cases} \qquad (5)$$

and $I_q(.)$ is the modified Bessel function of the first kind of order q defined as

$$I_q(z)=\left(\frac{z}{2}\right)^q\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+1)\Gamma(z+k+1)}\left(\frac{z}{2}\right)^{2k}. \quad (6)$$

Other classical models are Dothan, Rendleman-Barter, Courtadon and exponential Vasicek models. Recently, other models have proposed by SDEs on manifolds [6].

The organization of this paper is as follows. In Section 2, for convenience of the reader, some basic definitions, and mathematical preliminaries of the stochastic calculus are presented. Furthermore, the CIR model is solved in this section. Finally, in Section 3, the moments for solution of CIR model is obtained.

2. Basic Concepts of the Stochastic Calculus

In the following, some preliminary are reviewed in stochastic calculus [1,4,5,8,9]. In addition, stochastic CIR model is solved in this section.

Definition 1 *Brownian motion is a stochastic process* $\{W(t)\,|\,t\in[0,\infty]\}$ *with the following properties:*

1. $W(0)=0$.

2. It has a continuous path.

3. For all non-overlapping time interval $[t_1,t_2]$, $[t_3,t_4]$ *the random variables* $W(t_2)-W(t_1)$ *and* $W(t_4)-W(t_3)$ *are independent, i.e.* $W(t_2)-W(t_1)\perp W(t_4)-W(t_3)$.

4. For all s *and* t, $t>s$, *the increment* $W(t)-W(s)$ *is a normal variable, with zero mean and variance* $t-s$ *(i.e.* $W(t)-W(s)\sim N(0,t-s)$*).*

Definition 2 *A process* $X(t)$ *is called* $\mathcal{F}_i$ *-adapted, if for all* t, $X(t)$ *is* $\mathcal{F}_i$ *-measurable.*

Definition 3 *An Itô process has the form*

$$X(t)=X(0)=\int_0^t f(s)ds+\int_0^t g(s)dW(s),\ 0\le t\le T, \quad (7)$$

where $f(t)$ *and* $g(t)$ *are* $\mathcal{F}_t$ *-adapted, such that* $\int_0^T|f(t)|dt<\infty$ *and* $\int_0^T|g(t)|^2dt<\infty$. *It is said that the process* $X(t)$ *has the stochastic differential on* $[0,T]$

$$dX(t)=f(t)dt+g(t)dW(t),\ 0\le t\le T. \quad (8)$$

Let $X(t)$ and $Y(t)$ are Itô processes. Then the following properties are satisfied.

1. $\forall\alpha,\beta\in\mathbb{R}$,

$$\int_0^T(\alpha X(t)+\beta Y(t))dW(t) =\alpha\int_0^T X(t)dW(t)+\beta\int_0^T Y(t)dW(t). \quad (9)$$

2.

$$\int_0^T X(t)I_{(a,b]}dt=\int_0^T X(t)dW(t). \quad (10)$$

Furthermore, if $\int_0^T\mathbb{E}\left[X^2(t)\right]dt<\infty$, we have

3. **Zero mean property**

$$\mathbb{E}\left[\int_0^T X(t)dW(t)\right]=0. \quad (11)$$

4. **Isometry property**

$$\mathbb{E}\left[\left(\int_0^T X(t)dW(t)\right)^2\right]=\int_0^T\mathbb{E}\left[X^2(t)\right]dt. \quad (12)$$

In other words, the expectation of an Ito integral is zero. In the following, the CIR model is solved.

Theorem 1 *Let*

$$\begin{cases} dr(t)=\kappa(\theta-r(t))dt+\sigma\sqrt{r(t)}dW(t),\\ r(0)=r_0.\end{cases} \quad (13)$$

Then the exact solution is given by

$$r(t)=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)+\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s). \quad (14)$$

Proof. *The equation (13) changes to the following form*

$$dr(t)+\kappa r(t)dt=\kappa\theta dt+\sigma\sqrt{r(t)}dW(t). \quad (15)$$

Multiplying both sides of the relation (15) by $e^{\kappa t}$ *results in*

$$\begin{aligned} e^{\kappa t}dr(t)+\kappa e^{\kappa t}r(t)dt&=\kappa\theta e^{\kappa t}dt+\sigma e^{\kappa t}\sqrt{r(t)}dW(t),\\ d\left(e^{\kappa t}r(t)\right)&=\kappa\theta e^{\kappa t}dt+\sigma e^{\kappa t}\sqrt{r(t)}dW(t).\end{aligned} \quad (16)$$

Now, integrating both sides of the relation (16) on $[0,t]$ *gives us*

$$e^{\kappa t}r(t)-r_0=\kappa\theta\int_0^t e^{\kappa s}ds+\sigma\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s).$$

Consequently, the relation (14) is derived.

According to the above Theorem, the CIR model has no general explicit solution. However, its mean and variance can be calculated explicitly.

Theorem 2 *The expectation and variance of* $r(t)$ *are given by*

$$\begin{cases}\mathbb{E}\left[r(t)\right]=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right),\\ Var\left[r(t)\right]=\dfrac{\sigma^2}{\kappa}r_0\left(e^{-\kappa t}-e^{-2\kappa t}\right)+\dfrac{\theta\sigma^2}{2\kappa}\left(1-e^{-\kappa t}\right)^2.\end{cases} \quad (17)$$

Proof. *Taking the expectation from both sides of the relation (14) result in*

$$\begin{aligned}&\mathbb{E}\left[r(t)\right]\\ &=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\\ &\quad+\sigma e^{-\kappa t}\mathbb{E}\left[\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s)\right]\\ &=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\\ &\quad+\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r(s)}\mathbb{E}\left[dW(s)\right]\\ &=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right).\end{aligned} \quad (18)$$

It is interesting to note that $\lim_{t\to 0\infty}\mathbb{E}\left[r(t)\right]=\theta$. *Finally, the variance can be calculated as*

$$
\begin{aligned}
Var\left[r(t)\right] &= \mathbb{E}\left[r^2(t)\right]-\left(E\left[r(t)\right]\right)^2 \\
&= \mathbb{E}\left[\begin{pmatrix} e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right) \\ +\sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s)\end{pmatrix}^2\right] \\
&\quad -\left[e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\right]^2 \\
&= 2\left(e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\right)\sigma e^{-\kappa t} \\
&\quad \times\mathbb{E}\left[\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s)\right] \\
&\quad +\sigma^2 e^{-2\kappa t}\mathbb{E}\left[\left(\int_0^t e^{\kappa s}\sqrt{r(s)}dW(s)\right)\right] \\
&= 2\left(e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\right)\sigma \\
&\quad \times e^{-\kappa t}\left[\int_0^t e^{\kappa s}\sqrt{r(s)}\mathbb{E}\left[dW(s)\right]\right] \\
&\quad +\sigma^2 e^{-2\kappa t}\mathbb{E}\left[\int_0^t e^{2\kappa s}r(s)ds\right] \\
&= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa s}\mathbb{E}\left[r(s)\right]ds \\
&= \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa s}\left(e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)\right)ds \\
&= \frac{\sigma^2}{\kappa}r_0\left(e^{-\kappa t}-e^{-2\kappa t}\right) \\
&\quad +\frac{\theta\sigma^2}{2\kappa}\left(1-2e^{-\kappa t}+e^{-2\kappa t}\right) \\
&= \frac{\sigma^2}{\kappa}r_0\left(e^{-\kappa t}-e^{-2\kappa t}\right)+\frac{\theta\sigma^2}{2\kappa}\left(1-e^{-\kappa t}\right)^2.
\end{aligned}
\tag{19}
$$

3. The Moments of the Solution for CIR Model

In this Section, the expectation of $r^n(t)$ for $n=1,2,...$ (i.e. the moments of $r(t)$) is computed in Theorem 3.

Theorem 3 *For any $n\in\mathbb{N}$, we have*

$$
\mathrm{E}\left[r^n(t)\right]=\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{j}A^{n-2j}(t)B^{2j}(t)\left[\frac{1}{2\kappa}\left(e^{2\kappa t}-1\right)\right]^{2j}, \tag{20}
$$

where $A(t)=e^{-\kappa t}r_0+\theta\left(1-e^{-\kappa t}\right)$ and $B(t)=\sigma e^{-\kappa t}$.

Proof. *By using the relation (14) we have*

$$
r(t)=A(t)+B(t)I(t), \tag{21}
$$

where $I(t)=\int_0^t e^{\kappa t}dW(s)$. Therefore, we get

$$
r^n(t)=\sum_{j=0}^{n}\binom{n}{j}A^{n-j}B^j(t)I^j(t), \tag{22}
$$

Taking the expectation from both sides of the relation (22) results in

$$
\begin{aligned}
\mathbb{E}\left[r^n(t)\right] &= \mathbb{E}\left[\sum_{j=0}^{n}\binom{n}{j}A^{n-j}(t)B^j(t)I^j(t)\right] \\
&= \sum_{j=0}^{n}\binom{n}{j}A^{n-j}(t)B^j(t)\mathbb{E}\left[I^j(t)\right].
\end{aligned}
\tag{23}
$$

By noting the relation (12), for j = 2k, and $k\in\mathbb{N}$, we have

$$
\mathbb{E}\left[I^j(t)\right]=\left(\mathrm{E}\left[I^2(t)\right]\right)^k=\left[\frac{1}{2\kappa}\left(e^{2\kappa t}-1\right)\right]^k. \tag{24}
$$

Similarly, by noting the relation (11), for $j=2k+1$, and $k\in\mathbb{N}$, we have

$$
\begin{aligned}
\mathbb{E}\left[I^j(t)\right] &= \left(\mathbb{E}\left[I^2(t)\right]\right)^k\times\mathbb{E}\left[I(t)\right] \\
&= \left[\frac{1}{2\kappa}\left(e^{2\kappa t}-1\right)\right]^k\times 0 \\
&= 0.
\end{aligned}
\tag{25}
$$

Therefore, by using simultaneously the relations (24) and (25), the relation (23) simplifies as

$$
\mathbb{E}\left[r^n(t)\right]=\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{j}A^{n-2j}(t)B^{2j}(t)\left[\frac{1}{2\kappa}\left(e^{2\kappa t}-1\right)\right]^{2j}, \tag{26}
$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$. The following program, written by Maple, generates $\mathbb{E}\left[r^n(t)\right]$. In this program I supposed N = 10, t = 1, $r_0=1$, $\kappa=1$, $\theta=0.45$, and $\sigma=1$, which can easily be changed by the user.

Program 1. Computing the $\mathbb{E}\left[r^n(t)\right]$ using the general formula (26)

```
restart:
Digits:=8:
N:=10:
K:=floor(N/2):
t:=1:
r[0]:=1:
kappa:=1:
theta:=0.45:
sigma:=1:
A:=exp(1)**(-kappa*t)*r[0]+theta*(1-exp(1)**(-
kappa*t)):
B:=sigma*exp(1)**(-kappa*t):
expect:=evalf(add(binomial(N,j)*A**(K-2*j)*B**(2*j)*
((1/(2*kappa))*(exp(1)**(2*kappa*t)-1)),j=0..K));
```

The CIR model belongs to the family of diffusion processes, also known as square-root diffusions, and satisfies the following stochastic differential equation

$$
dX(t)=\left(a+bX(t)\right)dt+c\sqrt{X(t)}dW(t). \tag{27}
$$

The distribution function for $r(t)$ is Noncentral Chi-square, $\chi^2\left[2cr(s);2q+2,2u\right]$, with $2q+2$ degrees of freedom and parameter of non-centrality $2u$ proportional to the current spot rate [2]. This distribution function is given by

$$f\left(r(t);2q+2,2u\right) = \sum_{k=0}^{\infty}\frac{\exp\left(-u-\frac{r(t)}{2}\right)(u)^k\left(r(t)\right)^{q+k}}{k!2^{q+1+k}\Gamma(q+1+k)}. \quad (28)$$

Remark 1. If we set $\kappa=-b$, $\theta=-\frac{a}{b}$, and $c=\sigma$ in the relation (26), the moments of square-root diffusion process can easily be computed.

Remark 2. Note that by using the simultaneously the relations (26) and (28), For all $n\in\mathbb{N}$, we have

$$\mathbb{E}\left[r^n(t)\right] = \sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{j}A^{n-2j}(t)B^{2j}(t)\left[\frac{1}{2\kappa}\left(e^{2\kappa t}-1\right)\right]^{2j} \quad (29)$$
$$= \int_0^{\infty} r(t)^n\left(\sum_{k=0}^{\infty}\frac{\exp\left(-u-\frac{r(t)}{2}\right)(u)^k\left(r(t)\right)^{q+k}}{k!2^{q+1+k}\Gamma(q+1+k)}\right)dr.$$

4. Conclusion

In this paper, the moments of the solution for the CIR model are obtained explicitly. In addition, numerical implement is presented by Maple. Finally, we would like to emphasize that by using these moments, new identities can be resulted.

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