

An Order Level Inventory Model for Deteriorating Items with Time – Quadratic Demand and Partial Backlogging

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Abstract In this paper, a deterministic inventory model is developed for deteriorating items in which shortages are allowed and partially backlogged. The backlogging rate is assumed to be dependent on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. The deterioration rate is constant and demand rate is assumed to be time quadratic. Numerical example and sensitivity analysis are evaluated for validating the proposed model.

Keywords: *inventory, deteriorating items, shortages, time – quadratic demand, partial backlogging*

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1. Introduction

In the last three decades, the models for inventory replenishment policies involving time-varying demand patterns have received the attention of several researchers. The fundamental result in the development of economic order quantity model with time-varying demand patterns is that of Donaldson (1979) [11] who established the classical no-shortage inventory model with a linear trend in demand over a known and finite horizon. However, his procedure was too complex and tedious in computing. The complexity of Donaldson’s approach has led to the development of heuristic methods. Dave and Patel (1981) [12] first considered the inventory model for deteriorating items with time-varying demand. They considered a linear increasing demand over a finite horizon and a constant deterioration rate. Sachan (1984) [13] extended Dave and Patel’s model to allow shortages.

Researchers have developed economic order quantity models that focused on deteriorating items with time-varying demand and shortages. Silver et al. (1998) [1] explained inventory management and production planning and scheduling. Raafat (1991) [2] described survey of literature on continuously deteriorating inventory model. Chang and Dye (1999) [3] explored an EOQ model for deteriorating items with time varying demand and partial backlogging. Goyal and Giri (2001) [4] presented recent trends in modeling of deteriorating inventory. Teng and Chang (2005) [5] defined economic production quantity models for deteriorating items with price and stock dependent demand. Singh and Singh (2007) [6] presented an EOQ inventory model with Weibull distribution deterioration, Ramp type demand and Partial Backlogging.

Singh et al. (2009) [7] extended an EOQ model for perishable items with power demand and partial backlogging. You, (2005) [8] derived Inventory policy for products with price and time dependent demands. Pareek, Mishra, and Rani (2009) [9] presented an inventory model for time dependent deteriorating item with salvage value and shortages and Hamid (1989) [10] explored replenishment schedule for deteriorating items with time proportional demand. Lin et al. (2000) [14] explained an EOQ model for deteriorating items with time – varying demand and shortages. Hariga (1996) [15] defined optimal EOQ models for deteriorating items with time – varying demand. In practice, some customers would like to wait for backlogging during the shortage period, but the others would not. Consequently, the opportunity cost due to lost sales should be considered.

In this paper, backlogging rate is assumed to be a fixed fraction of demand rate during the shortage period. However, in some inventory system, for many stock such as fashionable commodities, the length of the waiting time for the next replenishment becomes a major factor for determining whether the backlogging will be accepted or not. The longer the waiting time is, the smaller the backlogging rate would be. Therefore, the backlogging rate is variable and is dependent on the waiting time for the next replenishment. Here an attempt has been made to model the situation where the demand rate is a time-quadratic function and the items deteriorate at a constant rate with partial backlogging.

2. Notations and Assumptions

The mathematical model in this paper is developed on the basis of the following notations and assumptions.

2.1. Notations

A – Ordering cost per order
 h – Inventory carrying cost per unit per unit time
 s – Shortage cost per unit per unit time
 l – Opportunity cost due to lost sales per unit
 r – Purchase cost per unit
 t_1 – the time at which the inventory level reaches zero,
 $t_1 \geq 0$
 t_2 – the length of period during which shortages are
 allowed, $t_2 \geq 0$
 T – (= $t_1 + t_2$) the length of cycle time
 $I_1(t)$ – the level of positive inventory at time t, $0 \leq t \leq t_1$
 $I_2(t)$ – the level of negative inventory at time t, $t_1 \leq t \leq t_1 + t_2$
 θ – Deterioration rate, a fraction of the on-hand
 inventory
 IM – Maximum inventory level during [0, T]
 IB – Maximum inventory level during shortage period
 R(t) – Demand rate at any time $t \geq 0$

2.2. Assumptions

A single item is considered with a constant rate of deterioration.

The replenishment occurs instantaneously at an infinite rate.

A deteriorated unit is neither repaired nor replaced during a given cycle

The deterministic demand rate R(t) varies quadratic with time, i.e., $R(t) = a + bt + ct^2$, where a, b and c are constants such that $a, b, c > 0$.

Shortages are allowed and partially backlogged.

During stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. The backlogging rate is assumed to be $\frac{1}{1 + \alpha(T - t)}$ where the backlogging

parameter α is a positive constant and (T - t) is waiting time ($t_1 \leq t \leq T$).

3. Mathematical Model

The rate of change of inventory during positive stock period [0, t_1] and shortage period [t_1 , T] is governed by the differential equations

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -(a + bt + ct^2), 0 \leq t \leq t_1 \quad (1)$$

$$\frac{dI_2(t)}{dt} = \frac{-(a + bt + ct^2)}{1 + \alpha(T - t)}, t_1 \leq t \leq T \quad (2)$$

With boundary conditions,

$$I_1(t) = I_2(t) = 0 \text{ at } t = t_1 \text{ and } I_1(t) = \text{IM at } t = 0.$$

4. Analytical Solution

During the period [0, t_1], the inventory depletes due to the deterioration and demand. Hence the inventory level at any time during [0, t_1] is described by equation (1) which is a linear ordinary differential equation of first order and its integrating factor is $e^{\theta t}$. Multiplying both sides of (1) by

$e^{\theta t}$ and then integrating over [0, t] and applying the boundary condition $I_1(t_1) = 0$ we have

$$I_1(t) = -\frac{a}{\theta} - \frac{b}{\theta} \left(t - \frac{1}{\theta} \right) - \frac{c}{\theta} \left[\left(t - \frac{1}{\theta} \right)^2 + \frac{1}{\theta^2} \right] + e^{\theta(t-t_1)} \left\{ \frac{a}{\theta} + \frac{b}{\theta} \left(t_1 - \frac{1}{\theta} \right) + \frac{c}{\theta} \left[\left(t_1 - \frac{1}{\theta} \right)^2 + \frac{1}{\theta^2} \right] \right\} \quad (3)$$

$$0 \leq t \leq t_1$$

During the interval [t_1 , T] the inventory level depends on demand and a fraction of demand is backlogged. The state of inventory during [t_1 , T] is represented by the equation (2) and its solution is given by

$$I_2(t) = \left\{ \frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right\} \log \left[\frac{1 + \alpha(t_1 + t_2 - t)}{1 + \alpha t_2} \right] + \frac{b(t - t_1)}{\alpha} + \frac{c(t - t_1)}{\alpha^2} \left[1 + \frac{3\alpha(t_1 + t_2)}{2} \right], \quad (4)$$

$$t_1 \leq t \leq t_1 + t_2$$

The total cost per replenishment cycle consists of the following components:

Inventory holding cost per cycle =

$$h \int_0^{t_1} I_1(t) dt \frac{h}{\theta^4} \left[a\theta^2 (e^{t_1\theta} - t_1\theta - 1) + \frac{b\theta}{2} (2e^{t_1\theta} (t_1\theta^2 - 1) - t_1^2\theta^2 + 1) + \frac{c}{3} (3e^{t_1\theta} (t_1^2\theta^2 - 2t_1\theta + 2) - t_1^3\theta^3 - 5) \right] \quad (5)$$

$$\text{Cost of deterioration} = \frac{rh}{\theta^3} =$$

$$\left[a\theta^2 (e^{t_1\theta} - t_1\theta - 1) + \frac{b\theta}{2} (2e^{t_1\theta} (t_1\theta^2 - 1) - t_1^2\theta^2 + 1) + \frac{c}{3} (3e^{t_1\theta} (t_1^2\theta^2 - 2t_1\theta + 2) - t_1^3\theta^3 - 5) \right] \quad (6)$$

Backordered cost per cycle

$$= s \int_{t_1}^{t_1+t_2} [-I_2(t)] dt = \left\{ \frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right\} \left\{ t_2 + \frac{1}{\alpha} \log \frac{1}{1 + \alpha t_2} \right\} - \frac{t_2^2}{6\alpha^2} [3b\alpha + 2c\alpha(3t_1 + 2t_2) + c] \quad (7)$$

Lost sales cost per cycle =

$$l \int_{t_1}^{t_1+t_2} \left[1 - \frac{1}{1 + \alpha(t_1 + t_2 - t)} \right] (a + bt + ct^2) dt =$$

$$l \left\{ at_2 + b \left(t_1 t_2 + \frac{t_2^2}{2} + \frac{t_2}{\alpha} \right) + c \left(t_1^2 t_2 + t_1 t_2^2 + \frac{t_2^3}{3} + \frac{2t_1 t_2}{\alpha} + \frac{3t_2^2}{2\alpha} + \frac{t_2}{\alpha^2} \right) + \frac{1}{\alpha} \left[a + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^2} \right] \log \frac{1}{1 + \alpha t_2} \right\} \quad (8)$$

When $t = 0$ the level of inventory is maximum and it is denoted by IM. Then from equation (3)

$$IM = -\frac{a}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3} + e^{\theta t_1} \left[\frac{a}{\theta} + \frac{b}{\theta} \left(t_1 - \frac{1}{\theta} \right) + \frac{c}{\theta} \left[\left(t_1 - \frac{1}{\theta} \right)^2 + \frac{1}{\theta^2} \right] \right] \quad (9)$$

The maximum backordered inventory is obtained at $t = t_1 + t_2$ and is denoted by IB. Thus from equation (4)

$$IB = -I_2(t_1 + t_2) = - \left\{ \frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right\} \log \frac{1}{1 + \alpha t_2} + \frac{bt_2}{\alpha} + \frac{3ct_2^2}{2\alpha} + \frac{2ct_1 t_2}{\alpha} + \frac{ct_2}{\alpha^2} \quad (10)$$

Thus the order size during total time interval (0,T) is given by

$$Q = IM + IB \quad (11)$$

Purchase cost per cycle = $rQ =$

$$r \left\{ -\frac{a}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3} + e^{\theta t_1} \left[\frac{a}{\theta} + \frac{b}{\theta} \left(t_1 - \frac{1}{\theta} \right) + \frac{c}{\theta} \left[\left(t_1 - \frac{1}{\theta} \right)^2 + \frac{1}{\theta^2} \right] \right] + \frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right\} \log(1 + \alpha t_2) + \frac{bt_2}{\alpha} - \frac{3ct_2^2}{2\alpha} - \frac{2ct_1 t_2}{\alpha} - \frac{ct_2}{\alpha^2} \quad (12)$$

Therefore the Total cost (TC) per time unit is given by,

$$TC = \frac{1}{(t_1 + t_2)} [\text{Ordering cost} + \text{Inventory holding cost} +$$

Cost of deterioration + Backordered cost + Lost sales cost + Purchase cost]

$$= \frac{1}{(t_1 + t_2)} \left[A + \frac{h}{\theta^4} + \frac{b\theta}{2} (2e^{t_1\theta} (t_1\theta^2 - 1) - t_1^2\theta^2 + 1) + \frac{c}{3} (3e^{t_1\theta} (t_1^2\theta^2 - 2t_1\theta + 2) - t_1^3\theta^3 - 5) \right]$$

$$+ \frac{rh}{\theta^3} \left[a\theta^2 (e^{t_1\theta} - t_1\theta - 1) + \frac{b\theta}{2} (2e^{t_1\theta} (t_1\theta^2 - 1) - t_1^2\theta^2 + 1) + \frac{c}{3} (3e^{t_1\theta} (t_1^2\theta^2 - 2t_1\theta + 2) - t_1^3\theta^3 - 5) \right]$$

$$+ s \left\{ \frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right\} \left\{ t_2 + \frac{1}{\alpha} \log \frac{1}{1 + \alpha t_2} \right\} - \frac{t_2^2}{6\alpha^2} [3b\alpha + 2c\alpha(3t_1 + 2t_2) + c]$$

$$+ l \left\{ at_2 + b \left(t_1 t_2 + \frac{t_2^2}{2} + \frac{t_2}{\alpha} \right) + c \left(t_1^2 t_2 + t_1 t_2^2 + \frac{t_2^3}{3} + \frac{2t_1 t_2}{\alpha} + \frac{3t_2^2}{2\alpha} + \frac{t_2}{\alpha^2} \right) + \frac{1}{\alpha} \left[a + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^2} \right] \log \frac{1}{1 + \alpha t_2} \right\}$$

$$+ r \left\{ -\frac{a}{\theta} + \frac{b}{\theta^2} - \frac{2c}{\theta^3} + e^{\theta t_1} \left[\frac{a}{\theta} + \frac{b}{\theta} \left(t_1 - \frac{1}{\theta} \right) + \frac{c}{\theta} \left[\left(t_1 - \frac{1}{\theta} \right)^2 + \frac{1}{\theta^2} \right] \right] \right\} \quad (13)$$

$$+ \left[\frac{a}{\alpha} + \frac{b[1 + \alpha(t_1 + t_2)]}{\alpha^2} + \frac{c[1 + \alpha(t_1 + t_2)]^2}{\alpha^3} \right] \log(1 + \alpha t_2) - \frac{bt_2}{\alpha} - \frac{3ct_2^2}{2\alpha} - \frac{2ct_1 t_2}{\alpha} - \frac{ct_2}{\alpha^2}$$

The necessary condition for the total cost per time unit to be a minimum is

$$\frac{\partial TC}{\partial t_1} = 0 \text{ and } \frac{\partial TC}{\partial t_2} = 0$$

Provided

$$\left(\frac{\partial^2 TC}{\partial t_1^2} \right) \left(\frac{\partial^2 TC}{\partial t_2^2} \right) - \left(\frac{\partial^2 TC}{\partial t_1 \partial t_2} \right) > 0$$

5. Numerical Example

Consider an inventory system with the following parameter in proper units: $A = 250,$

$h = 0.05, s = 1.2, l = 1.5, \alpha = 8, \theta = 0.005, a = 25, b = 20, c = 15.$ The computed optimal value of TC is 92.53.

6. Sensitivity Analysis

The variation in the parameters is as follows:

Table 1. Sensitivity Analysis of the Total Cost for the Parameters a, b, c, α and θ

Parameter	Value	t_1	t_2	TC	% Change in TC
a	20	5.32	0.05	92.42	0.12
	23	5.31	0.05	92.48	0.05
	25	5.3	0.05	92.53	0
	26	5.3	0.05	92.75	0.23
b	16	5.72	0.05	85.72	7.36
	18	5.52	0.05	89.16	3.64
	20	5.3	0.05	92.53	0
	21	5.2	0.05	93.12	0.63
c	15	5.88	0.05	82.63	10.70
	16	5.61	0.05	88.21	4.67
	17	5.3	0.05	92.53	0
	18	5.28	0.05	94.67	2.31
α	7.5	5.3	0.05	92.32	0.23
	8	5.3	0.05	92.53	0
	8.5	5.3	0.04	92.59	0.06
	9	5.31	0.04	92.66	0.14
θ	0.004	5.32	0.05	91.12	1.52
	0.0045	5.31	0.05	91.72	0.88
	0.005	5.3	0.05	92.53	0
	0.0055	5.3	0.05	92.95	0.45

From Table 1, it is observed that the total cost increases if the parameters a, b, c, α and θ are increased. Also the parameters a, b and c are more sensitive than α and θ .

7. Conclusion

In this paper, we have developed an order quantity model for deteriorating items with time – quadratic demand and partial backlogging. In particular, the backlogging rate is considered to be a decreasing function of the waiting time for the next replenishment. This assumption is more realistic in the market.

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