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Elzaki Substitution Method for Solving Nonlinear Partial Differential Equations with Mixed Partial Derivatives Using Adomian Polynomial

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Abstract In this paper we apply a new method, named Elzaki Substitution Method to solve nonlinear homogeneous and nonhomogeneous partial differential equations with mixed partial derivatives, which is based on Elzaki Transform. The proposed method introduces also Adomian polynomials and the nonlinear terms can be handled by the use of this polynomials. The proposed method worked perfectly to find the exact solutions of partial equations with mixed partial derivatives without any need of linearization or discretization in comparison with other methods such as Method of Separation of Variables (MSV) and Variation Iteration Method (VIM). Some illustrative examples are given to demonstrate the applicability and efficiency of proposed method.

Keywords: *Partial Differential Equations, Exact Solution, Mixed Partial Derivatives, Elzaki Transform, Elzaki Substitution Method, Adomian polynomial*

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1. Introduction

Partial differential equations have big importance in mathematics and other fields of science. Nonlinear partial differential equations (NLPDEs) involving mixed partial derivatives are mathematical models that are used to describe complex Phenomena arising in the world around us. The nonlinear partial differential equations appear in many applications of science and engineering such as fluid dynamics, plasma physics, hydrodynamics, solid state physics and other disciplines. Therefore, it is very important to know the methods to solve such nonlinear partial differential equations with mixed partial derivatives. In the recent years, many authors mainly had paid attention to study solutions of NLPDEs with mixed partial derivatives by using various methods such as Method of Separation of Variables, Variation Iteration Method, Laplace Substitution Method [15]. One of the most known method to solve such nonlinear partial differential equations with mixed partial derivatives is the Elzaki Substitution Method, which is based on the Elzaki transform [1,2,3,4] and Elzaki Transform is also modified transform of Laplace transform. Elzaki transform can be used to solve ordinary differential equations [5], partial differential equations [2], partial integro-differential equations [11,12], system of partial differential equations [6], Adomian decomposition method [14], Schrodinger equations [13] and wave equations [7]. The main advantage of this proposed method is that it eliminates the need of linearization, perturbation or any other transformation .The main goal of this proposed method is to find exact or approximate solution of nonlinear partial differential equations involving mixed partial derivatives with the help of Adomian polynomial. This powerful method will be proposed in section 3; in section 4 we will apply it to three coupled partial differential equations involving mixed partial derivatives out of them examples 1 and 2 are nonlinear homogeneous and nonhomogeneous partial differential equation involving mixed partial derivatives in which general linear term operator equal to zero i.e $(R(x,y)=0)$ and example 3 is of nonlinear nonhomogeneous partial differential equations involving mixed partial derivatives with $(R(x,y)\neq 0)$. In last section we give some conclusion.

2. Elzaki Transformation

A new transform called the Elzaki transform defined for function of exponential order we consider functions in the set A defined by:

$$A=\left\{ \begin{array}{l} f(t):\exists M,k_1,k_2>0,\left|f(t)\right|<Me^{\frac{|t|}{k_j}}, \\ t\in(-1)^j X[0,\infty) \end{array} \right\} \tag{1}$$

In a set A, M is constant must be finite, k_1, k_2 may be finite or infinite. The Elzaki transform denoted by the operator E $(\cdot)$ defined by the integral equation

$$E\left[f(t)\right] = T(v) = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt, \quad (2)$$
$$t > 0, k_1 \le v \le k_2$$

In this transform the variable v is used to factor the variable t in the argument of function f.

2.1. Fundamental Properties of Elzaki Transformation

Table 1

Sr .No	$f(t)$	$E[f(t)] = T(v)$
1	1	v^2
2	t	v^3
3	$t^n,\ n = 0,1,2\cdots\cdots$	$n!v^{n+2}$
4	e^{at}	$\frac{v^2}{1-av}$
5	te^{at}	$\frac{v^3}{(1-av)^2}$
6	$\frac{t^{n-1}e^{at}}{(n-1)!}, n = 1,2\cdots\cdots$	$\frac{v^{n+1}}{(1-av)^n}$
7	$\sin at$	$\frac{av^3}{1+a^2v^2}$
8	$\cos at$	$\frac{v^2}{1+a^2v^2}$
9	$\sinh at$	$\frac{av^3}{1-a^2v^2}$
10	$\cosh at$	$\frac{av^2}{1-a^2v^2}$
11	$e^{at}\sin bt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$
12	$e^{at}\cos bt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
13	$t\sin at$	$\frac{2av^4}{1+a^2v^2}$
14	$t\cos at$	$\frac{v^3}{1+a^2v^2}$

2.2. Partial Derivatives of Elzaki Transformation

Let $u(x,t)$ be a function of two independent variables u and t, then

i) $E\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{1}{v}T(x,v) - vu(x,0)$

ii) $E\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = \frac{1}{v^2}T(x,v) - u(x,0) - v\frac{\partial u(x,0)}{\partial t}$

iii) $E\left[\frac{\partial u(x,t)}{\partial x}\right] = \frac{d\left[T(x,v)\right]}{dx}$

iv) $E\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \frac{d^2\left[T(x,v)\right]}{dx^2}$

v) $E\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = \frac{E\left[u(x,t)\right]}{v^n} - \frac{u(x,0)}{v^{n-2}} - \frac{1}{v^{n-3}}\frac{\partial u(x,0)}{\partial t} - \cdots\cdots - \frac{\partial^{n-2}u(x,0)}{\partial t^{n-2}} - v\frac{\partial^{n-1}u(x,0)}{\partial t^{n-1}}$

Proof: To obtain the Elzaki transform of partial derivative we use integration by parts as follows:

i) $E\left[\frac{\partial u}{\partial t}(x,t)\right] = \int_0^\infty v\frac{\partial u}{\partial t}e^{\frac{-t}{v}}dt = \lim_{p\to\infty}\int_0^p ve^{\frac{-t}{v}}\frac{\partial u}{\partial t}dt$

$$= \lim_{p\to\infty}\left\{\left[ve^{\frac{-t}{v}}u(x,t)\right]_0^p - \int_0^p e^{\frac{-t}{v}}u(x,t)dt\right\}$$

$$= \frac{T(x,v)}{v} - vu(x,0)$$

$$\therefore\ E\left[\frac{\partial u}{\partial t}(x,t)\right] = \frac{T(x,v)}{v} - vu(x,0)$$

ii) $E\left[\frac{\partial^2 u}{\partial t^2}(x,t)\right]$

Let $\frac{\partial u}{\partial t} = g$, we have

$$E\left[\frac{\partial^2 u}{\partial t^2}(x,t)\right] = E\left[\frac{\partial g(x,t)}{\partial t}\right] = E\frac{\left[g(x,t)\right]}{v} - vg(x,0)$$

$$\therefore E\left[\frac{\partial^2 u}{\partial t^2}(x,t)\right] = \frac{1}{v^2}T(x,v) - u(x,0) - v\frac{\partial u}{\partial t}(x,0).$$

We assume that f is piecewise continuous and is of exponential order. Now

iii) $E\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty ve^{\frac{-t}{v}}\frac{\partial u(x,t)}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty ve^{\frac{-t}{v}}u(x,t)dt$

Using the Leibnitz rule we find,

$$E\left[\frac{\partial u}{\partial x}\right] = \frac{d}{dx}\left[T(x,v)\right].$$

The same method we find

iv) $E\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[T(x,v)\right]$

v) The proof is by the principle of mathematical induction on n. By direct differentiation (considering n=1)

$$E[\frac{\partial u(x,t)}{\partial t}] = \frac{1}{v}E[u(x,t)] - vu(x,0)$$

For n=2

$$E[\frac{\partial^2 u(x,t)}{\partial t^2}] = \frac{1}{v^2}E[u(x,t)] - u(x,0) - v\frac{\partial u(x,0)}{\partial t}.$$

We assume that the theorem is true for $n = m$

$$E\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] = \frac{E[u(x,t)]}{v^m} - \frac{u(x,0)}{v^{m-2}} - \frac{1}{v^{m-3}}\frac{\partial u(x,0)}{\partial t} - \dots\dots\dots - \frac{\partial^{m-2}u(x,0)}{\partial t^{m-2}} - v\frac{\partial^{m-1}u(x,0)}{\partial t^{m-1}}$$

Let, $f(x,t) = \frac{\partial^m u(x,0)}{\partial t^m}$

$$\therefore E[f(x,t)] = E[\frac{\partial^m u(x,0)}{\partial t^m}]$$

Differentiating, we get

$$E[\frac{\partial^{m+1}u(x,t)}{\partial t^{m+1}}] = E[\frac{\partial f(x,t)}{\partial t}] = \frac{1}{v}E[f(x,t)] - vf(x,0)$$

$$= \frac{1}{v}[\frac{E[u(x,t)]}{v^m} - \frac{u(x,0)}{v^{m-2}} - \frac{1}{v^{m-3}}\frac{\partial u(x,0)}{\partial t} - \dots\dots\dots - \frac{\partial^{m-2}u(x,0)}{\partial t^{m-2}} - v\frac{\partial^{m-1}u(x,0)}{\partial t^{m-1}}] - v\frac{\partial^m u(x,0)}{\partial t^m}$$

$$= \frac{E[u(x,t)]}{v^{m+1}} - \frac{u(x,0)}{v^{m-1}} - \frac{1}{v^{m-2}}\frac{\partial u(x,0)}{\partial t} - \dots\dots\dots - \frac{1}{v}\frac{\partial^{m-1}u(x,0)}{\partial t^{m-1}} - \frac{\partial^m u(x,0)}{\partial t^m} - v\frac{\partial^m u(x,0)}{\partial t^m}$$

$$\therefore E[\frac{\partial^{m+1}u(x,t)}{\partial t^{m+1}}] = \frac{E[u(x,t)]}{v^{m+1}} - \frac{u(x,0)}{v^{m-1}} - \frac{1}{v^{m-2}}\frac{\partial u(x,0)}{\partial t} - \dots\dots\dots - \frac{1}{v}\frac{\partial^{m-1}u(x,0)}{\partial t^{m-1}} - (v+1)\frac{\partial^m u(x,0)}{\partial t^m}$$

Therefore the theorem is true for $m+1$ and hence by principle of mathematical induction, the theorem true for any positive integer n.

3. Methodology

In this section, we will give the description of Elzaki Substitution Method for nonlinear partial differential equations with mixed partial derivatives. From that description we will remove the nonlinear part of use of Adomian polynomials.

3.1. Elzaki Substitution Method for Nonlinear Partial Differential Equations with Mixed Partial Derivatives

The aim of this section is to discuss the Elzaki transform substitution method for nonlinear partial differential equations involving mixed partial derivatives. We consider the general form of nonlinear nonhomogeneous partial differential equation with initial conditions is given below

$$Lu(x,y) + Ru(x,y) + Nu(x,y) = h(x,y) \quad (3)$$

$$u(x,0) = f(x), u_y(0,y) = g(y) \quad (4)$$

where $L = \frac{\partial^2}{\partial x \partial y}$, $Ru(x,y)$ is the remaining linear term in which contains only first order partial derivatives of $u(x,y)$ with respect to either x or y, $Nu(x,y)$ represents a general nonlinear differential operator and $h(x,y)$ is the source term. We can write the equation (3) in the following form

$$\frac{\partial^2 u}{\partial x \partial y} + Ru(x,y) + Nu(x,y) = h(x,y)$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) + Ru(x,y) + Nu(x,y) = h(x,y).$$

Putting $\frac{\partial u}{\partial y} = U$ in above equation, we get

$$\frac{\partial U}{\partial x} + Ru(x,y) + Nu(x,y) = h(x,y).$$

$$\frac{\partial U}{\partial x} + Ru(x,y) + Nu(x,y) = h(x,y).$$

Taking Elzaki transform in both sides with respect to x, we obtain

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) + E_x[Ru(x,y) + Nu(x,y)] = E_x[h(x,y)]$$

$$E_x[U(x,y)] = v^2 u_y(0,y) + vE_x[h(x,y) - Ru(x,y) - Nu(x,y)]$$

Using initial condition,

$$E_x[U(x,y)] = v^2 g(y) + vE_x[h(x,y) - Ru(x,y) - Nu(x,y)].$$

Taking inverse Elzaki transform in both sides with respect to x, we get

$$U(x,y) = g(y) + E_x^{-1}[vE_x[h(x,y) - Ru(x,y) - Nu(x,y)]].$$

Re-substitute the value of $U(x,y) = \frac{\partial u}{\partial y}$ in above equation, we get,

$$\frac{\partial u(x,y)}{\partial y} = g(y) + E_x^{-1}[vE_x[h(x,y) - Ru(x,y) - Nu(x,y)]] \quad (5)$$

This is a first order non-linear, non-homogeneous partial differential equation in the variables x and y.

Applying Elzaki transform of equation (5) with respect to y, then the equation becomes

$$\frac{1}{v}E_y[u(x,y)] - vu(x,o) = E_y[g(y) + E_x^{-1}[vE_x[h(x,y) - Ru(x,y) - Nu(x,y)]]]$$

Using initial condition, we have

$$E_y[u(x,y)] = v^2 f(x)$$
$$+vE_y\left[g(y)+E_x^{-1}\left[vE_x\left[h(x,y)-Ru(x,y)-Nu(x,y)\right]\right]\right]$$

Taking the inverse Elzaki transform with respect to y, the equation becomes

$$u(x,y) = f(x)+E_y^{-1}[vE_y[g(y)+E_x^{-1}[vE_x[h(x,y) -Ru(x,y)-Nu(x,y)]]]] \quad (6)$$

For solving nonlinear, nonhomogeneous partial differential equations involving mixed derivatives by Elzaki substitution method, let we consider solution of (3) is in series form.

Suppose that

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (7)$$

is a required solution of (3) is in series form. We know that nonlinear term $Nu(x,y)$ appear in equation (3), let we decompose it by using Adomian polynomial which is defined by the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\gamma^n}[N[\sum_{i=0}^{\infty}\gamma^i u_i]]\right]_{\gamma=0} \quad (8)$$

$$Nu(x,y) = \sum_{n=0}^{\infty} A_n \quad (9)$$

Where A_n is an Adomian polynomial of components $u_0(x,y), u_1(x,y), \dots\dots\dots, u_n(x,y),\ n \ge 0$ of series (7).

Substitute the value of equations (7) and (9) in equation (6), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = f(x)+E_y^{-1}[vE_y[g(y)+E_x^{-1} [vE_x[h(x,y)-R(\sum_{n=0}^{\infty} u_n(x,y))-\sum_{n=0}^{\infty} A_n]]]].$$

Comparing both sides of the above equation, we find the following relation

$$u_0(x,y) = f(x)+E_y^{-1}[vE_y[g(y)+E_x^{-1}[vE_x[h(x,y)]]]]$$

$$u_1(x,y) = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[Ru_0(x,y)+A_0]]]]$$

$$u_2(x,y) = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[Ru_1(x,y)+A_1]]]].$$

In general, we have the following required recursive relation

$$\left.\begin{aligned} &u_0(x,y) = f(x)+E_y^{-1}[vE_y[g(y)+E_x^{-1}[vE_x[h(x,y)]]]] \\ &u_{n+1}(x,y) = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[Ru_n(x,y)+A_n]]]], n \ge 0 \end{aligned}\right\}. \quad (10)$$

From this recursive relation we can calculate $u_i, i = 1, 2, 3, 4, \dots\ \dots\ \dots$ of $u(x,y)$. Substitute all values of u_i in equation (7), we get the required solution of equation (3).

4. Applications

In this section we apply the Elzaki Substitution Method for solving nonlinear homogeneous and nonhomogeneous partial differential equations with mixed partial derivatives.

Example 4.1.

Solve the following non-linear homogeneous partial differential equation with linear part $R(x,y)=0$

$$\frac{\partial^2 u}{\partial x \partial y} - (\frac{\partial u}{\partial y})^2 = 0$$

With initial conditions

$$u(x,0) = 0, u_y(0,y) = 1.$$

Solution: In the above initial value problem

$$Lu(x,y) = \frac{\partial^2 u}{\partial x \partial y}, Nu(x,y) = (\frac{\partial u}{\partial y})^2.$$

We can write the given equation in the following form

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) - (\frac{\partial u}{\partial y})^2 = 0.$$

Substituting $\frac{\partial u}{\partial y} = U$ in above equation, then the equation becomes

$$\frac{\partial U}{\partial x} - U^2 = 0.$$

Which is the first order partial differential equation.

Applying Elzaki transform on both sides of the above equation with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) - E_x[U^2] = 0.$$

Using initial condition, we have

$$E_x[U(x,y)] = v^2 + vE_x[U^2].$$

Taking inverse Elzaki transform with respect to x, we obtain the equation

$$U(x,y) = 1 + E_x^{-1}[vE_x[U^2]]$$

$$\frac{\partial u}{\partial y}(x,y) = 1 + E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]].$$

Taking Elzaki transform with respect to y, we obtain

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = v^2 + E_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]].$$

Using initial condition, we get

$$E_y[u(x,y)] = v^3 + vE_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]].$$

Taking inverse Elzaki transform of the above equation with respect to y, then we find

$$u(x,y) = y + E_y^{-1}[vE_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]]] \quad (11)$$

We know that in Elzaki substitution method, we represent solution in infinite series form. Let us suppose that

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (12)$$

be the required solution of given equation .The nonlinear term appear in given equation, we can decompose it by using Adomian polynomial defined by the equation (8).

$$(\frac{\partial u}{\partial y})^2 = \sum_{n=0}^{\infty} A_n \quad (13)$$

Where A_n is an Adomian polynomial of components $u_0, u_1, \dots \dots \dots, u_n, n \geq 0$. Let us find that the few Adomian polynomials,

$$A_0 = u^2_{0y}, A_1 = 2u_{0y}u_{1y}, A_2 = 2u_{0y}u_{2y} + u^2_{0y}, \dots\dots\dots$$

Using the value of equations (12) and (13), then the equation (11) reduces to

$$\sum_{n=0}^{\infty} u_n(x,y) = y + E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[\sum_{n=0}^{\infty} A_n]]]].$$

Comparing on both sides of above equation, we have the recursive relation

$$\left.\begin{aligned} u_0(x,y) &= y \\ u_{n+1} &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_n]]]] \end{aligned}\right\}.$$

From the above recursive relation, we get the following few components of $u(x,y)$

$$\begin{aligned} u_0(x,y) &= y \\ u_1(x,y) &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_0]]]] \\ &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[1]]]] \\ &= E_y{}^{-1}[vE_y[x]] \\ &= xy \\ u_2(x,y) &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_1]]]] \\ &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[2x]]]] \\ &= x^2 y \end{aligned}$$

$$u_3(x,y) = x^3 y$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$

Thus the solution of given equation is,

$$\begin{aligned} u(x,y) &= u_0(x,y) + u_1(x,y) + u_2(x,y) + u_3(x,y) + \dots\dots\dots \\ &= y + xy + x^2 y + x^3 y + \dots\dots\dots \\ &= y\sum_{n=0}^{\infty} x^n \end{aligned}$$

$\sum_{n=0}^{\infty} x^n$ is a geometric series converges to $\frac{1}{1-x}$, for $|x| < 1$. Consequently,

$$u(x,y) = \frac{y}{1-x}, |x| < 1.$$

which is the required solution with regarding initial conditions and verifying through the substitution. Also same as the solutions obtained by (LSM), (MSV) and (VIM).

Example 4.2.

Solve the following nonlinear nonhomogeneous partial differential equation with linear part $R(x,y) = 0$

$$\frac{\partial^2 u}{\partial x \partial y} + (\frac{\partial u}{\partial y})^2 = 1.$$

with initial conditions

$$u(x,0) = 0, u_y(0,y) = 0.$$

Solution: The given equation can be written as

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) + (\frac{\partial u}{\partial y})^2 = 1.$$

Putting $\frac{\partial u}{\partial y} = U$ in above equation, we get

$$\frac{\partial U}{\partial x} + U^2 = 1.$$

This is the first order nonlinear partial differential equation.

Applying Elzaki transform on both sides with respect to x and then using initial condition, finally the equation becomes

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) + E_x[U^2] = v^2$$

$$E_x[U(x,y)] = v^3 - vE_x[U^2].$$

Taking inverse Elzaki transform with respect to x, we get

$$U(x,y) = x - E_x{}^{-1}[vE_x[U^2]]$$

$$\frac{\partial u}{\partial y}(x,y) = x - E_x{}^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]].$$

Taking Elzaki transform of above equation with respect to y and then using initial condition, finally we obtain

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = xv^2 - E_y[E_x{}^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]]$$

$$E_y[u(x,y)] = xv^3 - vE_y[E_x{}^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]].$$

Applying inverse Elzaki transform with respect to y, the equation becomes

$$u(x,y) = xy - E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]]] \quad (14)$$

We know that in Elzaki substitution method, we represent solution in infinite series form. Let us suppose that

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (15)$$

be the required solution of given equation. The nonlinear term appear in given equation, we can decompose it by using Adomian polynomial defined by the equation (8) and the nonlinear term

$$(\frac{\partial u}{\partial y})^2 = \sum_{n=0}^{\infty} A_n \quad (16)$$

Where A_n is an Adomian polynomial of components $u_0, u_1, u_2, \dots \dots \dots u_n$. Let we find the few Adomian polynomials,

$$A_0 = u^2_{0y},\ A_1 = 2u_{0y}u_{1y},\ A_2 = 2u_{0y}u_{2y} + u^2_{0y}, \dots\dots\dots$$

Putting the value of equations (15) and (16) in equation (14), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = xy - E_y^{-1}[vE_y[E_x^{-1}[vE_x[\sum_{n=0}^{\infty} A_n]]]].$$

Comparing on both sides of above equation, we obtain the recursive relation

$$\left.\begin{aligned} & u_0(x,y) = xy \\ & u_{n+1} = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[A_n]]]] \end{aligned}\right\}$$

From the above recursive relation, we have find the following few components of $u(x,y)$

$$u_0(x,y) = xy$$

$$u_1(x,y) = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[A_0]]]] = -\frac{x^3 y}{3}$$

$$u_2(x,y) = -E_y^{-1}[vE_y[E_x^{-1}[vE_x[A_1]]]] = \frac{2x^5 y}{15}.$$

$$u_3(x,y) = -\frac{17x^7 y}{315}$$

Similarly, we can obtain the values of $u_4(x,y), u_5(x,y), \cdots \cdots \cdots, u_n(x,y)$

Substitute all the values of $u_n(x,y), n \geq 0$ in equation (15), we get

$$\begin{aligned} u(x,y) &= xy - \frac{x^3 y}{3} + \frac{2x^5 y}{15} - \frac{17x^7 y}{315} + \dots \dots \dots \\ &= y(x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots \dots \dots) \end{aligned}$$

Consequently

$$u(x,y) = y \tanh x$$

which is the required solution with regarding initial conditions and verifying through the substitution. Also same as the solutions obtained by (LSM), (MSV) and (VIM).

Example 4.3.

Solve the following nonlinear nonhomogeneous partial differential equation with linear part $R(x,y) \neq 0$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} - (\frac{\partial u}{\partial y})^2 = 1.$$

with initial conditions

$$u(x,0) = 0, u_y(0,y) = 0, u(0,y) = 0.$$

Solution: In the above example linear term $Ru(x,y) = \frac{\partial u}{\partial x}$ and non-linear term $Nu(x,y) = (\frac{\partial u}{\partial y})^2$.

The given equation can be written as

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) + \frac{\partial u}{\partial x} - (\frac{\partial u}{\partial y})^2 = 1.$$

Substituting $\frac{\partial u}{\partial y} = U$ in above equation, we have

$$\frac{\partial U}{\partial x} + \frac{\partial u}{\partial x} - U^2 = 1.$$

This is the first order nonlinear partial differential equation.

Applying Elzaki transform with respect to x and then taking initial conditions, finally the equation becomes

$$\frac{1}{v}E_x[U(x,y)] - vU(0,y) + \frac{1}{v}E_x[u(x,y)] - vu(0,y)$$
$$-E_x[U^2] = v^2$$

$$E_x[U(x,y)] + E_x[u(x,y)] = v^3 + vE_x[U^2].$$

Taking inverse Elzaki transform with respect to x, we obtain

$$U(x,y) = x + E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]] - u(x,y)$$

$$\frac{\partial u}{\partial y}(x,y) = x + E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]] - u(x,y).$$

Applying Elzaki transform on both sides with respect to y and then using initial conditions, we get

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = xv^2 + E_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]]$$
$$-E_y[u(x,y)]$$

$$E_y[u(x,y)] = xv^3 + vE_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]] - vE_y[u(x,y)]$$

Taking inverse Elzaki transform with respect to y, we obtain the equation

$$\begin{aligned} u(x,y) &= xy + E_y^{-1}[vE_y[E_x^{-1}[vE_x[(\frac{\partial u}{\partial y})^2]]]] \\ &-E_y^{-1}[vE_y[u(x,y)]] \end{aligned} \quad (17)$$

We know that in Elzaki substitution method, we represent solution in infinite series form. Let us suppose that

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (18)$$

be the required solution of given equation. The nonlinear term appear in equation, we can decompose it by using Adomian polynomial defined by the equation (8)

$$(\frac{\partial u}{\partial y})^2 = \sum_{n=0}^{\infty} A_n \quad (19)$$

Where A_n is an Adomain polynomial of components $u_0, u_1, u_2, \dots \dots \dots u_n$. Let us find that the few Adomian polynomials

$$A_0 = u^2{}_{0y},\ A_1 = 2u_{0y}u_{1y},\ A_2 = 2u_{0y}u_{2y} + u^2{}_{0y}, \dots\dots\dots$$

Putting the value of equations (18) and (19) in equation (17), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y) = xy + E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[\sum_{n=0}^{\infty} A_n]]]]$$
$$-E_y{}^{-1}[vE_y[\sum_{n=0}^{\infty} u_n(x,y)]].$$

Comparing on both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned} &u_0(x,y) = xy \\ &u_{n+1} = E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_n]]]] - E_y{}^{-1}[vE_y[u_n(x,y)]] \end{aligned}\right\}$$

From the above recursive relation, we get the following few components of $u(x,y)$

$$u_0(x,y) = xy$$
$$\begin{aligned} u_1(x,y) &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_1]]]] - E_y{}^{-1}[vE_y[u_0(x,y)]] \\ &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[x^2]]]] - E_y{}^{-1}[vE_y[xy]] \\ &= \frac{x^3 y}{3} - \frac{xy^2}{2} \end{aligned}$$
$$\begin{aligned} u_2(x,y) &= E_y{}^{-1}[vE_y[E_x{}^{-1}[vE_x[A_1]]]] - E_y{}^{-1}[vE_y[u_1(x,y)]] \\ &= \frac{2x^5 y}{15} - \frac{x^3 y^2}{2} + \frac{xy^3}{6} \end{aligned}$$

Similarly, we can find the values of $u_4(x,y), u_5(x,y), \dots\dots\dots, u_n(x,y)$

Substitute all the values of $u_n(x,y), n \geq 0$ in equation (17), we obtain the following solution

$$u(x,y) = xy + \frac{x^3 y}{3} - \frac{xy^2}{2} + \frac{2x^5 y}{15} - \frac{x^3 y^2}{2} + \frac{xy^3}{6} - \dots \dots$$

which is the required solution with regarding initial conditions and verifying through the substitution. Also same as the solutions obtained by (LSM), (MSV) and (VIM).

5. Conclusion

In this paper, we successfully apply the proposed Elzaki Substitution Method to solve nonlinear homogeneous and nonhomogeneous partial differential equations in which involves mixed partial derivatives with general linear term $Ru(x,y) = 0$ or $Ru(x,y) \neq 0$ without using linearization, perturbation, or restrictive assumption. The utilization of this method is simple in use, economic, time saving and exquisite. Another advantage of this method is that it gives the solution in series form. This method giving better solution than the other existing methods. In future, we plan to generalize this method to solve higher order nonlinear homogeneous and nonhomogeneous partial differential equations with mixed partial derivatives in nonlinear terms.

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