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Optimization of Wealth Investment Strategies for a DC Pension Fund with Stochastic Salary and Extra Contributions

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Abstract We studied optimal investment strategies for a plan contributor in a defined pension scheme, with stochastic salary and extra contributions, under the affine interest rate model. We considered two cases; where the extra contribution rates are stochastic and constant. We considered investment in three different assets namely risk free asset (cash), zero coupon bonds and the risky asset (stock). Using Legendre transformation method and dual theory, we obtained the optimal investment strategies the three investments using exponential utility function for the two cases. The result shows that the strategies for the respective investments used when there is no extra contribution can be used when the extra contribution rate is constant as in [1] but cannot be used when it is stochastic. Clearly this gives the member and the fund manager good insight on how to invest to maximize profit with minimal risk once this condition arises.

Keywords: *CRRA, DC Pension fund, Optimal Investment Strategies, Legendre Transform, constant rate, stochastic rate, stochastic salary, extra contribution*

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1. Introduction

Generally there are basically two different approaches in which pension scheme is designed namely defined benefit (DB) and defined contribution (DC). In defined benefit, the benefit of the member is fixed by his employer and the contribution are adjusted to maintain the fund balance but in defined contribution the contributions are fixed and the benefit depends on the on the outcome of the investment on the various assets the member embarks upon. It should be noted that in DB pension fund, the risks in the business are on the employer while in DC the risks are on the member. Recently the practice of DC pension scheme has grown rapidly in most countries since it has provided a convenience plat form for members to involve in it. One of the most attractive features of this scheme is that the member has an idea of what to expect at the time of retirement. Since the member knows that he benefit depend on the various investment which may include investment in one or more of riskless asset (cash), coupon bond, stock, there is need to understand the risk involve and the possible way to minimize the risk form the base of this research. The essence of this research is to determine the optimal investment strategies associated with the various investments embarked upon by member. These strategies help the member to minimize risk and maximize profit. Considering the fact that a member is given the liberty to decide which investment he will like to invest in and knowing that in DC pension scheme the risk is borne by the member, hence there is need to solve the problem of optimal investment strategies associated with the investment.

Over the years a lot of work have been done in connection with the study of optimal investment strategies some of which include [1,6,7,8,9]. The optimal portfolio problem in a DC pension fund with stochastic interest rate was studied in [6,7], they assume the interest rate to be of Vasicek model, [8,9] studied the same problem but assume the interest rate to have an affine structure i.e a model with Vasicek and Cox – Ingeroll – Ross (CIR) model. [6] considered risk incurred from the salary and inflation using vasicek model to determine the optimal allocation strategy. [1] Studied optimal investment strategies with stochastic salary under affine interest rate. The effect of extra contribution on stochastic optimal investment strategies in a DC pension with stochastic salary under affine interest rate model was studied in [2]; they considered a case where the extra contribution rate was constant.

In solving the portfolio problem, the need to understand how investors react to risk is very necessary, this has lead to the study of CARA and CRRA utility functions. Some authors use the power utility function which exhibit CRRA see [7,8,9,10,11] while other use exponential utility function which exhibit CARA see [6,12] also some make use of the two see [1,12].

In this paper, we extend the work of [1,2], by study the optimal investment strategies with extra contribution where member have the liberty to contribute extra proportion of his salary to the pension purse, we considered two cases, where the extra contribution rate is stochastic and when it is constant. By applying Legendre transformation method, we obtain explicit solutions of the optimal investment strategies for the three assets for both cases.

2. Preliminaries

We start with a complete and frictionless financial market that is continuously open over the fixed time interval [0, T], for $T > 0$ representing the retirement time of a given shareholder.

Financial Market: We assume that the market is made up of risk free asset (cash) a zero coupon bond and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{W_r(t), W_s(t): t \geq 0\}$ is a standard two dimensional motion such that they orthogonal to each other. F is the filtration and denotes the information generated by the Brownian motion $\{W_r(t), W_s(t)\}$.

Risk free Asset (Cash): Let $B(t)$ denote the price of the risk free asset at time t and it is modeled as follows

$$\frac{dC(t)}{C(t)} = r(t)dt, C(0) = 0 \tag{1}$$

$r(t)$ is the short interest rate process and is given as

$$\begin{aligned} dr(t) &= (a - br(t))dt - \sigma_r dW_r(t), \\ \sigma_r &= \sqrt{c_1 r(t) + c_2}, t \geq 0, \end{aligned} \tag{2}$$

Where a, b, (0), c_1, and c_2 are positive real numbers. If c_1 (resp., c_2) is equal to zero, we have a special case, as in [3,4]. So under these dynamics, the term structure of the short interest rates is affine, which has been studied in [1,5,8,9].

Risky Asset (stock): Let $S(t)$ denote the price of the risky asset and its dynamics is given based on a continuous time stochastic process at $t \geq 0$ and the price process described as follows

$$\begin{aligned} \frac{dS(t)}{S(t)} &= r(t)dt + \sigma_s\left(dW_s(t) + \lambda_1 dt\right) \\ &\quad + n_1\sigma_r\left(dW_r(t) + \lambda_2\sigma_r dt\right), \\ S(0) &= S_0 \end{aligned} \tag{3}$$

with λ_1, λ_2 (resp., σ_s, n_1) being constants (resp., positive constants), [1,8,9]

Zero Coupon Bond: A zero-coupon bond with maturity T, whose price at time t is denoted by $B(t,T), t \geq 0$, and its dynamics is given by the SDE below)

$$\begin{aligned} \frac{dB(t,T),)}{B(t,T),} &= r(t)dt + \sigma_B(T-t, r(t))\left(dW_r(t) + \lambda_2\sigma_r dt\right), \\ B(T,T) &= 1 \end{aligned} \tag{4}$$

Where $\sigma_B(T-t, r(t)) = f(T-t)\sigma_r$ and

$$\begin{aligned} f(t) &= \frac{2(e^{mt}-1)}{m - (b - c_1\lambda_2) + e^{mt}(m + b - c_1\lambda_2)}, \\ m &= \sqrt{((b - c_1\lambda_2)^2 + 2c_1}. \end{aligned} \tag{5}$$

Stochastic Salary: Based on the works of [1,6,7,9], we denote the salary at time t by $L(t)$ which is described by

$$\begin{aligned} \frac{dL(t)}{L(t)} &= \mu_L(t, r(t))dt + n_2\sigma_r dW_r(t) \\ &\quad + n_3\sigma_s dW_s(t), L(0) = L_0 \end{aligned} \tag{6}$$

Where n_2, n_3 are real constants, which are two volatility scale factors measuring how the risk sources of interest rate and stock affect the salary. That is to say, the salary volatility is supposed to a hedgeable volatility whose risk source belongs to the set of the financial market risk sources. This is in accordance with the assumption [9] but is differs from those of [6,7] who also suggest that the salary was influence by non hedgeable risk source (i.e., non-financial market). Also [1] assume that the instantaneous mean of the salary is such that $\mu_L(t, r(t)) = r(t) + m_L$ where m_L is a real constant.

3. Methodology

3.1. Hamilton-Jacobi-Bellman (HJB) Equation

Assume we represent $\pi = (\pi_B, \pi_S)$ as the strategy and we define the utility attained by the contributor from a given state x at time t as

$$H_\pi(t, r, x) = E_\pi\left[U(X(T)) \mid r(t) = r, X(t) = x\right], \tag{7}$$

Where t is the time, S is the price of the risky asset and x is the wealth. Our interest here is to find the optimal value function

$$H(t, r, x) = \sup_\pi H_\pi(t, r, x) \tag{8}$$

and the optimal strategy $\pi^* = (\pi_B{}^*, \pi_S{}^*)$ such that

$$H_{\pi^*}(t, r, x) = H(t, r, x). \tag{9}$$

3.2. Legendre Transformation

The Legendre transform and dual theory help to transform non linear partial differential equation to a linear partial differential equation.

Theorem 3.1: Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform

$$L(z) = \max_x\{f(x) - zx\}, \tag{10}$$

where $L(z)$ is the Legendre dual of $f(x)$. [13]

Since $f(x)$ is convex, from theorem 3.1 we defined the Legendre transform

$$\hat{G}(t,r,z)=\sup\{G(t,r,x)-zx \mid 0<x<\infty\}, 0<t<T. \quad (11)$$

Where $\hat{G}$ is the dual of G and $z > 0$ is the dual variable of x. The value of x where this optimum is attained is denoted by $h(t, r, z)$, so that

$$h(t,r,z)=\inf\left\{x \mid G(t,r,x)\ge zx+\hat{G}(t,r,z)\right\}, 0<t<T. \quad (12)$$

The function h and $\hat{G}$ are closely related and can be refers to as the dual of G. These functions are related as follows

$$\hat{G}(t,r,z) = G(t,r,h) - zh. \quad (13)$$

Where

$$h(t,r,z) = x, G_x = z, h = -\hat{G}_z. \quad (14)$$

At terminal time, we denote

$$\hat{V}(z) = \sup\{ V(x) - zx \mid 0 < x < \infty \},$$

and

$$H(z) = \sup\{x \mid V(x) \ge zx + \hat{V}(z).$$

As a result

$$H(z) = (V')^{-1}(z), \quad (15)$$

where H is the inverse of the marginal utility V and note that $G(T, r, x) = V(x)$

At terminal time T, we can define

$$h(T,r,z)=\inf_{x>0}\left\{xV(x)\ge zx+\hat{G}(t,r,z)\right\}$$

$$\text{and } \hat{G}(t,r,z)=\sup_{x>0}\{V(x)-zx\}$$

so that

$$h(T,r,z) = (V')^{-1}(z). \quad (16)$$

4. Model Formulation

4.1. Pension Wealth with Stochastic Extra Rate of Contribution

We assume that the contributions are continuously paid into the pension fund at the rate of

$$\left(e_1 dt + e_2 dW_s(t)\right)L(t)$$

where e_1 is the mandatory rate of contribution and e_2 is the extra contribution rate which is assume to be stochastic. Let $U(t)$ denote the wealth of pension fund at time $t \in [0,T]$. $\pi_{B_1}(t)$ and $\pi_{s_1}(t)$ represent the proportion of the pension fund invested in the bond and the stock respectively. This implies that the proportion of the pension fund invested in the risk-free asset $\pi_{c_1}(t) = 1 - \pi_{B_1}(t) - \pi_{s_1}(t)$.

The dynamics of the pension wealth are given by

$$dU(t)=\pi_{c_1}U(t)\frac{dC(t)}{C(t)}+\pi_{B_1}U(t)\frac{dB(t,T)}{B(t,T)} +\pi_{s_1}U(t)\frac{dS(t)}{S(t)}+\left(e_1dt+e_2dW_s(t)\right)L(t). \quad (17)$$

Substituting (1), (3) and (4) in to (7) we have

$$\begin{aligned} dU(t)&=(1-\pi_{B_1}(t)-\pi_{s_1}(t))U(t)r(t)dt \\ &+\pi_{B_1}U(t)r(t)dt \\ &+\sigma_B(T-t,r(t))\left(dW_r(t)+\lambda_2\sigma_r dt\right) \\ &+\pi_{s_1}U(t)\begin{bmatrix} r(t)dt+\sigma_s\left(dW_s(t)+\lambda_1 dt\right) \\ +n_1\sigma_r\left(dW_r(t)+\lambda_2\sigma_r dt\right)\end{bmatrix} \\ &+\left(e_1dt+e_2dW_s(t)\right)L(t). \end{aligned} \quad (18)$$

Simplifying (18) we have

$$\begin{aligned} dU(t)&=U(t)\begin{bmatrix} r(t)+\lambda_2\sigma_r\sigma_B\pi_{B_1} \\ +\pi_{s_1}\left(\lambda_1\sigma_s+\lambda_2\sigma_r^2 n_1\right)\end{bmatrix}dt \\ &+\left(e_1dt+e_2dW_s(t)\right)L(t) \\ &+U(t)(\sigma_B\pi_{B_1}+\pi_{s_1}\sigma_r n_1)dW_r(t) \\ &+U(t)\sigma_S\pi_{s_1}dW_S(t) \end{aligned} \quad (19)$$

When it is time for retirement the contributor will be interested in preserving his standard of living and will be interested in his retirement income relative to his predetermine salary. If we consider the contributor salary as a numeraire. Let the relative wealth be defined as follows

$$X(t)=\frac{U(t)}{L(t)}. \quad (20)$$

Applying product rule and Ito's formula to (10) and making use of (6) and (9) we arrive at the following equation

$$\begin{aligned} &dX(t) \\ &= X(r(t)-\mu_L+n_2^2\sigma_r^2+n_3^2\sigma_S^2 \\ &+\sigma_B\pi_{B_1}\sigma_r(\lambda_2-n_2) \\ &+\pi_{s_1}\begin{pmatrix}\lambda_1\sigma_s+\lambda_2\sigma_r^2 n_1 \\ +\sigma_S^2 n_3-n_1n_2\sigma_r^2\end{pmatrix}dt \\ &+\left(e_1-\sigma_S n_3 e_2\right)dt \\ &+X\left(\sigma_B\pi_{B_1}+n_1\sigma_r\pi_{s_1}-n_2\sigma_r\right)dW_r(t) \\ &+\left(\frac{e_2}{X}+X\left(\pi_{s_1}-n_3\right)\sigma_s\right)dW_S(t) \end{aligned} \quad (21)$$

$$\begin{aligned} &dX(t) \\ &= X(\theta_1+\theta_2\pi_{B_1}+\theta_3\pi_{s_1})dt+\left(e_1-\sigma_S n_3 e_2\right)dt \\ &+X\left(\sigma_B\pi_{B_1}+n_1\sigma_r\pi_{s_1}-n_2\sigma_r\right)dW_r(t) \\ &+\left(\frac{e_2}{X}+X\left(\pi_{s_1}-n_3\right)\sigma_s\right)dW_S(t) \end{aligned} \quad (22)$$

$$\begin{aligned} \theta_1 &= r(t)-\mu_L+n_2^2\sigma_r^2+n_3^2\sigma_S^2 \\ \theta_2 &= \sigma_B\sigma_r(\lambda_2-n_2) \\ \theta_3 &= \lambda_1\sigma_s+\lambda_2\sigma_r^2 n_1+\sigma_S^2 n_3-n_1n_2\sigma_r^2. \end{aligned} \quad (23)$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$G_t + (a-br)G_r + \frac{1}{2}\sigma_r{}^2 G_{rr} + \sup_\pi\{[(x(\theta_1 + \pi_{B_1}\theta_2 + \pi_{s_1}\theta_3)) + (e_1 - \sigma_S n_3 e_2)]G_x + \frac{1}{2}(\frac{e_2}{X} + x(\pi_{s_1} - n_3)\sigma_s)^2 G_{xx} + \frac{1}{2}x^2(\sigma_B\pi_{B_1} + n_1\sigma_r\pi_{s_1} - n_2\sigma_r)^2 G_{xx} - x\sigma_r(\sigma_B\pi_{B_1} + n_1\sigma_r\pi_{s_1} - n_2\sigma_r)G_{rx}\} = 0. \quad (24)$$

where G_t, G_r, G_x, G_{rx}, G_{rx}, and G_{xx} denote partial derivatives of first and second orders with respect to time, short interest rate, and relative wealth.

Differentiating (24) with respect to π_{B_1} and π_{s_1}, we obtain the first-order maximizing conditions for the optimal strategies $\pi_{B_1}{}^*$ *and* $\pi_{s_1}{}^*$ as

$$\theta_2 G_x + x\sigma_B(\sigma_B\pi_{B_1} + n_1\sigma_r\pi_{s_1} - n_2\sigma_r)G_{xx} - \sigma_r\sigma_B G_{rx} = 0 \quad (25)$$

$$\theta_3 G_x + xn_1\sigma_r(\sigma_B\pi_{B_1} + n_1\sigma_r\pi_{s_1} - n_2\sigma_r)G_{xx} + x(\pi_{s_1} - n_3)\sigma_S{}^2 G_{xx} + \frac{e_2\sigma_S}{x}G_{xx} - n_1\sigma_r{}^2 G_{rx} = 0 \quad (26)$$

Solving (25) and (26) simultaneously we have

$$\pi_{s_1}{}^* = n_3 - \frac{e_2\sigma_S}{x^2} - \left(\frac{\lambda_1 + n_3\sigma_S{}^2}{x\sigma_S}\right)\frac{G_x}{G_{xx}} \quad (27)$$

$$\pi_{B_1}{}^* = \frac{\sigma_r(n_2 - n_1n_3 + \frac{e_2n_1}{\sigma_S x^2})}{\sigma_B} + \left(\frac{\theta_4\sigma_r}{x\sigma_B}\right)\frac{G_x}{G_{xx}} + \left(\frac{\sigma_r}{x\sigma_B}\right)\frac{G_{rx}}{G_{xx}} \quad (28)$$

$$\theta_4 = \frac{\sigma_S n_2 + n_1\lambda_1 + n_1n_3\sigma_S{}^2 - \lambda_2\sigma_S}{\sigma_S}. \quad (29)$$

Substituting (23), (27), (28), (29) into (17) we have

$$G_t + (a-br)G_r + \frac{1}{2}\sigma_r{}^2 G_{rr} + (\tau_0 + (\tau_1 + \tau_2)x)G_x + \left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2\sigma_r{}^2\right)\frac{G_x{}^2}{G_{xx}} + (\lambda_2 - n_2)\sigma_r{}^2\frac{G_xG_{rx}}{G_{xx}} - \frac{1}{2}\sigma_r{}^2\frac{G_{rx}}{G_{xx}} = 0 \quad (30)$$

$$\tau_0 = e_1 - e_2\sigma_S n_3$$
$$\tau_1 = n_2{}^2\sigma_r{}^2 + n_3{}^2\sigma_S{}^2 + n_3\lambda_1\sigma_s + \lambda_2\sigma_r{}^2 n_1n_3 + \sigma_S{}^2 n_3{}^2 - n_1n_2n_3\sigma_r{}^2 - m_L - \frac{n_1n_3\sigma_r}{\sigma_B} \quad (31)$$
$$\tau_2 = \frac{n_1\sigma_r}{\sigma_B} - \lambda_1\sigma_s - \lambda_2\sigma_r{}^2 n_1 - \sigma_S{}^2 n_3 + n_1n_2\sigma_r{}^2$$
$$\tau_3 = -\left(\frac{1}{2}\sigma_S{}^2 n_3{}^2 + n_3\sigma_s\lambda_1 + \frac{1}{2}\lambda_1{}^2\right).$$

Differentiating (13) with respect to $t, r, and\ x$ and substituting into (30)

$$G_t = \hat{G}_t, G_r = \hat{G}_r, G_x = z, G_{rx} = \frac{-\hat{G}_{rz}}{\hat{G}_{zz}},$$

$$G_{xx} = \frac{-1}{\hat{G}_{zz}}, G_{rr} = \hat{G}_{rr} - \frac{\hat{G}_{rz}{}^2}{\hat{G}_{zz}}.$$

$$\hat{G}_t + (a-br)\hat{G}_r + \frac{1}{2}\sigma_r{}^2\hat{G}_{rr} + (\tau_0 + (\tau_1 + \tau_2)x)z - \left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2\sigma_r{}^2\right)z^2\hat{G}_{zz} + (\lambda_2 - n_2)\sigma_r{}^2 z\hat{G}_{rz} = 0. \quad (32)$$

$$\tau_0 = e_1 - e_2\sigma_S n_3$$
$$\tau_1 = n_2{}^2\sigma_r{}^2 + n_3{}^2\sigma_S{}^2 + n_3\lambda_1\sigma_s + \lambda_2\sigma_r{}^2 n_1n_3 - n_1n_2n_3\sigma_r{}^2 - m_L - \frac{n_1n_3\sigma_r}{\sigma_B}$$
$$\tau_2 = \frac{n_1\sigma_r}{\sigma_B} - \lambda_1\sigma_s - \lambda_2\sigma_r{}^2 n_1 - \sigma_S{}^2 n_3 + n_1n_2\sigma_r{}^2$$
$$\tau_3 = -\left(\frac{1}{2}\sigma_S{}^2 n_3{}^2 + n_3\sigma_s\lambda_1 + \frac{1}{2}\lambda_1{}^2\right).$$

Using $x = h = \hat{G}_z$ and differentiating equation (32) for $\hat{G}$ with respect to z we obtain a linear PDE in terms of h and its derivatives.

$$h_t + (a-br)h_r + \frac{1}{2}\sigma_r{}^2 h_{rr} - \tau_0 - \varphi h - \varphi z h_z + (\lambda_2 - n_2)\sigma_r{}^2 h_r + (\lambda_2 - n_2)\sigma_r{}^2 z h_{rz} - 2\left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2\sigma_r{}^2\right)zh_z - \left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2\sigma_r{}^2\right)z^2 h_{zz} = 0 \quad (33)$$

Where

$$\varphi = \tau_1 + \tau_2$$
$$\tau_0 = e_1 - e_2\sigma_S n_3$$
$$\tau_1 = n_2{}^2\sigma_r{}^2 + n_3{}^2\sigma_S{}^2 + n_3\lambda_1\sigma_s + \lambda_2\sigma_r{}^2 n_1n_3 - n_1n_2n_3\sigma_r{}^2 - m_L - \frac{n_1n_3\sigma_r}{\sigma_B}$$
$$\tau_2 = \frac{n_1\sigma_r}{\sigma_B} - \lambda_1\sigma_s - \lambda_2\sigma_r{}^2 n_1 - \sigma_S{}^2 n_3 + n_1n_2\sigma_r{}^2$$
$$\tau_3 = -\left(\frac{1}{2}\sigma_S{}^2 n_3{}^2 + n_3\sigma_s\lambda_1 + \frac{1}{2}\lambda_1{}^2\right).$$

$$\pi_{c_1}{}^* = 1 - \pi_{B_1}{}^* - \pi_{s_1}{}^*$$

$$\pi_{s_1}{}^* = n_3 - \left(\frac{\lambda_1 + n_3\sigma_S{}^2}{h\sigma_S}\right)zh_z - \frac{e_2\sigma_S}{h^2} \quad (34)$$

$$\pi_{B_1}{}^{*} = \frac{(n_2 - n_1 n_3 + \frac{e_2 n_1}{\sigma_S h^2})}{f(T-t)} + \left(\frac{\theta_4}{h}\right) z h_z - \left(\frac{h_r}{hf(T-t)}\right) \quad (35)$$

$$\theta_4 = \frac{\sigma_S n_2 + n_1\lambda_1 + n_1 n_3 \sigma_S{}^2 + \lambda_2 \sigma_S}{\sigma_S},$$

$$f(t) = \frac{2(e^{mt}-1)}{m - (b - c_1\lambda_2) + e^{mt}(m + b - c_1\lambda_2)},$$

$$m = \sqrt{((b - c_1\lambda_2)^2 + 2c_1}.$$

We will now solve (33) for h and substitute into (34) and (35) to obtain the optimal investment strategies.

Assume the investor takes a power utility function

$$V(x) = \frac{x^p}{p},\ p < 1, p \neq 0. \quad (36)$$

The relative risk aversion of an investor with utility described in (36) is constant and (36) is a CRRA utility.

From (16) we have $h(T,r,z) = (V')^{-1}(z)$ and from (36) we have

$$h(T,r,z) = z^{\frac{1}{p-1}}. \quad (37)$$

We assume a solution to (37) with the following form

$$h(t,r,z) = v(t,r)\left[z^{\frac{1}{p-1}}\right] + y(t),\ y(T) = 0, v(T,s) = 1.$$

Then

$$h_t = v_t z^{\frac{1}{p-1}} + y', h_z = -\frac{v}{1-p} z^{\left(\frac{1}{p-1}-1\right)},$$
$$h_{rz} = -\frac{v_r}{1-p} z^{\left(\frac{1}{p-1}-1\right)}, h_{zz} = \frac{(2-p)v}{(1-p)^2} z^{\left(\frac{1}{p-1}-1\right)}, \quad (38)$$
$$h_r = v_r z^{\frac{1}{p-1}}, h_{rr} = v_{rr} z^{\frac{1}{p-1}}.$$

Substituting (38) into (33) we have

$$\left\{\begin{matrix} v_t + (a - br)v_r - \frac{(\lambda_2 - n_2)p\sigma_r{}^2}{1-p} v_r + \frac{1}{2}\sigma_r{}^2 v_{rr} \\ + \frac{\varphi p v}{1-p} - \frac{pv}{(1-p)^2}\left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2 \sigma_r{}^2\right) \end{matrix}\right\} z^{\frac{1}{p-1}} \quad (39)$$
$$+ y^I(t) - \varphi y(t) - \tau_0 = 0.$$

Splitting (39) we have

$$y^I(t) - \varphi y(t) - \tau_0 = 0 \quad (40)$$

$$v_t + (a - br)v_r - \frac{(\lambda_2 - n_2)p\sigma_r{}^2}{1-p} v_r + \frac{1}{2}\sigma_r{}^2 v_{rr}$$
$$+ \frac{\varphi p v}{1-p} - \frac{pv}{(1-p)^2}\left(\tau_3 - \frac{1}{2}(\lambda_2 - n_2)^2 \sigma_r{}^2\right) = 0. \quad (41)$$

Solving (40) with the given boundary condition $y(T) = 0$

$$y(t) = -\frac{\tau_0}{\varphi}(1 - e^{-\varphi(T-t)}). \quad (42)$$

To solve (41), we conjecture a solution of the form

$$v(t,r) = M(t)e^{N(t)r}, M(T) = 1, N(T) = 0,$$
$$v_r = M(t)N(t)e^{N(t)r},$$
$$v_{rr} = M(t)N^2(t)e^{N(t)r} \quad (43)$$
$$and\ v_t = rM(t)N^I(t)e^{N(t)r} + M^I(t)e^{N(t)r}.$$

Substituting (43) into (41), we have

$$\frac{M^I}{M} + \frac{a - ((\lambda_2 - n_2)c_1 + a)p}{1-p} N$$
$$+ \frac{1}{2}c_2 N^2 + \frac{p(\varphi - \tau_3 - p\varphi)}{(1-p)^2} + \frac{1}{2}\frac{(\lambda_2 - n_2)^2 pc_2}{(1-p)^2}$$
$$+ r(N^I - \frac{b + ((\lambda_2 - n_2)c_1 - b)p}{1-p} N \quad (44)$$
$$+ \frac{1}{2}c_1 N^2 + \frac{1}{2}\frac{(\lambda_2 - n_2)^2 pc_1}{(1-p)^2}).$$

Splitting (43) we have

$$\frac{M^I}{M} + \frac{a - ((\lambda_2 - n_2)c_1 + a)p}{1-p} N + \frac{1}{2}c_2 N^2$$
$$+ \frac{p(\varphi - \tau_3 - p\varphi)}{(1-p)^2} + \frac{1}{2}\frac{(\lambda_2 - n_2)^2 pc_2}{(1-p)^2} = 0 \quad (45)$$

$$N^I - \frac{b + ((\lambda_2 - n_2)c_1 - b)p}{1-p} N$$
$$+ \frac{1}{2}c_1 N^2 + \frac{1}{2}\frac{(\lambda_2 - n_2)^2 pc_1}{(1-p)^2} = 0 \quad (46)$$

$$N(t) = \frac{d_1 - d_2 e^{\frac{1}{2}c_1(d_1 - d_2)(T-t)}}{1 - \frac{d_1}{d_2} e^{\frac{1}{2}c_1(d_1 - d_2)(T-t)}} \quad (47)$$

$$M(t) = e^{\left\{\begin{matrix} \frac{((\lambda_2 - n_2)c_1 + a)p - a}{1-p}(\int N(t)dt - \frac{1}{2}c_2 \int N^2(t)dt) \\ - \frac{p(\varphi - \tau_3 - p\varphi)}{(1-p)^2} t + C \end{matrix}\right\}} \quad (48)$$
$$M(T) = 1$$

Where

$$d_1 = \frac{\left(\begin{matrix} b + ((\lambda_2 - n_2)c_1 - b)p \\ + \sqrt{(b + ((\lambda_2 - n_2)c_1 - b)p)^2 - (\lambda_2 - n_2)^2 pc_1{}^2} \end{matrix}\right)}{(1-p)c_1} \quad (49)$$

$$d_2=\frac{\begin{pmatrix}b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\\-\sqrt{\left(b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\right)^2-\left(\lambda_2-n_2\right)^2pc_1^{\,2}}\end{pmatrix}}{(1-p)c_1}\quad(50)$$

$$v(r,t)$$

$$\begin{aligned}=\exp\Bigg\{&\frac{\left(\left(\lambda_2-n_2\right)c_1+a\right)p-a}{1-p}\int\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)dt\\&-\frac{1}{2}c_2\int\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)^2dt\\&-\frac{p\left(\varphi-\tau_3-p\varphi\right)}{(1-p)^2}t+C\\&+\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)r\Bigg\}.\end{aligned}\quad(51)$$

Hence the solution of (33) is given as

$$h(t,r,z)$$

$$\begin{aligned}=\exp\Bigg\{&\frac{\left(\left(\lambda_2-n_2\right)c_1+a\right)p-a}{1-p}\int\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)dt\\&-\frac{1}{2}c_2\int\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)^2dt\\&-\frac{p\left(\varphi-\tau_3-p\varphi\right)}{(1-p)^2}t+C\\&+\left(\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}\right)r\Bigg\}\left[z^{\frac{1}{p-1}}\right]-\frac{\tau_0}{\varphi}\left(1-e^{-\varphi(T-t)}\right)\end{aligned}\quad(52)$$

Proposition 4.1

The optimal investment strategies for cash, bond and stock is given as follows

$$\pi_{c_1}{}^{*}=1-\pi_{B_1}{}^{*}-\pi_{s_1}{}^{*}$$

$$\pi_{s_1}{}^{*}=n_3-\left(\frac{\lambda_1+n_3\sigma_S{}^2}{h\sigma_S}\right)zh_z-\frac{e_{2\sigma_S}}{h^2}$$

$$\pi_{B_1}{}^{*}=\frac{\left(n_2-n_1n_3+\frac{e_2n_1}{\sigma_Sh^2}\right)}{f(T-t)}+\left(\frac{\theta_4}{h}\right)zh_z-\left(\frac{h_r}{hf(T-t)}\right)$$

$$\theta_4=\frac{\sigma_Sn_2+n_1\lambda_1+n_1n_3\sigma_S{}^2+\lambda_2\sigma_S}{\sigma_S}$$

$$h(t,r,z)=v(t,r)\left[z^{\frac{1}{p-1}}\right]+y(t),$$

$$y(t)=-\frac{\gamma_0}{\varphi}\left(1-e^{-\varphi(T-t)}\right)$$

$$\varphi=\tau_1+\tau_2$$

$$\tau_0=e_1-e_2\sigma_Sn_3$$

$$\begin{aligned}\tau_1=\;&n_2{}^2\sigma_r{}^2+n_3{}^2\sigma_S{}^2+n_3\lambda_1\sigma_s+\lambda_2\sigma_r{}^2n_1n_3\\&-n_1n_2n_3\sigma_r{}^2-m_L-\frac{n_1n_3\sigma_r}{\sigma_B}\end{aligned}$$

$$\tau_2=\frac{n_1\sigma_r}{\sigma_B}-\lambda_1\sigma_s-\lambda_2\sigma_r{}^2n_1-\sigma_S{}^2n_3+n_1n_2\sigma_r{}^2$$

$$\tau_3=-\left(\frac{1}{2}\sigma_S{}^2n_3{}^2+n_3\sigma_s\lambda_1+\frac{1}{2}\lambda_1{}^2\right)$$

$$h_r=v_rz^{\frac{1}{p-1}},$$

$$h_Z=-\frac{v}{1-p}z^{\left(\frac{1}{p-1}-1\right)}$$

$$v(t,r)=M(t)e^{N(t)r}$$

$$v_r=M(t)N(t)e^{N(t)r}$$

$$N(t)=\frac{d_1-d_2e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1\left(d_1-d_2\right)(T-t)}}$$

$$M(t)=e^{\left\{\begin{matrix}\frac{\left(\left(\lambda_2-n_2\right)c_1+a\right)p-a}{1-p}\int N(t)dt\\-\frac{1}{2}c_2\int N^2(t)dt-\frac{p\left(\varphi-\tau_3-p\varphi\right)}{(1-p)^2}t+C\end{matrix}\right\}}$$

Where

$$d_1=\frac{\begin{pmatrix}b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\\+\sqrt{\begin{pmatrix}\left(b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\right)^2\\-\left(\lambda_2-n_2\right)^2pc_1^{\,2}\end{pmatrix}}\end{pmatrix}}{(1-p)c_1}$$

$$d_2=\frac{\begin{pmatrix}b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\\-\sqrt{\begin{pmatrix}\left(b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\right)^2\\-\left(\lambda_2-n_2\right)^2pc_1^{\,2}\end{pmatrix}}\end{pmatrix}}{(1-p)c_1}$$

$$v(r,t) = \frac{\left((\lambda_2 - n_2)c_1 + a\right)p - a}{1-p}\int\left(\frac{d_1 - d_2 e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}\right)dt$$

$$-\frac{1}{2}c_2\int\left(\frac{d_1 - d_2 e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}\right)^2 dt$$

$$-\frac{p(\varphi - \tau_3 - p\varphi)}{(1-p)^2}t + C + \left(\frac{d_1 - d_2 e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}\right).$$

$$f(t) = \frac{2(e^{mt}-1)}{m - (b - c_1\lambda_2) + e^{mt}(m + b - c_1\lambda_2)},$$

$$m = \sqrt{((b - c_1\lambda_2)^2 + 2c_1}.$$

4.2. Pension Wealth with Constant Extra Rate of Contribution

Here the contributions are continuously paid into the pension fund at the rate of $(e_1 + e_2)L(t)$ where e_1 is the mandatory rate of contribution and e_2 is the extra contribution rate which is assume to be at constant rate. Let $U(t)$ denote the wealth of pension fund at time $t \in [0,T]$. π_{B_2} and π_{s_2} represent the proportion of the pension fund invested in the bond and the stock respectively. This implies that the proportion of the pension fund invested in the risk-free asset $\pi_{c_2} = 1 - \pi_{B_2} - \pi_{s_2}$. The dynamics of the pension wealth is given by

$$dU(t) = \pi_{c_2}U(t)\frac{dC(t)}{C(t)} + \pi_{B_2}U(t)\frac{dB(t,T)}{B(t,T)} + \pi_{s_2}U(t)\frac{dS(t)}{S(t)} + (e_1 + e_2)L(t)dt \quad (53)$$

Substituting (1), (3) and (4) in to (53) we have

$$\begin{aligned} dU(t) &= U(t)\begin{bmatrix} r(t) + \lambda_2\sigma_r\sigma_B\pi_{B_2} \\ +\pi_{s_2}\left(\lambda_1\sigma_s + \lambda_2\sigma_r^2 n_1\right)\end{bmatrix}dt \\ &+ (e_1 + e_2)L(t)dt \\ &+ U(t)(\sigma_B\pi_{B_2} + \pi_{s_2}\sigma_r n_1)dW_r(t) \\ &+ U(t)\sigma_S\pi_{s_2}dW_S(t). \end{aligned} \quad (54)$$

Let the relative wealth be defined as follows

$$X(t) = \frac{U(t)}{L(t)}. \quad (55)$$

Applying product rule and Ito's formula to (55) and making use of (6) and (54) we arrive at the following equation

$$\begin{aligned} dX(t) &= X(r(t) - \mu_L + n_2^2\sigma_r^2 + n_3^2\sigma_S^2 \\ &+ \sigma_B\pi_{B_2}\sigma_r(\lambda_2 - n_2) \\ &+ \pi_{s_2}\left(\lambda_1\sigma_s + \lambda_2\sigma_r^2 n_1 + \sigma_S^2 n_3 - n_1 n_2\sigma_r^2\right)dt \\ &+ (e_1 + e_2)dt + X\left(\sigma_B\pi_{B_2} + n_1\sigma_r\pi_{s_2} - n_2\sigma_r\right)dW_r(t) \\ &+ X\left(\pi_{s_2} - n_3\right)\sigma_s)dW_S(t) \end{aligned} \quad (56)$$

$$\begin{aligned} dX(t) &= X(\theta_1 + \theta_2\pi_{B_2} + \theta_3\pi_{s_2})dt + (e_1 + e_2)dt \\ &+ X\left(\sigma_B\pi_{B_2} + n_1\sigma_r\pi_{s_2} - n_2\sigma_r\right)dW_r(t) \\ &+ (X\left(\pi_{s_2} - n_3\right)\sigma_s)dW_S(t) \end{aligned} \quad (57)$$

$$\begin{aligned} \theta_1 &= r(t) - \mu_L + n_2^2\sigma_r^2 + n_3^2\sigma_S^2 \\ \theta_2 &= \sigma_B\sigma_r(\lambda_2 - n_2) \\ \theta_3 &= \lambda_1\sigma_s + \lambda_2\sigma_r^2 n_1 + \sigma_S^2 n_3 - n_1 n_2\sigma_r^2. \end{aligned} \quad (58)$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with (57) is

$$\begin{aligned} &G_t + (a - br)G_r + \frac{1}{2}\sigma_r^2 G_{rr} \\ &+ \sup_\pi\{[(x(\theta_1 + \pi_{B_2}\theta_2 + \pi_{s_2}\theta_3)) \\ &+ (e_1 + e_2)]G_x + \frac{1}{2}\left(x\left(\pi_{s_2} - n_3\right)\sigma_s\right)^2 G_{xx} \\ &+ \frac{1}{2}(x\left(\sigma_B\pi_{B_2} + n_1\sigma_r\pi_{s_2} - n_2\sigma_r\right))^2 G_{xx} \\ &- x\sigma_r(\sigma_B\pi_{B_2} + n_1\sigma_r\pi_{s_2} - n_2\sigma_r)G_{rx}\} = 0. \end{aligned} \quad (59)$$

With G_t, G_r, G_x, G_{rx}, G_{rx}, and G_{xx} all partial derivatives of first and second orders with respect to time, short interest rate, and relative wealth.

Differentiating (59) with respect to π_{B_2} and π_{s_2}, we obtain the first-order maximizing conditions for the optimal strategies $\pi_{B_2}{}^*$ *and* $\pi_{s_2}{}^*$ as

$$\theta_2 G_x + x\sigma_B\begin{pmatrix}\sigma_B\pi_{B_2}{}^* \\ +n_1\sigma_r\pi_{s_2}{}^* - n_2\sigma_r\end{pmatrix}G_{xx} - \sigma_r\sigma_B G_{rx} = 0 \quad (60)$$

$$\begin{aligned} &\theta_3 G_x + xn_1\sigma_r\left(\sigma_B\pi_{B_2}{}^* + n_1\sigma_r\pi_{s_2}{}^* - n_2\sigma_r\right)G_{xx} \\ &+ x\left(\pi_{s_2}{}^* - n_3\right)\sigma_S G_{xx} - n_1\sigma_r^2 G_{rx} = 0. \end{aligned} \quad (61)$$

Solving (60) and (61) simultaneously we have

$$\pi_{s_2}{}^* = n_3 - \left(\frac{\lambda_1 + n_3\sigma_S^2}{x\sigma_S}\right)\frac{G_x}{G_{xx}} \quad (62)$$

$$\pi_{B_2}{}^* = \frac{\sigma_r(n_2 - n_1 n_3)}{\sigma_B} + \left(\frac{\theta_4\sigma_r}{x\sigma_B}\right)\frac{G_x}{G_{xx}} + \left(\frac{\sigma_r}{x\sigma_B}\right)\frac{G_{rx}}{G_{xx}} \quad (63)$$

$$\theta_4 = \frac{\sigma_S n_2 + n_1\lambda_1 + n_1 n_3\sigma_S^2 - \lambda_2\sigma_S}{\sigma_S}.$$

Substituting (62) and (63) into (59) we have

$$G_t+(a-br)G_r+\frac{1}{2}\sigma_r{}^2G_{rr}+(\rho_0+\rho_1 x)G_x$$
$$+\left(\rho_2-\frac{1}{2}(\lambda_2-n_2)^2\sigma_r{}^2\right)\frac{G_x{}^2}{G_{xx}} \tag{64}$$
$$+(\lambda_2-n_2)\sigma_r{}^2\frac{G_xG_{rx}}{G_{xx}}-\frac{1}{2}\sigma_r{}^2\frac{G_{rx}}{G_{xx}}=0$$

$$\rho_0=e_1+e_2$$
$$\rho_1=n_3\lambda_1\sigma_s+\lambda_2\sigma_r{}^2n_2+2\sigma_S{}^2n_3{}^2-m_L \tag{65}$$
$$\rho_2=\frac{1}{2}\sigma_S{}^4n_3{}^2-\sigma_S{}^3n_3{}^2-n_3\sigma_s\lambda_1-\frac{1}{2}\lambda_1{}^2.$$

Applying Legendre transform to (64) we have

$$\hat{G}_t+(a-br)\hat{G}_r+\frac{1}{2}\sigma_r{}^2\hat{G}_{rr}+(\rho_0+\rho_1 x)z$$
$$-\left(\rho_2-\frac{1}{2}(\lambda_2-n_2)^2\sigma_r{}^2\right)z^2\hat{G}_{zz} \tag{66}$$
$$+(\lambda_2-n_2)\sigma_r{}^2z\hat{G}_{rz}=0$$

$$\rho_0=e_1+e_2$$
$$\rho_1=n_3\lambda_1\sigma_s+\lambda_2\sigma_r{}^2n_2+2\sigma_S{}^2n_3{}^2-m_L$$
$$\rho_2=\frac{1}{2}\sigma_S{}^4n_3{}^2-\sigma_S{}^3n_3{}^2-n_3\sigma_s\lambda_1-\frac{1}{2}\lambda_1{}^2.$$

Differentiating equation (66) for $\hat{G}$ with respect to z we obtain a linear PDE in terms of h and its derivatives and using $x=h=\hat{H}_z$, we have

$$h_t+(a-br)h_r+\frac{1}{2}\sigma_r{}^2h_{rr}-\rho_0-\rho_1h-\rho_1zh_z$$
$$+(\lambda_2-n_2)\sigma_r{}^2h_r+(\lambda_2-n_2)\sigma_r{}^2zh_{rz}$$
$$-2\left(\rho_2-\frac{1}{2}(\lambda_2-n_2)^2\sigma_r{}^2\right)zh_z \tag{67}$$
$$-\left(\rho_2-\frac{1}{2}(\lambda_2-n_2)^2\sigma_r{}^2\right)z^2h_{zz}=0$$

Where

$$\rho_0=e_1+e_2$$
$$\rho_1=n_3\lambda_1\sigma_s+\lambda_2\sigma_r{}^2n_2+2\sigma_S{}^2n_3{}^2-m_L$$
$$\rho_2=\frac{1}{2}\sigma_S{}^4n_3{}^2-\sigma_S{}^3n_3{}^2-n_3\sigma_s\lambda_1-\frac{1}{2}\lambda_1{}^2$$
$$\pi_{c_2}=1-\pi_{B_2}-\pi_{s_2}$$
$$\pi_{s_2}{}^*=n_3-\left(\frac{\lambda_1+n_3\sigma_S{}^2}{h\sigma_S}\right)zh_z$$
$$\pi_{B_2}{}^*=\frac{(n_2-n_1n_3)}{f(T-t)}+\left(\frac{\theta_4}{h}\right)zh_z-\left(\frac{h_r}{hf(T-t)}\right)$$
$$\theta_4=\frac{\sigma_S n_2+n_1\lambda_1+n_1n_3\sigma_S{}^2+\lambda_2\sigma_S}{\sigma_S}$$
$$f(t)=\frac{2(e^{mt}-1)}{m-(b-c_1\lambda_2)+e^{mt}(m+b-c_1\lambda_2)},$$
$$m=\sqrt{((b-c_1\lambda_2)^2+2c_1}.$$

Next we solve for the strategies for CRRA utility function, to achieve this we substitute (37) into (67) and obtain the result below

$$h(t,r,z)$$
$$=\exp\left\{\frac{((\lambda_2-n_2)c_1+a)p-a}{1-p}\int\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}\right)dt\right.$$
$$-\frac{1}{2}c_2\int\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}\right)^2dt$$
$$-\frac{p(\rho_1-\rho_2-p\rho_1)}{(1-p)^2}t+C \tag{68}$$
$$\left.+\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}\right)r\right\}\left[z^{\frac{1}{p-1}}\right]$$
$$-\frac{\rho_0}{\rho_1}(1-e^{-\rho_1(T-t)})$$

Proposition 5.2.

The optimal investment strategies for cash, bond and stock is given as follows

$$\pi_{c_2}=1-\pi_{B_2}-\pi_{s_2}$$
$$\pi_{s_2}{}^*=n_3-\left(\frac{\lambda_1+n_3\sigma_S{}^2}{h\sigma_S}\right)zh_z$$
$$\pi_{B_2}{}^*=\frac{(n_2-n_1n_3)}{f(T-t)}+\left(\frac{\theta_4}{h}\right)zh_z-\left(\frac{h_r}{hf(T-t)}\right)$$
$$\theta_4=\frac{\sigma_S n_2+n_1\lambda_1+n_1n_3\sigma_S{}^2+\lambda_2\sigma_S}{\sigma_S}$$
$$h(t,r,z)=v(t,r)\left[z^{\frac{1}{p-1}}\right]+y(t),$$
$$y(t)=-\frac{\rho_0}{\rho_1}(1-e^{-\rho_1(T-t)})$$
$$\rho_0=e_1+e_2$$
$$\rho_1=n_3\lambda_1\sigma_s+\lambda_2\sigma_r{}^2n_2+2\sigma_S{}^2n_3{}^2-m_L$$
$$\rho_2=\frac{1}{2}\sigma_S{}^4n_3{}^2-\sigma_S{}^3n_3{}^2-n_3\sigma_s\lambda_1-\frac{1}{2}\lambda_1{}^2$$
$$h_r=v_rz^{\frac{1}{p-1}},$$
$$h_z=-\frac{v}{1-p}z^{\left(\frac{1}{p-1}-1\right)}$$

$$v(t,r)=M_2(t)e^{N_2(t)r}$$

$$v_r=M_2(t)N_2(t)e^{N(t)r}$$

$$N_2(t)=\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}.$$

$$M_2(t)=e^{\left\{\begin{array}{l}\frac{((\lambda_2-n_2)c_1+a)p-a}{1-p}\int N_2(t)dt \\ -\frac{1}{2}c_2\int N_2{}^2(t)dt-\frac{p(\rho_1-\rho_2-p\rho_1)}{(1-p)^2}t+C\end{array}\right\}}$$

Where

$$r_1=\frac{\left(\begin{array}{l}b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p \\ +\sqrt{\left(\begin{array}{l}\left(b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\right)^2 \\ -\left(\lambda_2-n_2\right)^2 pc_1{}^2\end{array}\right)}\end{array}\right)}{(1-p)c_1}$$

$$r_2=\frac{\left(\begin{array}{l}b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p \\ -\sqrt{\left(\begin{array}{l}\left(b+\left(\left(\lambda_2-n_2\right)c_1-b\right)p\right)^2 \\ -\left(\lambda_2-n_2\right)^2 pc_1{}^2\end{array}\right)}\end{array}\right)}{(1-p)c_1}$$

$$v(r,t)$$

$$=\exp\left\{\frac{\left(\left(\lambda_2-n_2\right)c_1+a\right)p-a}{1-p}\int\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{d_1}{d_2}e^{\frac{1}{2}c_1(d_1-d_2)(T-t)}}\right)dt\right.$$

$$-\frac{1}{2}c_2\int\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}\right)^2 dt$$

$$-\frac{p\left(\rho_1-\rho_2-p\rho_1\right)}{(1-p)^2}t+C$$

$$\left.+\left(\frac{r_1-r_2e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}{1-\frac{r_1}{r_2}e^{\frac{1}{2}c_1(r_1-r_2)(T-t)}}\right)r\right\}$$

$$f(t)=\frac{2(e^{mt}-1)}{m-\left(b-c_1\lambda_2\right)+e^{mt}(m+b-c_1\lambda_2)},$$

$$m=\sqrt{((b-c_1\lambda_2)^2+2c_1}.$$

5. Conclusion

We studied optimal investment strategy for a plan contributor in a defined pension scheme, with stochastic salary and extra contributions, under the affine interest rate model. We took into consideration the Nigerian Pension Reform Act of 2004 which allow members to contribute another proportion of their income into the pension account different from the compulsory contribution. We also considered three different investment which include risk free asset (cash), risk less asset (zero coupon bond) and the risky asset (stock). We assume for two cases; where the extra contribution rate are (1) stochastic and (2) constant. We obtained two optimized problems for both (1) and (2) using the Hamilton Jacobi equation then went on to solve these equations by Legendre transformation method to obtained explicit solutions of the optimal investment strategies for the three investments using the exponential utility function for the two cases. We observed a disparity between the two strategies. This result shows that the strategies used when there is no extra contribution can be used for (2) but cannot be used for (1).

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