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The Modified Bi-quintic B-spline Base Functions: An Application to Diffusion Equation

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Abstract In this paper, the bi-quintic B-spline base functions are modified on a general 2-dimensional problem and then they are applied to two-dimensional Diffusion problem in order to obtain its numerical solutions. The computed results are compared with the results given in the literature.

Keywords: *Galerkin Finite Element Method, Bi-quintic B-splines, Two-dimensional B-splines, Modified bi-quintic B-splines, Two-dimensional Diffusion Equation*

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1. Introduction

In this paper, we will consider two dimensional Diffusion equation of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{K}\frac{\partial u}{\partial t},\ (x,y)\in D, t>0 \qquad (1)$$

with the initial condition

$$u(x,y,0)=u_0(x,y)=\sin(\pi x)\sin(\pi y) \qquad (2)$$

and boundary conditions

$$u(x,y_0,t)=g_1(x,t)=0,\ x_0=0\le x\le x_n=1, t>0 \qquad (3)$$

$$u(x,y_m,t)=g_2(x,t)=0,\ x_0=0\le x\le x_n=1, t>0 \qquad (4)$$

$$u(x_0,y,t)=h_1(y,t)=0,\ y_0=0\le y\le y_m=1, t>0 \qquad (5)$$

$$u(x_n,y,t)=h_2(y,t)=0,\ y_0=0\le y\le y_m=1, t>0 \qquad (6)$$

on the region $D=[0,1]\times[0,1]$. The exact solution of this problem is [1]

$$u(x,y,t)=e^{-2\pi^2 Kt}\sin(\pi x)\sin(\pi y) \qquad (7)$$

where K is a diffusion coefficent.

The finite element method has been widely applied to physics, solid and fluid mechanics, engineering, medicine and so on [2-12]. Particularly, Kasi and Koneru [13] have used the Galerkin finite element method for onedimensional and two-dimensional time dependent problems by modifying bicubic B-spline base functions. Moon et al. [1] have obtained a non-separable solution of the diffusion equation based on the Galerkin's method using cubic splines. Tavakoli and Davami [14] have developed a parallel unconditionally stable fully explicit finite difference scheme for solution of the diffusion equation. Ang [15] has proposed a boundary integral equation method for the numerical solution of the two-dimensional diffusion equation subject to a non-local condition. Velivelli and Bryden [16] have examined cache optimization for the Lattice Boltzmann method in both serial and parallel implementations by utilizing the two-dimesional diffusion equation. Aboanber and Hamada [17] have developed a generalized Runge-Kutta method for the numerical integration of the stiff space-time diffusion equations. Bhaskar et al. [18] have developed Heatlets, the fundamental solutions of heat equation using wavelets, for numerically solving inhomogeneous and homogeneous initial value problems of diffusion equation on unbounded domains. Sterk and Trobec [19] have given the derivation and implementation of a numerical solution of a time-dependent diffusion equation in detail, based on the meshless local Petrov-Galerkin method.

In this paper, we have first modified bi-quintic b-spline functions on the boundary of a general two dimensional dimensional problem and used them to obtain numerical solutions of the Diffusion problem by the Galerkin finite element method. The modified bi-quintic B-splines are used as basis functions and rectangles as element shapes.

We try a bi-quintic B-spline function of the form

$$u_{nm}(x,y,t)=\sum_{i=-2}^{n+2}\sum_{j=-2}^{m+2}\alpha_{ij}(t)B_i(x)B_j(y)$$

as an approximation solution to two dimensional diffusion problem. In order to find an approximate solution in the above form to the problem by the Galerkin method, first of all we have to redefine the B-spline basis functions into a new set of functions, namely modified bi-quintic B-spline functions. This redefining process was successfully applied to cubic B-spline functions to obtain modified bi-cubic B-spline functions [13]. The newly obtained set of modified functions are identically zero on the boundary of the given problem. It should be pointed out that this process is necessary, since B-spline basis functions $B_i(x)(i = -2(1)(n + 2)$ in the x-direction and the B-spline basis functions $B_j(y)(j = -2(1)(m + 2)$ in the y-direction are not zero on the boundary of the problem. After the redefining process of the basis functions, we can now try the modified bi-quintic B-spline functions of the form

$$u_{nm}(x,y,t) = \varphi(x,y,t) + \sum_{i=-1}^{n+1}\sum_{j=-1}^{m+1} \alpha_{ij}(t)\tilde{B}_i(x)\tilde{B}_j(y)$$

as its approximate solution, where $\tilde{B}_i(x)$ and $\tilde{B}_j(y)$ are the quintic B-splines to be modified only on the boundary in the x and y- directions, respectively. Since all of the modified quintic B-splines are zero on the boundary of the problem, non-homogenous boundary conditions are satisfied by the term $\varphi(x,y,t)$.

2. Derivation of the Modified Bi-quintic B-splines

2.1. Bi-quintic B-spline Element

The rectangular region D of the problem is subdivided into a number of uniform rectangular finite elements of sides h_x and h_y by the knots (x_i, y_i) where $0 \le i \le n$, $0 \le j \le m$. An approximation $u_{nm}(x,y,t)$ with quintic B-spline functions to $u(x,y,t)$ is taken of the form

$$u_{nm}(x,y,t) = \sum_{i=-2}^{n+2}\sum_{j=-2}^{m+2} \alpha_{ij}(t) B_{ij}(x,y) \tag{8}$$

where $\alpha_{ij}(t)$'s are the amplitudes of bi-quintic B-splines $B_{ij}(x,y)$ given by

$$B_{ij}(x,y) = B_i(x)B_j(y) \tag{9}$$

and $B_i(x)$ is defined as

$$B_i(x) = \frac{1}{h^5}\begin{cases} (x-x_{i-3})^5, & [x_{i-3}, x_{i-2}], \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5, & [x_{i-2}, x_{i-1}], \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & \\ +15(x-x_{i-1})^5, & [x_{i-1}, x_i], \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & \\ +15(x-x_{i-1})^5 - 20(x-x_i)^5, & [x_i, x_{i+1}], \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & \\ +15(x-x_{i-1})^5 - 20(x-x_i)^5 & \\ +15(x-x_{i+1})^5, & [x_{i+1}, x_{i+2}], \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & \\ +15(x-x_{i-1})^5 - 20(x-x_i)^5 & \\ +15(x-x_{i+1})^5 - 6(x-x_{i+2})^5, & [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases}$$

[20]. $B_j(y)$ can easily be found by replacing i with j and x with y. Figure 1 depicts a region, where $h_x = h_y = 1$, so that it is divided into finite elements by the integer knots (i, j), and a single bi-quintic B-spline B_{33} which peaks on the point (3, 3) and also covers a total of 36 square elements. When the entire set of bi-quintic splines B_{ij}, each of which peaks on a knot (i, j), where $0 \le i \le 6$, $0 \le j \le 6$, are added to this figure, a total of 36 splines cover each finite element [21].

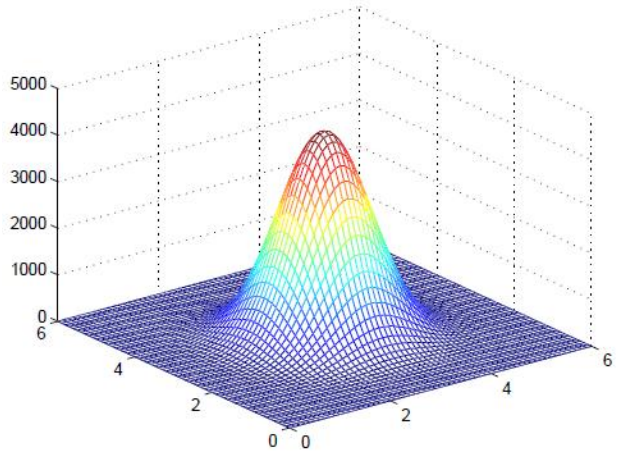

Figure 1. The bi-quintic B-spline B_{33} centered on (3,3) covering 36 finite elements of side 1

2.2. A Modified Bi-quintic B-spline Element

To show how to modify bi-quintic spline functions on the boundary, we consider the two-dimensional general linear equation of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= a(x,y,t)\frac{\partial^2 u}{\partial x^2} + b(x,y,t)\frac{\partial^2 u}{\partial y^2} + c(x,y,t)\frac{\partial u}{\partial t} \\ &+ d(x,y,t)\frac{\partial u}{\partial y} + e(x,y,t)u + f(x,y,t)\end{aligned} \tag{10}$$

subject to the initial condition

$$u(x,y,0) = u_0(x,y) \tag{11}$$

and boundary conditions

$$u(x,y_0) = g_1(x), \quad x_0 \le x \le x_n \tag{12}$$

$$u(x,y_m) = g_2(x), \quad x_0 \le x \le x_n \tag{13}$$

$$u(x_0,y) = h_1(y), \quad y_0 \le y \le x_m \tag{14}$$

$$u(x_n,y) = h_2(y), \quad y_0 \le y \le x_m \tag{15}$$

where $x \in [x_0, x_n]$, $y \in [y_0, y_m]$ and $g_1(x)$, $g_2(x)$, $h_1(y)$, $h_2(y)$ are given functions, D is a rectangular region in R^2 with boundary ∂D.

Now, it is supposed that both the x-space variable domain and y-space variable domain of the system (10)-(15) are divided into n and m subintervals, respectively, by the set of $n + 1$ distinct grid points $x_i\,(i = 0(1)n)$ and $m + 1$ distinct grid points $y_j\,(j = 0(1)m)$ such that

$$0 = x_0 < x_1 < \ldots < x_n = 1$$
$$\text{and } 0 = y_0 < y_1 < \ldots < y_m = 1.$$

Since a quintic B-spline function covers six consecutive elements, we add ten additional grid points x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5} in the x-direction and ten additional grid points y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_{m+1}, y_{m+2}, y_{m+3}, y_{m+4}, y_{m+5} in the y-direction such that

$$h_x = x_{n+1} - x_n, \quad n = -5,-4,-3,-2,-1$$
$$h_y = y_{m+1} - y_m, \quad m = -5,-4,-3,-2,-1$$
$$h_x = x_{n+1} - x_n, \quad n = n+4, n+3, n+2, n+1, n, n-1$$
$$h_y = y_{m+1} - y_m, \quad m = m+4, m+3, m+2, m+1, m, m-1.$$

To find an approximate solution in the form of Eq. (8) to the problem given by Eqs. (10)-(15) with the Galerkin method, we do need to redefine the basis functions into a new set of basis functions which all vanish on ∂D. The redefining process of the basis functions is done in the following three steps.

Step 1. The approximate solution $u_{nm}(x,y,t)$ given by Eq. (8) can also be written as [13]

$$u_{nm}(x,y,t) = \sum_{i=-2}^{n+2} \gamma_i(y) B_i(x) \tag{16}$$

where

$$\gamma_i(y) = \sum_{j=-2}^{m+2} \alpha_{ij}(t) B_j(y). \tag{17}$$

Allowing the approximate solution $u_{nm}(x,y,t)$ given by Eq. (16) to satisfy the boundary conditions (14) and (15) and eliminating $\gamma_{-2}(y)$ and $\gamma_{n+2}(y)$ from the resulting equations, we obtain

$$u_{nm}(x,y,t) = \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \sum_{i=-1}^{n+1} \gamma_i(y) \tilde{B}_i(x) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) \tag{18}$$

where

$$\tilde{B}_i(x) = B_i(x) - \frac{B_i(x_0)}{B_{-2}(x_0)} B_{-2}(x), \ for\ i = -1,0,1,2 \tag{19}$$

$$\tilde{B}_i(x) = B_i(x), \ for\ i = 3(1)n-3 \tag{20}$$

$$\tilde{B}_i(x) = B_i(x) - \frac{B_i(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), \quad for\ i = n-2, n-1, n, n+1. \tag{21}$$

Step 2. By evaluating the expression $\gamma_i(y)$ given by Eq. (17) at y_0 and y_m and eliminating $\alpha_{i,-2}$ and $\alpha_{i,m+2}$ from the resulting equations, we obtain

$$\gamma_i(y) = \frac{B_{-2}(y)}{B_{-2}(y_0)} \gamma_i(y_0) + \sum_{j=-1}^{m+1} \alpha_{ij} \tilde{B}_j(y) + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \gamma_i(y_m) \tag{22}$$

where

$$\tilde{B}_j(y) = B_j(y) - \frac{B_j(y_0)}{B_{-2}(y_0)} B_{-2}(y), \ for\ j = -1,0,1,2 \tag{23}$$

$$\tilde{B}_j(y) = B_j(y), \ for\ j = 3(1)m-3 \tag{24}$$

$$\tilde{B}_j(y) = B_j(y) - \frac{B_j(y_m)}{B_{m+2}(y_m)} B_{m+2}(y), \quad for\ j = m-2, m-1, m, m+1. \tag{25}$$

Step 3. Finally, substituting $\gamma_i(y)$ in Eq. (22) into Eq. (18) and allowing the resulting equation to satisfy the boundary conditions (12) and (13), we obtain

$$u(x,y,t) = \varphi_1(x,y,t) + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij}(t) \tilde{B}_i(x) \tilde{B}_j(y) \tag{26}$$

where

$$\begin{aligned} \varphi_1(x,y,t) &= \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) \\ &+ \frac{B_{-2}(y)}{B_{-2}(y_0)} g_1(x) + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} g_2(x) \\ &- \frac{B_{-2}(y)}{B_{-2}(y_0)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y_0) \right] \\ &- \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y_m) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y_m) \right] \end{aligned} \tag{27}$$

Applying exactly the same three steps, but now writing the approximate solution (8) as [13]

$$u_{nm}(x,y,t) = \sum_{j=-2}^{m+2} \delta_j(x) B_j(y) \tag{28}$$

where

$$\delta_j(x) = \sum_{j=-2}^{n+2} \alpha_{ij}(t) B_i(x) \tag{29}$$

we obtain

$$u(x,y,t) = \varphi_2(x,y,t) + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij}(t) B_i(x) \tilde{B}_j(y) \tag{30}$$

where

$$\begin{aligned} \varphi_2(x,y,t) &= \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) \\ &+ \frac{B_{-2}(y)}{B_{-2}(y_0)} g_1(x) + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} g_2(x) \\ &- \frac{B_{-2}(y)}{B_{-2}(y_0)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} g_1(x_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} g_1(x_n) \right] \\ &- \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} g_2(x_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} g_2(x_n) \right] \end{aligned} \tag{31}$$

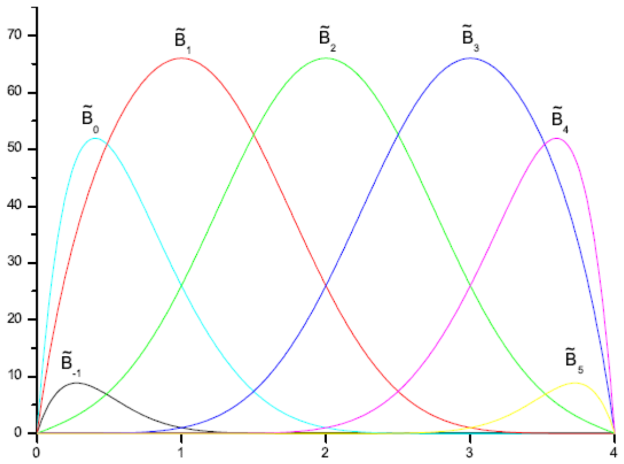

Figure 2. Modified B-spline functions $\tilde{B}_i(x), i=-1(1)5$

By taking the average of Eq. (27) and Eq. (31), we obtain the general approximation $u_{nm}(x,y,t)$ to $u(x,y,t)$ of the form

$$\begin{aligned} u_{nm}(x,y,t) &= \frac{\varphi_1(x,y,t)+\varphi_2(x,y,t)}{2} \\ &+ \sum_{i=-1}^{n+1}\sum_{j=-1}^{m+1}\alpha_{ij}(t)\tilde{B}_i(x)\tilde{B}_j(y) \qquad (32) \\ &= \varphi(x,y,t)+\sum_{i=-1}^{n+1}\sum_{j=-1}^{m+1}\alpha_{ij}(t)\tilde{B}_i(x)\tilde{B}_j(y) \end{aligned}$$

where the new set of basis functions are $\tilde{B}_i(x)\tilde{B}_j(y)$ for $i=-1(1)n+1$, $j=-1(1)m+1$, which all vanish on ∂D and $\varphi(x,y,t)$ given in Eq. (32) satisfies the boundary conditions given by Eqs. (12)-(15). The profiles of the modified quintic B-splines are shown in Figure 2 for 4 elements.

3. Numerical Example and Results

In this section, we will try to obtain the numerical solutions of the diffusion problem given by Eqs. (1)-(6) using the Galerkin finite element method with the modified bi-quintic B-spline base functions. The diffusion equation is integrated in space variables x and y. To apply the method to the problem, first of all we need to construct the weak form of the problem.

3.1. Weak Form of the Model Problem

For this purpose, all terms in Eq. (1) are taken to the right hand side of the equation and then multiplied by the weight function $\Psi(x,y)$. Finally, by integrating the resulting equation over the region D and setting it to zero, we get

$$\iint_D \left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{K}\frac{\partial u}{\partial t}\right\}\Psi dxdy=0 \qquad (33)$$

where $\Psi=\tilde{B}_k(x)\tilde{B}_l(y)$ for $k = 0(1)n$ and $l = 0(1)m$. By applying the Green Theorem (see, e.g. Reddy [22]) to Eq. (33), the weak form of the model problem in the global coordinate system is obtained as follows

$$\iint_D \left\{\frac{1}{K}\frac{\partial u}{\partial t}\Psi+\frac{\partial u}{\partial x}\frac{\partial \Psi}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial \Psi}{\partial y}-\right\}dxdy=0. \qquad (34)$$

To change from the global coordinate system into the local one, we use the transformations $h_x\xi=x-x_n$ and $h_y\eta=y-y_m$. Thus, the weak form (34) transforms to the form

$$\frac{h^2}{K}\int_0^1\int_0^1\frac{\partial u}{\partial t}\Psi d\xi d\eta+\int_0^1\int_0^1\left(\frac{\partial u}{\partial \xi}\frac{\partial \Psi}{\partial \xi}+\frac{\partial u}{\partial \eta}\frac{\partial \Psi}{\partial \eta}\right)d\xi d\eta=0. \quad (35)$$

3.2. Galerkin Finite Element Solutions of the Model Problem

In this subsection, the numerical solutions of the model problem are obtained by the Galerkin finite element method using the modified bi-quintic Bspline basis functions. The numerical scheme is implemented by dividing the region $D=[0,1]\times[0,1]$ into $16(h_x=h_y=1/4)$ and $100(h_x=h_y=1/10)$ elements. The obtained solutions are compared with those existing in the literature and tabulated with error norms L_2 and L_∞. In all numerical calculations, the coefficient K in Eq. (1) is taken as 10^{-6}.

For this, the approximate solution $u_{nm}(\xi,\eta,t)$ for each element is written in the weak form (35) and then a coefficient matrix is obtained for each element. By combining the coefficient matrix for each element, we obtain an algebraic equation in the form

$$ZA(0)=g \qquad (36)$$

and an iterative equation in the form

$$G\frac{dA(t)}{dt}=PA(t) \qquad (37)$$

$$\left[G-\frac{1}{2}\Delta tP\right]A(t_{r+1})=\left[G+\frac{1}{2}\Delta tP\right]A(t_r) \qquad (38)$$

$$\begin{aligned} &G=\left[g_{i_l j_l}\right], \\ &g_{i_k j_l}=h^2\int_0^1\int_0^1\tilde{B}_i(\xi)\tilde{B}_j(\eta)\tilde{B}_k(\xi)\tilde{B}_l(\eta)d\eta d\xi \end{aligned} \qquad (39)$$

$$\begin{aligned} &P=\left[p_{i_l j_l}\right], \\ &p_{i_l j_l}=-\int_0^1\int_0^1\frac{\partial\tilde{B}_i(\xi)}{\partial \xi}\tilde{B}_j(\eta)\frac{\partial\tilde{B}_k(\xi)}{\partial \xi}\tilde{B}_l(\eta)d\eta d\xi \quad (40a) \\ &-\int_0^1\int_0^1\tilde{B}_i(\xi)\frac{\partial\tilde{B}_j(\eta)}{\partial \eta}\tilde{B}_k(\xi)\frac{\partial\tilde{B}_l(\eta)}{\partial \eta}d\eta d\xi \end{aligned}$$

$$A_s(t)=\left[A_1(t)A_2(t)\ldots A_{(m+3)(n+3)}(t)\right]^t; \qquad (41)$$

where $i, j, k, l = 0(1)n$, $i_l = j(n+1)+i$ and $j_l = l(n+1)+k$

$$Z=\left[z_{i_l j_l}\right],\ z_{i_l j_l}=\int_0^1\int_0^1 \tilde{B}_i(\xi)\tilde{B}_j(\eta)\tilde{B}_k(\xi)\tilde{B}_l(\eta)\,d\eta\,d\xi \quad (42)$$

where $i, j, k, l = 0(1)n$, $i_l = j(n+1)+1$ and $j_l = l(n+1)+k$

$$g=\left[g_{i_l}\right],\ g_{i_l}=\int_0^1\int_0^1 u_0(\xi,\mu)\tilde{B}_i(\xi)\tilde{B}_j(\eta)\,d\eta\,d\xi. \quad (43)$$

where $i, j = 0(1)1$, $i_l = j(n+1)+1$.

The division of the region into 4×4=16 elements

If the solution domain D of the problem is equally divided into 4×4 = 16 elements, 16 squares having the sides of $h_x = h_y = h = 1/4$ are obtained. If the approximate solution $u_{nm}(\xi,\eta,t)$ is constructed for each element, a total number of 7 × 7 = 49 global element parameters over the region D are obtained depending on the local element parameters $\alpha_i(t)$ and $\beta_j(t)$, $(i, j = -1(1)5)$. It is obvious that we need to find the global element parameters $A_i(t)$, $(i = 1(1)49)$ in order to obtain the approximate solution $u_{nm}(\xi, \eta, t)$ for each element. By solving these algebraic equations, element parameters $A_i(t)$, $(i=1(1)49)$ are obtained at times $t = 0.0$ and $t = 1.0$.The obtained element parameters are put in their places in the element equations, and then the approximate solution $u_{nm}(\xi, \eta, t)$ for each element at times $t = 0.0$ and $t = 1.0$ is found.

The division of the region into 10×10=100 elements

Now the solution domain D of the problem is equally divided into 10 × 10 = 100 elements, 100 squares having the sides of $h_x = h_y = h = 1/10$ are obtained.

As in 4 × 4 = 16 elements, if the approximate solution $u_{nm}(\xi,\eta,t)$ is constructed for each element, a total number of 13× 13 = 169 global element parameters over the region D are obtained depending on the local element parameters $\alpha_i(t)$ and $\beta_i(t)$, $(i = -1(1)11)$. It is obvious that we need to find the global element parameters $A_i(t)$, $(i = 1(1)49)$ in order to obtain the approximate solution $u_{nm}(\xi,\eta,t)$ for each element. For this, the approximate solution $u_{nm}(\xi,\eta,t)$ for each element is written in the weak form (35) and then a coefficient matrix is obtained for each element. By combining the coefficient matrices for each element, first we obtain an algebraic equation in the form Eq. (36) then an iterative equation in the form of Eq. (38) by applying forward difference and Crank-Nicolson finite difference formula to Eq. (37). By solving these equations with the help of a computer program we obtain the global element parameters $A_i(t)$, $(i = 1(1)169)$ at times $t = 0.0$ and $t = 1.0$, respectively. These obtained element parameters are put in their places in the element equations, and then the approximate solutions $u_{nm}(\xi,\eta,t)$ for each element at times $t = 0.0$ and $t = 1.0$ are found.

The obtained numerical results by the present method using the modified bi-quintic B-splines have been displayed and also compared with its exact ones in Table 3 – Table 6. It is obviously seen from the tables that the numerical results are in good agreement with the exact ones. Figure 3 and Figure 4 show graphically how closely the approximate solution $u_{nm}(x,y,t)$ matches with the exact solution $u(x,y,t)$ at times $t = 0.0$ and $t = 0.05$, respectively. Since the numerical solution is very close to the exact solution, their graphs are indiscriminately similar to each other. In Table 5 and Table 6, the numerical solutions obtained by the Galerkin finite element method with modified biquintic B-spline basis functions are compared with the exact ones at times $t = 0.0$ and $t = 1.0$, respectively.

Table 1. Numerical and exact solutions of the model problem for 4 × 4 = 16 elements at t = 0.0

(x, y)	Numerical Solution (u_{nm}) 4 × 4	Exact Solution (u)
(0.25,0.25)	0.500009	0.500000
(0.25,0.50)	0.707119	0.707107
(0.25,0.75)	0.500009	0.500000
(0.50,0.25)	0.707119	0.707107
(0.50,0.50)	1.000017	1.000000
(0.50,0.75)	0.707119	0.707107
(0.75,0.25)	0.500009	0.500000
(0.75,0.50)	0.707119	0.707107
(0.75,0.75)	0.500009	0.500000

Table 2. Numerical and exact solutions of the model problem for 4 × 4 = 16 elements and Δt = 0.05 at t = 1.0

(x, y)	Numerical Solution (u_{nm}) 4 × 4	Exact Solution (u)
(0.25,0.25)	0.499999	0.499990
(0.25,0.50)	0.707105	0.707093
(0.25,0.75)	0.499999	0.499990
(0.50,0.25)	0.707105	0.707093
(0.50,0.50)	0.999997	0.999980
(0.50,0.75)	0.707105	0.707093
(0.75,0.25)	0.499999	0.499990
(0.75,0.50)	0.707105	0.707093
(0.75,0.75)	0.499999	0.499990

Table 3. Numerical and exact solutions of the problem for 10 × 10 = 100 elements at t = 0.0

(x, y)	Numerical Solution (u_{nm}) 4 × 4	Exact Solution (u)
(0.25,0.25)	0.500000	0.500000
(0.25,0.50)	0.707107	0.707107
(0.25,0.75)	0.500000	0.500000
(0.50,0.25)	0.707107	0.707107
(0.50,0.50)	1.000000	1.000000
(0.50,0.75)	0.707107	0.707107
(0.75,0.25)	0.500000	0.500000
(0.75,0.50)	0.707107	0.707107
(0.75,0.75)	0.500000	0.500000

Table 4. Numerical and exact solutions of the problem for 10 × 10 = 100 elements at t = 1.0 with Δt = 0.05

(x, y)	Numerical Solution (u_{nm}) 4 × 4	Exact Solution (u)
(0.25,0.25)	0.499990	0.499990
(0.25,0.50)	0.707093	0.707093
(0.25,0.75)	0.499990	0.499990
(0.50,0.25)	0.707093	0.707093
(0.50,0.50)	0.999980	0.999980
(0.50,0.75)	0.707093	0.707093
(0.75,0.25)	0.499990	0.499990
(0.75,0.50)	0.707093	0.707093
(0.75,0.75)	0.499990	0.499990

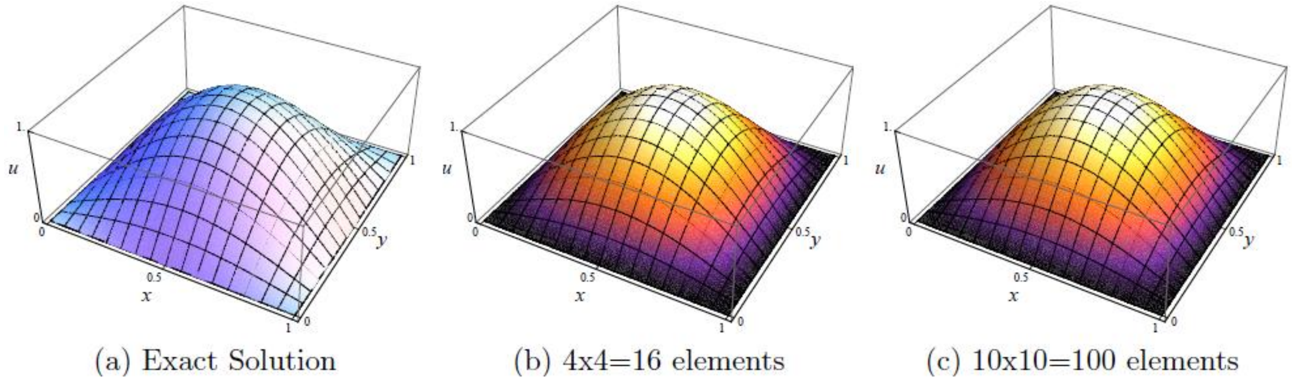

Figure 3. (a) Exact solution (b) Numerical solution for 4 × 4 = 16 elements and (c) Numerical solution for 10 × 10 = 100 elements at $t = 0$

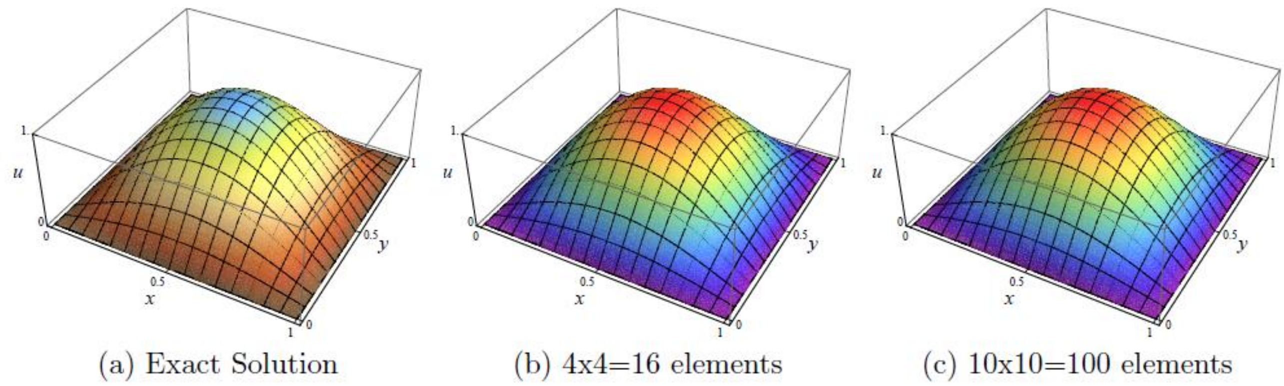

Figure 4. (a) Exact solution (b) Numerical solution for 4 × 4 = 16 elements and (c) Numerical solution for 10 × 10 = 100 elements at $t = 1.0$ with $\Delta t = 0.05$

Table 5. Numerical and exact solutions of the problem for 4 × 4 = 16 and 10 × 10 = 100 elements at $t = 0.0$

	Numerical Solution (u_{nm})		Exact Solution (u)
(x, y)	4 × 4	10 × 10	
(1.0,0.0)	0.000000	0.000000	0.000000
(0.9,0.1)	0.095489	0.095492	0.095492
(0.8,0.2)	0.345495	0.345492	0.345492
(0.7,0.3)	0.654510	0.654509	0.654508
(0.6,0.4)	0.904497	0.904508	0.904508
(0.5,0.5)	1.000017	1.000000	1.000000
(0.4,0.6)	0.904497	0.904508	0.904508
(0.3,0.7)	0.654510	0.654509	0.654508
(0.2,0.8)	0.345495	0.345492	0.345492
(0.1,0.9)	0.095489	0.095492	0.095492
(0.0,1.0)	0.000000	0.000000	0.000000
(0.0,0.0)	0.00000	0.000000	0.000000
(0.1,0.1)	0.095489	0.095492	0.095492
(0.2,0.2)	0.345495	0.345492	0.345492
(0.3,0.3)	0.654510	0.654509	0.654508
(0.4,0.4)	0.904497	0.904508	0.904508
(0.5,0.5)	1.000017	1.000000	1.000000
(0.6,0.6)	0.904497	0.904508	0.904508
(0.7,0.7)	0.654510	0.654509	0.654508
(0.8,0.8)	0.345495	0.345492	0.345492
(0.9,0.9)	0.095489	0.095492	0.095492
(1.0,1.0)	0.000000	0.000000	0.000000

Table 6. Numerical and exact solutions of the problem for 4 × 4 = 16 and 10 × 10 = 100 elements at $t = 1.0$ with $\Delta t = 0.05$

	Numerical Solution (u_{nm})		Exact Solution (u)
(x, y)	4 × 4	10 × 10	
(1.0,0.0)	0.000000	0.000000	0.000000
(0.9,0.1)	0.095487	0.095490	0.095490
(0.8,0.2)	0.345488	0.345485	0.345485
(0.7,0.3)	0.654497	0.654496	0.654496
(0.6,0.4)	0.904480	0.904491	0.904491
(0.5,0.5)	0.999997	0.999980	0.999980
(0.4,0.6)	0.904480	0.904491	0.904491
(0.3,0.7)	0.654497	0.654496	0.654496
(0.2,0.8)	0.345488	0.345485	0.345485
(0.1,0.9)	0.095487	0.095490	0.095490
(0.0,1.0)	0.000000	0.000000	0.000000
(0.0,0.0)	0.00000	0.000000	0.000000
(0.1,0.1)	0.095487	0.095490	0.095490
(0.2,0.2)	0.345488	0.345485	0.345485
(0.3,0.3)	0.654497	0.654496	0.654496
(0.4,0.4)	0.904480	0.904491	0.904491
(0.5,0.5)	0.999997	0.999980	0.999980
(0.6,0.6)	0.904480	0.904491	0.904491
(0.7,0.7)	0.654497	0.654496	0.654496
(0.8,0.8)	0.345488	0.345485	0.345485
(0.9,0.9)	0.095487	0.095490	0.095490
(1.0,1.0)	0.000000	0.000000	0.000000

Table 7. Comparison of the error norms L_2 and L_∞ of the model problem with results from [23]

	Number of Elements	t=0.0		t=1.0	
		L_2	L_∞	L_2	L_∞
Cubic [23]	2 × 2	4.57634×10^{-3}	7.24231×10^{-3}	4.57634×10^{-3}	7.24234×10^{-3}
	4 × 4	6.17439×10^{-4}	1.30443×10^{-3}	6.17436×10^{-4}	1.30444×10^{-3}
	10 × 10	3.20014×10^{-5}	5.42266×10^{-5}	3.28112×10^{-5}	5.41446×10^{-5}
Quintic	4 × 4	9.47159×10^{-6}	1.68070×10^{-5}	9.47156×10^{-6}	1.68088×10^{-5}
	10 × 10	3.07748×10^{-7}	3.51590×10^{-7}	3.37081×10^{-8}	6.86755×10^{-8}

In order to measure how good the numerical solutions obtained by the Galerkin finite element method with the bi-quintic B-spline basis functions, the error norms L_2 and L_∞ defined as

$$L_2 = \|u-u_{nm}\|_2 = \frac{\sqrt{\sum_{i=1}^{n_{ip}} |u_i-(u_{nm})_i|^2}}{\sqrt{\sum_{i=1}^{n_{ip}} |u_i|^2}},$$

$$L_\infty = \|u-u_{nm}\|_\infty = \max_{0<i<n_{ip}} |u_i-(u_{nm})_i|$$

are computed and tabulated in Table 7. In L_2 and L_∞, n_{ip} is the number of inner points, u_i and $(u_{nm})_i$ are the exact and approximate solutions at the point *i*., respectively. The error norms L_2 and L_∞ are computed by taking the values u_i and $(u_{nm})_i$ at $41\times 41 = 1681(=n_{ip})$ points obtained by dividing the region $D = [0, 1] \times [0, 1]$ into 40 equal elements in the directions x and y. As seen from the table, the approximate solutions become better as the number of elements increase.

4. Conclusion

In this paper, a modified bi-quintic B-spline finite element method is proposed and successfully applied to two dimensional Diffusion problem to obtain its numerical solutions. The agreement between our numerical results and the exact solution is satisfactorily good. The obtained numerical results showed that the present method is a remarkably successful numerical technique and can also be applied to a large number of physically important two dimensional non-linear problems.

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