International Journal of Partial Differential Equations and Applications, 2017, Vol. 5, No. 1, 10-18
Available online at http://pubs.sciepub.com/ijpdea/5/1/2

DOI:10.12691/ijpdea-5-1-2

On Instability of Steady–State Three–Dimensional Flows of an Ideal Compressible Fluid

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Abstract The problem on linear stability of stationary spatial flows of an inviscid compressible fluid entirely occupying a certain volume with quiescent solid impenetrable boundary in absence of external mass forces is studied. Applying the direct Lyapunov method, such flows are proved to be absolutely unstable under small three–dimensional (3D) perturbations. Constructive conditions for linear practical instability are obtained. The a priori exponential lower estimate for the growth of the considered perturbations in time is found.

Keywords: *an ideal compressible fluid, steady–state spatial flows, small 3D perturbations, the direct Lyapunov method, absolute linear theoretical instability, conditions for linear practical instability, a priori exponential lower estimate*

Cite This Article: Yuriy G. Gubarev, “On Instability of Steady–State Three–Dimensional Flows of an Ideal Compressible Fluid.” *International Journal of Partial Differential Equations and Applications*, vol. 5, no. 1 (2017): 10-18. doi: 10.12691/ijpdea-5-1-2.

1. Introduction

The problem on linear stability of stationary spatial flows of an inviscid compressible fluid continues to remain one of classical problems for modern mathematical theory of hydrodynamic stability (e.g. [1-6]).

Scientific literature devoted to the problem is so extensive and diverse that the author does not even think about any detailed review of it. Nevertheless, the author gives further a short analysis of [1,3]. These references show a current level of development in solving the problem stated and explain the actuality of this paper.

In [1], the variational principle for the integral of the total energy for steady–state 3D flows of an ideal compressible fluid entirely occupying a certain uniformly rotating volume with solid impenetrable boundary under gravity field is formulated. By means of the variational principle from [1], the Lyapunov functional is calculated as the second variation for the sum of integral of the total energy and integral of the product of density at arbitrary function of two dependent variables (entropy and potential vortex) within the following conditions: 1) level surfaces of the stationary field of an entropy are not tangent anywhere to level surfaces of the stationary field of a potential vortex; 2) solid impenetrable boundary of the studied uniformly rotating volume entirely occupied with the fluid is one of a constant entropy surfaces. Unfortunately, in the case of small spatial perturbations, the Lyapunov functional from [1] doesn’t satisfy property of sign–defineteness/sign–constantness for the studied steady–state 3D flows of an inviscid compressible fluid. That’s why, such stationary flows being stable under the perturbations are not found in [1].

In [3], the variational principle for integral of the total energy for steady–state spatial flows of an ideal compressible fluid entirely occupying a certain volume with quiescent solid impenetrable boundary in absence of external mass forces is formulated. By means of the variational principle from [3], the Lyapunov functional is calculated as the second variation for the bundle of integrals of the total energy, the product of density at arbitrary function of two dependent variables (entropy and potential vortex) and some constant value multiplied by the product of density at potential vortex. The calculation is made under the conditions 1), 2) of [1] (without uniform rotation) and two additional conditions: 3) admissible values of density are limited, 4) modulus of the gradient of perturbations for entropy field is not more than modulus of the same perturbations multiplied by the given positive constant (see (8BD) and (8k+) in [[3], p. 79 and 81]). For the particular class of small 3D perturbations considered in [3], the Lyapunov functional satisfies property of sign–defineteness/sign–constantness under some assumptions on the stationary spatial flows of an inviscid compressible fluid. Hence, steady–state flows studied in [3] are stable under the perturbations. Unfortunately, as it is mentioned in [[3], p. 79], the conditions 3) and 4) are dynamically contradictory and they will not be satisfied during the whole time.

Other results on the problem of linear stability for stationary spatial flows of an ideal compressible fluid known to the author are close to the results of [1] or [3].

We will consider the problem on linear stability of steady–state 3D flows of an inviscid compressible fluid in formulation of [3]. The main goal of the article is to prove absolute instability of stationary spatial flows of an ideal compressible fluid entirely occupying a certain volume with quiescent solid impenetrable boundary under small 3D perturbations in absence of external mass forces. Also, we will receive constructive conditions for linear practical instability and, as a sequence, the a priori exponential lower estimate for the growth of the studied perturbations in time.

To state the above results, we will apply the original analytical technique of [7]. This technique has been used successfully for a wide range of linear problems in mathematical theory of hydrodynamic stability both states of equilibrium (rest) and steady–state flows of gases, fluids, and plasma (e.g. [6,8-20]). The heart of the technique (e.g. [6-20]) is an algorithmic construction of the Lyapunov functionals increasing along the solutions to the mixed problems for small perturbations in time. Applying the technique, we obtain results about theoretical (on semi–infinite temporal intervals) (e.g. [21,22,23]) and practical (on finite time intervals) (e.g. [24,25]) instability of equilibrium (rest) states and stationary flows of gases, fluids, and plasma under small perturbations. To state this, we don't need to know explicit form of solutions to the linearized initial–boundary value problems.

In section 2, we formulate exact mixed problem and consider its stationary solutions corresponding to steady–state spatial flows of an inviscid compressible fluid entirely occupying a certain volume with quiescent solid impenetrable boundary in absence of external mass forces. In section 3, we state some properties of the initial–boundary value problem – linearization of exact mixed problem in the neighborhood of its stationary solutions. Finally, the absolute instability on linear approximation for steady–state spatial flows of an ideal compressible fluid entirely occupying a certain volume with quiescent solid impenetrable boundary in absence of external mass forces under 3D perturbations is proved in section 4 by direct construction of the a priori exponential lower estimate for the growth of small spatial perturbations in time.

2. Formulation of Exact Problem

We study 3D flows of an inviscid compressible fluid satisfying the state equation in general form and entirely occupying a certain volume τ with quiescent solid impenetrable boundary $\partial\tau$ in absence of external mass forces.

The flows are described by the solutions to the initial–boundary value problem in the form [3]:

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{1}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0,\quad \frac{\partial s}{\partial t}+u_i\frac{\partial s}{\partial x_i}=0$$

$$e=e(\rho,s):\, de=Tds+\frac{p}{\rho^2}d\rho \ \text{ in } \ \tau$$

$$u_i n_i=0 \text{ on } \partial\tau;\, u_i(\mathbf{x},0)=u_{i0}(\mathbf{x})$$

$$\rho(\mathbf{x},0)=\rho_0(\mathbf{x}),\quad s(\mathbf{x},0)=s_0(\mathbf{x})$$

where $\mathbf{u}=(u_1,u_2,u_3)$, ρ, p, s, e, and T are fields of velocity, density, pressure, entropy, internal energy, and temperature, respectively; they are functions of independent variables: Cartesian coordinates $\mathbf{x}=(x_1,x_2,x_3)$ and the time t; $\mathbf{n}=(n_1,n_2,n_3)$ is the unit outward normal to the boundary $\partial\tau$ of the volume τ; $\mathbf{u}_0(\mathbf{x})=(u_{10},u_{20},u_{30})$, ρ_0, and s_0 are initial fields of velocity, density, and entropy of the investigated ideal compressible fluid. We assume the vector function $\mathbf{u}_0$ to satisfy boundary condition of the mixed problem (1) and all functions concerned with the initial–boundary value problem (1) to possess necessary degree of smoothness. Hereinafter, there is a summation from one to three on all repeating vector and tensor lower indices.

Further, we suppose that exact stationary solutions to the mixed problem (1) are of the form

$$u_i=U_i(\mathbf{x}),\quad \rho=\rho^0(\mathbf{x}),\quad p=P(\mathbf{x}) \tag{2}$$

$$s=s^0(\mathbf{x})$$

satisfying the relations

$$U_k\frac{\partial U_i}{\partial x_k}=-\frac{1}{\rho^0}\frac{\partial P}{\partial x_i},\quad \frac{\partial}{\partial x_i}\left(\rho^0 U_i\right)=0 \tag{3}$$

$$U_i\frac{\partial s^0}{\partial x_i}=0$$

$$e=e^0\left(\rho^0,s^0\right):\, de^0=T^0ds^0+\frac{P}{\rho^{02}}d\rho^0$$

inside volume τ and condition of impermeability

$$U_i n_i=0 \tag{4}$$

on its quiescent solid impenetrable boundary $\partial\tau$. Here $\mathbf{U}(\mathbf{x})=(U_1,U_2,U_3)$, ρ^0, P, s^0, e^0, and $T^0(\mathbf{x})$ are steady–state fields of velocity, density, pressure, entropy, internal energy, and temperature of the inviscid compressible fluid, respectively.

The exact stationary solutions to the mixed problem (1) correspond to stationary spatial flows of an ideal compressible fluid satisfying general equation of state and entirely occupying a certain volume τ with quiescent solid impenetrable boundary $\partial\tau$ in absence of external mass forces. We will study a stability of such solutions under small 3D perturbations.

Let us remark that the class of exact stationary solutions (2)–(4) to the initial–boundary value problem (1) isn't an empty set. Indeed, as volume τ could be either bounded or unbounded closed domain, there exist stationary solutions corresponding to shearing, rotating, screw, and the like steady–state spatial flows of the investigated inviscid compressible fluid (e.g. [26,27]).

3. Formulation of Linearized Problem

To find out whether the exact stationary solutions (2)–(4) to the mixed problem (1) are stable under small 3D perturbations, we study a linearized version of the problem in the neighborhood of these solutions:

$$\rho^0\left(\frac{\partial u_i'}{\partial t}+U_k\frac{\partial u_i'}{\partial x_k}+u_k'\frac{\partial U_i}{\partial x_k}\right)+\rho' U_k\frac{\partial U_i}{\partial x_k}$$
$$=-\frac{\partial p'}{\partial x_i},\quad \frac{\partial \rho'}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho^0 u_i'+\rho' U_i\right)=0 \tag{5}$$
$$\frac{\partial s'}{\partial t}+u_i'\frac{\partial s^0}{\partial x_i}+U_i\frac{\partial s'}{\partial x_i}=0;$$
$$e'=\left.\frac{\partial e}{\partial \rho}\right|_{\left(\rho^0,s^0\right)}\rho'+\left.\frac{\partial e}{\partial s}\right|_{\left(\rho^0,s^0\right)}s'$$
$$\text{in } \tau;\quad u_i' n_i=0 \text{ on } \partial\tau$$
$$u_i'(\mathbf{x},0)=u_{i0}'(\mathbf{x}),\quad \rho'(\mathbf{x},0)=\rho_0'(\mathbf{x})$$
$$s'(\mathbf{x},0)=s_0'(\mathbf{x})$$

where $\mathbf{u}'=(u_1',u_2',u_3')$, ρ', p', s' , and e' are small perturbations for fields of velocity, density, pressure, entropy, and internal energy, respectively; they are functions of independent variables: Cartesian coordinates $\mathbf{x}$ and the time t; $\mathbf{u}_0'(\mathbf{x})=(u_{10}',u_{20}',u_{30}')$, ρ_0', and s_0' are initial small perturbations for fields of velocity, density, and entropy of the ideal compressible fluid, respectively. We assume the vector function $\mathbf{u}_0'$ to satisfy boundary condition of the mixed problem (5) and the functions ρ', p', s', and the vector function $\mathbf{u}'$ to possess necessary degree of smoothness.

Unfortunately, as it is known to the author, an analogue of the total energy integral for the initial–boundary value problem (5) is still not found. There is only the analogue of the functional of the total energy for a subclass of the solutions to the mixed problem (5) satisfying the following conditions: 1) quiescent solid impenetrable boundary $\partial\tau$ of the considered volume τ entirely occupying by an inviscid compressible fluid is a constant entropy surface; 2) admissible values of density are limited; 3) modulus of the gradient of perturbations for entropy field is not more than modulus of the same perturbations multiplied by the given positive constant [3]. Under the conditions 1)–3), the analogue of integral of the total energy (see (8SV4) in [[3], p. 80]) turns out to be positive definite for the particular class (see (8FS1)–(8FS4) in [[3], p. 81 and 82]) of the exact stationary solutions (2)–(4) to the initial–boundary value problem (1). However, the relations (8FS1)–(8FS4) couldn't be considered as sufficient conditions for linear stability, since the conditions 2), 3) are dynamically contradictory and they will not be satisfied during the whole time.

In light of the above, we can state the hypothesis of absolute instability in the linear approximation under 3D perturbations (5) for the exact stationary solutions (2)–(4) to the initial–boundary value problem (1) corresponding to steady–state spatial flows of an ideal compressible fluid under study.

To prove the linear instability for any of exact stationary solutions (2)–(4) to the mixed problem (1) under spatial perturbations (5), we will find a perturbation growing exponentially or faster in time. The required perturbation will be found further not among all evolutionary solutions to the initial–boundary value problem (5) but only among so–called incompressible "isovortical" small 3D perturbations (e.g. [6,7,9,16,19]).

Incompressible "isovortical" small spatial perturbations (5) can be well expressed by the Lagrangian displacements field $\boldsymbol{\xi}(\mathbf{x},t)=(\xi_1,\xi_2,\xi_3)$ (e.g. [28]):

$$\frac{\partial \xi_i}{\partial t}=u_i'+\xi_k\frac{\partial U_i}{\partial x_k}-U_k\frac{\partial \xi_i}{\partial x_k}$$
$$\frac{\partial \xi_i}{\partial x_i}=0;\quad \frac{\partial u_i'}{\partial x_i}=-\xi_i\frac{\partial^2 U_k}{\partial x_i\partial x_k} \tag{6}$$

Taking equations (6) into account, the mixed problem (5) can be reformulated as

$$\rho^0\left[\frac{\partial^2\xi_i}{\partial t^2}+2U_k\frac{\partial^2\xi_i}{\partial t\partial x_k}+U_k\frac{\partial}{\partial x_k}\left(U_m\frac{\partial \xi_i}{\partial x_m}\right)\right]$$
$$=\frac{\partial \xi_m}{\partial x_i}\frac{\partial P}{\partial x_m};\quad \rho'=-\xi_i\frac{\partial\rho^0}{\partial x_i},\ s'=-\xi_i\frac{\partial s^0}{\partial x_i} \tag{7}$$
$$p'=-\xi_i\frac{\partial P}{\partial x_i},\quad e'=-\xi_i\frac{\partial e^0}{\partial x_i}\text{ in }\tau;\ \xi_i n_i=0\text{ on }\partial\tau$$
$$\xi_i(\mathbf{x},0)=\xi_{i0}(\mathbf{x}),\quad \frac{\partial\xi_i}{\partial t}(\mathbf{x},0)=\left(\frac{\partial\xi_i}{\partial t}\right)_0(\mathbf{x}).$$

Here $\boldsymbol{\xi}_0(\mathbf{x})=(\xi_{10},\xi_{20},\xi_{30})$ is initial field of Lagrangian displacements and $(\partial\boldsymbol{\xi}/\partial t)_0(\mathbf{x})$ is initial partial time derivative of the first order from the Lagrangian displacements field. We assume the vector function $(\partial\boldsymbol{\xi}/\partial t)_0$ to satisfy the third equality of (6) and the vector function $\boldsymbol{\xi}_0$ to satisfy the second and the third equations of (6) and all but the first equalities of (7).

Although the incompressibility assumption for "isovortical" small 3D perturbations (6) redefines the initial–boundary value problem (7), it does not make the problem unsolvable. Moreover, the subclass of incompressible "isovortical" small spatial perturbations (6), (7) is not empty, it includes, f.e., sound and entropy–vortex waves (e.g. [26]).

4. Lyapunov Functional

Let us introduce into the study an auxiliary integral in the form [6]

$$M=M(t)\equiv\int_\tau \rho^0\xi_i\xi_i d\tau\geq 0 \tag{8}$$

where $d\tau\equiv dx_1dx_2dx_3$ is an element of the flow domain τ of an inviscid compressible fluid under consideration. This functional (virial in terminology of [28]) is the volume integral over domain τ from the product of steady–state field of density ρ^0 (2) at the squared distance ξ_i^2 (6) between the dislocation places of the same fluid particles and at the same time points on trajectories of the perturbed (6), (7) and current lines of the steady–state (2)–(4) flows of the studied ideal compressible fluid in the phase space of the solutions to the linearized initial–boundary value problem (5), respectively. The volume τ being an unbounded closed domain, the incompressible "isovortical" small 3D perturbations (6), (7) are considered as either periodic or damped towards infinity points, so that the functional M (8) and other integrals arising in the further research exist.

Further, the virial $M(t)$ is differentiated one and two times in its argument along the solutions to the mixed problem (6), (7):

$$M' = 2\int_\tau \rho^0 \xi_i \left(\frac{\partial \xi_i}{\partial t} + U_k \frac{\partial \xi_i}{\partial x_k} \right) d\tau \tag{9}$$

$$M'' = 2\int_\tau \left[\rho^0 \left(\frac{\partial \xi_i}{\partial t} + U_k \frac{\partial \xi_i}{\partial x_k} \right)^2 - \xi_i \xi_k \frac{\partial^2 P}{\partial x_i \partial x_k} \right] d\tau$$

(hereinafter, the prime denotes ordinary time derivative).

Now, by means of (8), (9), we write down an equality of the form

$$M'' - 2\lambda M' + 2\lambda^2 M = 2\int_\tau \rho^0 \left(\frac{\partial \xi_i}{\partial t} + U_k \frac{\partial \xi_i}{\partial x_k} - \lambda \xi_i \right)^2 d\tau - 2\int_\tau \xi_i \xi_k \frac{\partial^2 P}{\partial x_i \partial x_k} d\tau \tag{10}$$

where λ is a constant. Discarding the first integral (due to its non–negativity) in RHS of (10), we transform the equality into the inequality:

$$M'' - 2\lambda M' + 2\lambda^2 M \geq -2\int_\tau \xi_i \xi_k \frac{\partial^2 P}{\partial x_i \partial x_k} d\tau \tag{11}$$

It is easy to see that, without loss of generality, the ratios

$$-\alpha \xi_m^2 \leq \xi_i \xi_k \frac{\partial^2 P}{\partial x_i \partial x_k} \leq \alpha \xi_m^2 \tag{12}$$

are valid for the exact stationary solutions (2)–(4) to the initial–boundary value problem (1) (here α is a positive constant). Using (11) and the right inequality of (12), we obtain the key relation — the basic differential inequality (e.g. [7,9]):

$$\begin{aligned} &M'' - 2\lambda M' + 2\lambda^2 M \\ &\geq -2\int_\tau \xi_i \xi_k \frac{\partial^2 P}{\partial x_i \partial x_k} d\tau \geq -2\alpha M \\ &M'' - 2\lambda M' + 2\lambda^2 M \geq -2\alpha M \\ &M'' - 2\lambda M' + 2\left(\lambda^2 + \alpha\right) M \geq 0. \end{aligned} \tag{13}$$

Proposition. The exact stationary solutions (2)–(4) to the mixed problem (1) are absolutely unstable under incompressible "isovortical" small spatial perturbations (6), (7).

Proof of the **Proposition** is carried out by the direct integration of (13) under the condition that $\lambda > 0$ (e.g. [7,13,16,19]).

Unfortunately, the differential operator of (13) is not positive on the time half–interval $[0, +\infty)$ (e.g. [29]). Thus, the relation (13) couldn't be integrated on the half–interval by the Chaplygin method [30].

Our proof strategy is the following: 1) To represent the time half–interval $[0, +\infty)$ as a countable set of pairwise disjoint half–intervals $[a, b)$ of positivity for the differential operator of (13) (a and b are two points in time, $b > a$); 2) To integrate (13) on each time half–interval from 1) by the Chaplygin method; 3) Analyzing the conditions at the left ends of the time half–intervals from 1), to characterize the initial data for growing solutions to the initial–boundary value problem (6), (7); 4) Using the results from 2), to calculate the a priori lower estimate for growing solutions to the mixed problem (6), (7) on the time half–interval $[0, +\infty)$. The latter implies the exponential (or faster) growth of the solutions.

The inequality (13) can be formally integrated, f.e., on the following time half–intervals

$$t \in \left[\frac{2\pi n}{\sqrt{\lambda^2 + 2\alpha}}, \frac{\pi}{2\sqrt{\lambda^2 + 2\alpha}} + \frac{2\pi n}{\sqrt{\lambda^2 + 2\alpha}} \right) \\ n = 0, 1, 2, \ldots \tag{14}$$

Let us consider some substitutions for the functional M (8):

1) $M_1(t) \equiv \exp(-\lambda t) M(t)$: $M_1''(t) + \left(\lambda^2 + 2\alpha\right) M_1(t) \geq 0$

2) $M_2(t) \equiv \dfrac{M_1(t)}{\cos t\sqrt{\lambda^2 + 2\alpha}}$

$$\frac{d}{dt}\left[M_2'(t) \cos t\sqrt{\lambda^2 + 2\alpha} \right] \\ -\sqrt{\lambda^2 + 2\alpha} M_2'(t) \times \sin t\sqrt{\lambda^2 + 2\alpha} \geq 0$$

3) $M_3(t) \equiv M_2'(t) \cos^2 t\sqrt{\lambda^2 + 2\alpha}$: $M_3'(t) \geq 0$.

By the integrating the differential inequality for M_3 and the inverse substitutions, we deduce the inequality

$$M(t) \geq \left(\begin{aligned} &A_{1n} \sin t\sqrt{\lambda^2 + 2\alpha} \\ &+ A_{2n} \cos t\sqrt{\lambda^2 + 2\alpha} \end{aligned} \right) \times \exp \lambda t \tag{15}$$

where A_{1n} and A_{2n} are certain constants.

Non–strictness of the inequality (15) allows us to express the constants A_{1n} and A_{2n} $(n = 0, 1, 2, \ldots)$ in terms of the values of M and $M'(t)$ in time points $t_n \equiv 2\pi n \big/ \sqrt{\lambda^2 + 2\alpha}$. Thus, the inequality (15) could be rewritten as

$$M(t) \geq f(t) \tag{16}$$

$$f(t) \equiv \left\{ \begin{aligned} &M(t_n) \cos t\sqrt{\lambda^2 + 2\alpha} \\ &+ \frac{1}{\sqrt{\lambda^2 + 2\alpha}} \times \left[\begin{aligned} &M'(t_n) \\ &-\lambda M(t_n) \end{aligned} \right] \sin t\sqrt{\lambda^2 + 2\alpha} \end{aligned} \right\} \\ \times \exp \lambda (t - t_n).$$

To explain while the integration (13) on the time half–intervals (14) is correct, we have to calculate ordinary derivative of f at t:

$$f'(t)=\begin{Bmatrix} M'(t_n)\cos t\sqrt{\lambda^2+2\alpha} \\ +\begin{Bmatrix} \frac{\lambda}{\sqrt{\lambda^2+2\alpha}}\left[M'(t_n)-\lambda M(t_n)\right] \\ -\sqrt{\lambda^2+2\alpha}M(t_n) \end{Bmatrix} \\ \times\sin t\sqrt{\lambda^2+2\alpha} \end{Bmatrix} \times\exp\lambda(t-t_n). \quad (17)$$

Taking relations (16) and (17) into account, we can say that the function $f(t)$ is positive and strictly increasing on the time half–intervals (14) if and only if the inequalities

$$M(t_n)>0, \quad M'(t_n)\geq 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(t_n) \quad (18)$$

are hold (e.g. [31]). The inequalities guarantee the correctness of the integrating (13) on the time half–intervals (14).

Since the time half–intervals (14) are mutually disjoint, the values of M and $M'(t)$ on its left ends can be chosen relatively arbitrarily. In particular, these values can be taken in the form

$$M(t_n)\equiv M(0)\exp\lambda t_n, \quad M'(t_n)\equiv M'(0)\exp\lambda t_n.$$

Hence, the inequalities (18) are valid if and only if

$$M(0)>0, \quad M'(0)\geq 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(0)$$

and the function $f(t)$ is equal to

$$f(t)=\begin{bmatrix} M(0)\cos t\sqrt{\lambda^2+2\alpha} \\ +\frac{1}{\sqrt{\lambda^2+2\alpha}}\begin{Bmatrix} M'(0) \\ -\lambda M(0)\end{Bmatrix}\sin t\sqrt{\lambda^2+2\alpha}\end{bmatrix}\exp\lambda t.$$

Analogous arguments could be applied if one needs to integrate (13) on all other time half–intervals. The integrating (13) on the remaining time half–intervals is written below in the form of illustrative calculations without any comments $\left(t_* \equiv \pi/2\sqrt{\lambda^2+2\alpha}\right)$:

a) $t\in\left[t_*+t_n, 2t_*+t_n\right); \quad n=0,1,2,\ldots$

1) $M_1(t)=\exp(-\lambda t)M(t)$

$$M_1''(t)+\left(\lambda^2+2\alpha\right)M_1(t)\geq 0$$

2) $M_2(t)=\dfrac{M_1(t)}{\cos t\sqrt{\lambda^2+2\alpha}}$

$$\frac{d}{dt}\left[M_2'(t)\cos t\sqrt{\lambda^2+2\alpha}\right]-\sqrt{\lambda^2+2\alpha}M_2'(t)\times\sin t\sqrt{\lambda^2+2\alpha}\geq 0$$

3) $M_3(t)=M_2'(t)\cos^2 t\sqrt{\lambda^2+2\alpha}: \quad M_3'(t)\leq 0$

4) $M(t)\geq\begin{pmatrix} A_{3n}\sin t\sqrt{\lambda^2+2\alpha} \\ +A_{4n}\cos t\sqrt{\lambda^2+2\alpha}\end{pmatrix}\exp\lambda t;$

A_{3n}, A_{4n} — const

5) $M(t)\geq f_1(t)$

$$f_1(t)\equiv\Big\{M(t_*+t_n)\sin t\sqrt{\lambda^2+2\alpha}-\frac{1}{\sqrt{\lambda^2+2\alpha}}\times\left[M'(t_*+t_n)-\lambda M(t_*+t_n)\right]\cos t\sqrt{\lambda^2+2\alpha}\Big\}\times\exp\lambda(t-t_*-t_n),$$

$$f_1'(t)=\Bigg(M'(t_*+t_n)\times\sin t\sqrt{\lambda^2+2\alpha}-\begin{Bmatrix}\frac{\lambda}{\sqrt{\lambda^2+2\alpha}}\left[M'(t_*+t_n)-\lambda M(t_*+t_n)\right] \\ -\sqrt{\lambda^2+2\alpha}M(t_*+t_n)\end{Bmatrix}\times\cos t\sqrt{\lambda^2+2\alpha}\Bigg)\exp\lambda(t-t_*-t_n)$$

6) $M(t_*+t_n)>0, M'(t_*+t_n)\geq 2\left(\lambda+\frac{\alpha}{\lambda}\right)\times M(t_*+t_n)$

7) $M(t_*+t_n)\equiv M(0)\exp\lambda(t_*+t_n)$

$M'(t_*+t_n)\equiv M'(0)\exp\lambda(t_*+t_n); \quad M(0)>0$

$M'(0)\geq 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(0);$

$$f_1(t)=\begin{pmatrix} M(0)\times\sin t\sqrt{\lambda^2+2\alpha} \\ -\frac{1}{\sqrt{\lambda^2+2\alpha}}\left\{M'(0)-\lambda M(0)\right\} \\ \times\cos t\sqrt{\lambda^2+2\alpha}\end{pmatrix}\exp\lambda t$$

b) $t\in\left[2t_*+t_n, 3t_*+t_n\right); \quad n=0,1,2,\ldots$

1) $M_1(t)=\exp(-\lambda t)M(t)$

$$M_1''(t)+\left(\lambda^2+2\alpha\right)M_1(t)\geq 0$$

2) $M_2(t)=\dfrac{M_1(t)}{\cos t\sqrt{\lambda^2+2\alpha}}$

$$\frac{d}{dt}\left[M_2'(t)\cos t\sqrt{\lambda^2+2\alpha}\right]-\sqrt{\lambda^2+2\alpha}M_2'(t)\times\sin t\sqrt{\lambda^2+2\alpha}\geq 0$$

3) $M_3(t)=M_2'(t)\cos^2 t\sqrt{\lambda^2+2\alpha}: \quad M_3'(t)\leq 0$

4) $M(t)\geq\begin{pmatrix} A_{5n}\sin t\sqrt{\lambda^2+2\alpha} \\ +A_{6n}\times\cos t\sqrt{\lambda^2+2\alpha}\end{pmatrix}\exp\lambda t;$

A_{5n}, A_{6n} — const

5) $M(t)\geq f_2(t)$

$$f_2(t) \equiv -\left\{\begin{array}{l} M(2t_* + t_n)\cos t\sqrt{\lambda^2+2\alpha} \\ +\frac{1}{\sqrt{\lambda^2+2\alpha}}\left[\begin{array}{l} M'(2t_*+t_n) \\ -\lambda M(2t_*+t_n)\end{array}\right] \\ \sin t\sqrt{\lambda^2+2\alpha}\end{array}\right\}$$
$$\times \exp\lambda(t-2t_*-t_n),$$

$$f_2'(t) = -\left(\begin{array}{l} M'(2t_*+t_n)\cos t\sqrt{\lambda^2+2\alpha} \\ +\left\{\begin{array}{l}\frac{\lambda}{\sqrt{\lambda^2+2\alpha}}\left[\begin{array}{l} M'(2t_*+t_n) \\ -\lambda M(2t_*+t_n)\end{array}\right] \\ -\sqrt{\lambda^2+2\alpha}M(2t_*+t_n)\end{array}\right\} \\ \sin t\sqrt{\lambda^2+2\alpha}\end{array}\right)$$
$$\times \exp\lambda(t-2t_*-t_n)$$

6) $M(2t_*+t_n) > 0,$

$$M'(2t_*+t_n) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(2t_*+t_n)$$

7) $M(2t_*+t_n) \equiv M(0)\exp\lambda(2t_*+t_n)$

$$M'(2t_*+t_n) \equiv M'(0)\exp\lambda(2t_*+t_n);\ M(0) > 0$$

$$M'(0) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(0);$$

$$f_2(t) = -\left(\begin{array}{l} M(0)\times\cos t\sqrt{\lambda^2+2\alpha} \\ +\frac{1}{\sqrt{\lambda^2+2\alpha}}\{M'(0)-\lambda M(0)\} \\ \times\sin t\sqrt{\lambda^2+2\alpha}\end{array}\right)\exp\lambda t$$

c) $t \in [3t_*+t_n, 4t_*+t_n);\quad n = 0,1,2,...$

1) $M_1(t) = \exp(-\lambda t)M(t)$

$$M_1''(t) + \left(\lambda^2+2\alpha\right)M_1(t) \ge 0$$

2) $M_2(t) = \dfrac{M_1(t)}{\cos t\sqrt{\lambda^2+2\alpha}}$

$$\frac{d}{dt}\left[M_2'(t)\cos t\sqrt{\lambda^2+2\alpha}\right]$$
$$-\sqrt{\lambda^2+2\alpha}M_2'(t)\sin t\sqrt{\lambda^2+2\alpha} \ge 0$$

3) $M_3(t) = M_2'(t)\cos^2 t\sqrt{\lambda^2+2\alpha} : M_3'(t) \ge 0$

4) $M(t) \ge \left(\begin{array}{l} A_{7n}\sin t\sqrt{\lambda^2+2\alpha} \\ +A_{8n}\times\cos t\sqrt{\lambda^2+2\alpha}\end{array}\right)\exp\lambda t;$

A_{7n}, A_{8n} — const

5) $M(t) \ge f_3(t)$

$$f_3(t) \equiv \left\{\begin{array}{l} -M(3t_*+t_n)\sin t\sqrt{\lambda^2+2\alpha} \\ +\frac{1}{\sqrt{\lambda^2+2\alpha}}\times\left[\begin{array}{l} M'(3t_*+t_n) \\ -\lambda M(3t_*+t_n)\end{array}\right]\cos t\sqrt{\lambda^2+2\alpha}\end{array}\right\}$$
$$\times \exp\lambda(t-3t_*-t_n),$$

$$f_3'(t) = \left(\begin{array}{l} -M'(3t_*+t_n)\times\sin t\sqrt{\lambda^2+2\alpha} \\ +\left\{\begin{array}{l}\frac{\lambda}{\sqrt{\lambda^2+2\alpha}}\left[\begin{array}{l} M'(3t_*+t_n) \\ -\lambda M(3t_*+t_n)\end{array}\right] \\ -\sqrt{\lambda^2+2\alpha}M(3t_*+t_n)\end{array}\right\}\cos t\sqrt{\lambda^2+2\alpha}\end{array}\right)$$
$$\times \exp\lambda(t-3t_*-t_n)$$

6) $M(3t_*+t_n) > 0,$

$$M'(3t_*+t_n) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)\times M(3t_*+t_n)$$

7) $M(3t_*+t_n) \equiv M(0)\exp\lambda(3t_*+t_n)$

$$M'(3t_*+t_n) \equiv M'(0)\exp\lambda(3t_*+t_n);\quad M(0) > 0$$

$$M'(0) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(0);$$

$$f_3(t) = \left(\begin{array}{l} -M(0)\sin t\sqrt{\lambda^2+2\alpha} \\ +\frac{1}{\sqrt{\lambda^2+2\alpha}}\{M'(0)-\lambda M(0)\} \\ \times\cos t\sqrt{\lambda^2+2\alpha}\end{array}\right)\exp\lambda t$$

Analyzing final expressions for the functions $f(t),\ f_i(t)\ (i = 1,2,3)$, we conclude that graphs of the functions on corresponding time intervals are curves lying inside the half–strip directed exponentially in time to infinity. Left ends of the curves are located at the lower boundary of the half–strip

$$g(t) \equiv M(0)\exp\lambda t$$

and right ones lie on its upper boundary

$$g_1(t) \equiv g_1(0)\exp\lambda t;\quad g_1(0) \equiv \frac{M'(0)-\lambda M(0)}{\sqrt{\lambda^2+2\alpha}}$$

(Figure 1).

Hence, the virial M (8) grows in time exponentially or faster.

Let us consider the key differential relation (13) together with the following conditions

$$M(nt_*) > 0;\quad n = 0,1,2,... \tag{19}$$

$$M'(nt_*) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(nt_*),$$
$$M(nt_*) \equiv M(0)\exp\lambda nt_*,$$
$$M'(nt_*) \equiv M'(0)\exp\lambda nt_*$$
$$M(0) > 0, M'(0) \ge 2\left(\lambda+\frac{\alpha}{\lambda}\right)M(0)$$

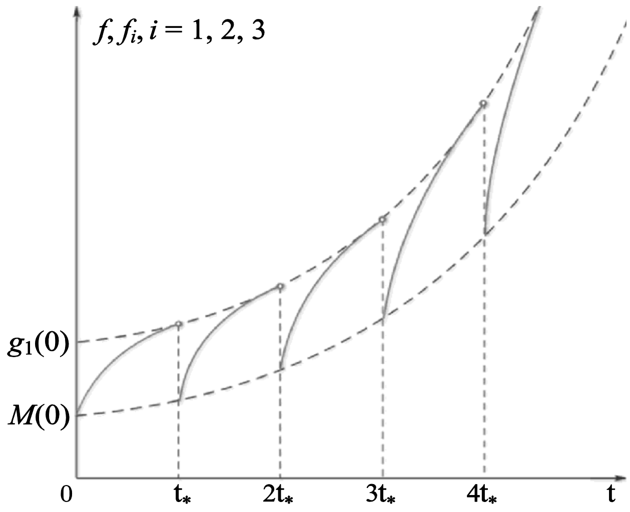

Figure 1. Graphical representation of the procedure for integrating key differential relation (13)

Summarizing the results on integrating basic differential inequality (13) on half–intervals of time $\left[nt_*, (n+1)t_*\right) (n = 0,1,2,...)$, and taking conditions (19) into account, we get the a priori exponential lower estimate

$$M(t) \geq C \exp \lambda t \tag{20}$$

of the growth in time of incompressible "isovortical" small 3D perturbations (6), (7) of the exact stationary solutions (2)–(4), (12) to the initial–boundary value problem (1). Here C is the known positive constant.

Now, let us clarify the connection between the executed integration of the basic differential inequality (13) and the properties of the solutions to the linearized mixed problem (6), (7).

Choosing the special initial data (see the identities (19)) at the left ends of the time half–intervals, one can find unified initial conditions (see the last pair of the inequalities (19)) for the key differential relation (13) and incompressible "isovortical" small spatial perturbations (6), (7) of the exact stationary solutions (2)–(4), (12) to the initial–boundary value problem (1). Moreover, such unified initial conditions prove the necessity of the positivity and the strict growth of the functions $f(t), f_i(t) (i = 1,2,3)$ (see the first pair of the inequalities (19)) on all considered half–intervals of time.

The particular class of the solutions to the linearized mixed problem (6), (7) growing in time due to the estimate (20) and considering with additional conditions

$$M(0) > 0, \quad M'(0) \geq 2\left(\lambda + \frac{\alpha}{\lambda}\right)M(0) \tag{21}$$

for initial data $\xi_0(\mathbf{x})$ and $(\partial\xi/\partial t)_0(\mathbf{x})$ is not an empty set.

Indeed, the initial–boundary value problem (6), (7) is linear, so it is solvable under incompressible "isovortical" small 3D perturbations in the form of normal waves (e.g. [28,32]). Further, since positively definite linear analogue of the total energy functional has been found to date only for one incomplete open subclass of the solutions to the mixed problem (5) [3], the initial–boundary value problem (6), (7) is also solvable under growing in time incompressible "isovortical" small spatial perturbations in the form of normal waves (e.g. [7,16,19]). Finally, due to arbitrariness of positive constant λ, every solution to the mixed problem (6), (7) growing in time and corresponding to incompressible "isovortical" small 3D perturbation in the form of normal wave will satisfy (13), (19), and (20) identically and automatically.

Hence, there are growing in time solutions to the linearized initial–boundary value problem (6), (7), (21) corresponding to incompressible "isovortical" small spatial perturbations in the form of normal waves.

All in all, according to the Lyapunov definition of unstable solution to a system of differential equations (e.g. [23]), the a priori exponential lower estimate (20) guarantees that at least one incompressible "isovortical" small 3D perturbation (6), (7) with initial conditions (21) of steady–state spatial flows (2)–(4), (12) of an inviscid compressible fluid will grow in time exponentially or faster. The estimate is obtained without any additional conditions on steady–state 3D flows (2)–(4), (12), so we state an absolute instability of the latter under incompressible "isovortical" small spatial perturbations (6), (7), (21).

Thereby, **Proposition is proved** and the conditional result on stability [3] is converted.

Let us discuss the results obtained. The first pair of inequalities (19) can be regarded as sufficient conditions for linear practical instability of stationary 3D flows (2)–(4), (12) of an ideal compressible fluid under incompressible “isovortical” small spatial perturbations (6), (7), (21) and as necessary and sufficient ones — under incompressible "isovortical" small 3D perturbations (6), (7), (21) in the form of normal waves (because λ is in the rest arbitrary). It is important that the conditions for linear practical instability are constructive. Therefore, one can apply them as testing and control mechanism in the course of conducting physical experiments, performing numerical calculations, and implementing technological processes (e.g. [7,16,19]).

At last, we remark that the virial M (8) is the required Lyapunov functional growing in time due to the equations of the linearized mixed problem (6), (7), (21). Relative freedom in the choice of positive constant λ in the exponent from RHS of (20) is an important property of the growth. Applying the property, any solution to the initial–boundary value problem (6), (7), (21) growing in time due to (20) can be considered as an analogue of Hadamard’s example of ill–posedness (e.g. [33]).

5. Conclusion

We have studied in the article the problem on linear stability of steady–state spatial flows (2)–(4), (12) of an inviscid compressible fluid entirely occupying a certain volume τ with quiescent solid impenetrable boundary $\partial\tau$ in absence of external mass forces.

Applying the direct Lyapunov method, it is proved that the flows are absolutely unstable under incompressible "isovortical" small 3D perturbations (6), (7), (21). Constructive conditions for linear practical instability are given (see first pair of inequalities (19)). The growth of incompressible "isovortical" small spatial perturbations in time follows from the constructed a priori exponential lower estimate (20). We have shown conditional nature of the result [3] on stability for the subclass of exact stationary solutions (2)–(4) to the initial–boundary value problem (1).

We have to remark that from the mathematical point of view the proved proposition is a priori, since the question of existence for solutions to the mixed (1), the boundary value (3), (4), the initial–boundary value (5) and the mixed (6), (7) problems for the systems of partial differential equations have not yet been investigated.

Finally, we want to list some points clarifying the connection between the results obtained in the article and the results received earlier by other authors:

1) Since there is no information on exact stationary solutions (2)–(4) to the initial–boundary value problem (1) in (13), we reasonably expect that the inequality (13) or its analogue will arise in studying other mathematical models of hydrodynamic type (e.g. [7-20]);

2) Depending on whether we should require additional conditions on the studied steady–state flows to deduce the differential relation (13) or its analogue, we can state the kind of instability for the flows: if yes then the conditions will be sufficient for instability of stationary flows; if not then the studied steady–state flows will be absolutely unstable (e.g. [6-20]);

3) The existence of differential inequalities of the form (13) in itself leads to immediate conversion of known sufficient conditions for linear stability of some stationary flows under consideration; in other words, one can find only necessary and sufficient conditions for linear stability in presence of basic differential relations of the type (13) (e.g. [7,8,10,11,12,13,15,17,18,20]);

4) Inequalities of the form (13) cannot be constructed if analogues of energy integrals for some linearized mixed problems in terms of the Lagrangian displacements field (e.g. [28]) are either sign–definite or sign–constant (e.g. [6-20]);

5) The obtained conditions for linear practical instability (see first pair of inequalities (19)) can be interpreted as computational algorithm constructed without discretization of the studied systems of differential equations; applying the conditions, it is possible to get numerical results which are in no way inferior to corresponding analytical results in accuracy, reliability, and veracity; therefore, any differences between numerical and analytical results are erased.

The author expects that the described in the article method for constructing Lyapunov functionals growing in time along the solutions to the studied linearized initial–boundary value problems will be useful for not yet solved linear problems in the mathematical theory of hydrodynamic stability.

Acknowledgements

The author expresses deep gratitude to A.M. Blokhin, S.K. Godunov, and V.Yu. Gubarev for fruitful discussion of this paper and useful advice.

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