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Study of a System of Convection-Diffusion-Reaction

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Abstract In this article, we are interested in the study of the existence of weak solutions of boundary value problem for the nonlinear elliptic system $\begin{cases}-div\left(a_1\left(x,v(x)\nabla u\right)\right)-div\left(G_1\left(x,\varphi_1(v)\right)\right)=f(x,u,v)\ in\ \Omega,\\ -div\left(a_2\left(x,u(x)\nabla v\right)\right)-div\left(G_2\left(x,\varphi_2(u)\right)\right)=g(x,u,v)\ in\ \Omega,\\ u=v=0\ on\ \partial\Omega\end{cases}$, where Ω is a bounded domain in $\mathbb{R}^N$ and a_1, a_2 are continuous functions . We use the Leray-Schauder degree theory under not linear for the three reasons: the terms of diffusion, convection and reaction, and the following condition on the last term f and g : $\begin{cases}\lim\limits_{s,|t|\to\infty} f(x,s,t)=\zeta_1^+,\ \lim\limits_{-s,|t|\to\infty} f(x,s,t)=\zeta_1^-,\\ \zeta_1^-,\zeta_1^+\in L^2(\Omega),\ \zeta_1^-\le f(x,s,t)\le\zeta_1^+,\end{cases}$ and $\begin{cases}\lim\limits_{s,|t|\to\infty} g(x,s,t)=\zeta_2^+,\ \lim\limits_{-s,|t|\to\infty} g(x,s,t)=\zeta_2^-,\\ \zeta_2^-,\zeta_2^+\in L^2(\Omega),\ \zeta_2^-\le g(x,s,t)\le\zeta_2^+.\end{cases}$.

Keywords: *topological degree, elliptic systems, homotopy*

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1. Introduction

This article is devoted to presenting the results of existence of solution for a nonlinear elliptic systems of partial differential equations, in a bounded domain of $\mathbb{R}^N$, with zero Dirichlet boundary conditions. These results are obtained by using Leray-Schauders topological degree and some tools of functional analysis. The corresponding scalar case considered in [6] wich has shown the existence of solutions to the problem $Au=\alpha u^+-\beta u^-+f(x,u)+h,$ where A is a self-adjoint operator with compact resolvent in $L^2(\Omega)$, $f(.,.)$ maps $\Omega\times\mathbb{R}$ into $\mathbb{R}$, such that $\lim\limits_{s\to\infty}\frac{f(x,s)}{s}=0$ and $[\alpha,\beta]\cap\sigma(A)=\lambda$, ($\lambda$ a simple eigenvalue of A). In this paper we establish the existence of weak solutions for the problem

$$\begin{cases}\begin{bmatrix}-div\left(a_1\left(x,v(x)\nabla u\right)\right)\\ -div\left(G_1\left(x,\varphi_1(v)\right)\right)\end{bmatrix}=f(x,u,v)\quad in\quad \Omega,\\ \begin{bmatrix}-div\left(a_2\left(x,u(x)\nabla v\right)\right)\\ -div\left(G_2\left(x,\varphi_2(u)\right)\right)\end{bmatrix}=g(x,u,v)\quad in\quad \Omega,\\ u=v=0\quad on\quad \partial\Omega,\end{cases}\tag{1.1}$$

Where Ω is a bounded domain in $\mathbb{R}^N\ (N\ge 1)$ wiht smooth boundary $\partial\Omega$ and a_1, a_2 are continuous functions satisfying the condition below:

$$\begin{cases}\alpha_i,\beta_i>0\\ \alpha_i\le a_i(x,s)\le\beta_i\quad \forall s\in\mathbb{R}\ \ a.e\ x\in\Omega\ \ for\ i=1,2.\end{cases}\tag{1.2}$$

In the case where $a_1=a_2=1$, $\varphi_1=\varphi_2=0$, we have the existence of the solution if l is a simple eigenvalue of $(-\Delta)$, see [9]. And in [10] studied the case where l is not an eigenvalue i.e $\lambda\notin\sigma(-\Delta)$, $\sigma(-\Delta)$ denotes the spectrum of $-\Delta$. The case no resonance was treated by Lakhal and Khodja (see [11]).

For the rest of this article, we suppose that

$$\begin{cases}G_i\in C^1\left(\bar{\Omega},\mathbb{R}^N\right),\\ divG_i=0\ \ for\ i=1,2\end{cases}\tag{1.3}$$

and

$$\begin{cases}\varphi_i\in C(\mathbb{R},\mathbb{R}),\\ |\varphi_i(s)|\le C_i|s|\quad for\ i=1,2\end{cases}\tag{1.4}$$

where C_i, are real positive constants.

We assume that f, g: $\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the carathéodory conditions, and verifying also the growth restriction defined below:

$$\begin{cases}|f(x,s,t)|\le C_1'(1+|s|+|t|),\\ |g(x,s,t)|\le C_2'(1+|s|+|t|),\end{cases}\tag{1.5}$$

where C_1', C_2' are real positive constants.

$$\begin{cases} \lim\limits_{s,|t|\to\infty} f(x,s,t)=\zeta_1^+, \quad \lim\limits_{-s,|t|\to\infty} f(x,s,t)=\zeta_1^-, \\ \zeta_1^-, \zeta_1^+ \in L^2(\Omega), \; \zeta_1^- \le f(x,s,t) \le \zeta_1^+, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \lim\limits_{s,|t|\to\infty} g(x,s,t)=\zeta_2^+, \quad \lim\limits_{-s,|t|\to\infty} g(x,s,t)=\zeta_2^-, \\ \zeta_2^-, \zeta_2^+ \in L^2(\Omega), \; \zeta_2^- \le g(x,s,t) \le \zeta_2^+. \end{cases} \tag{1.7}$$

Theorem 1.1. *Under the assumptions* (1.2), (1.3), (1.4), (1.5), (1.6) *and* (1.7), *there exists a solution to the problem* (1.1).

2. Preliminaries

Let us consider the space

$$U = H_0^1(\Omega)\times H_0^1(\Omega),$$

which is a Banach space endowed with the norm

$$\|(u,v)\|_U^2 = \|(u)\|_{H_0^1(\Omega)}^2 + \|(v)\|_{H_0^1(\Omega)}^2,$$

such as U^* its dual, and let us take $V = L^2(\Omega)\times L^2(\Omega)$. In the sequel, $\|.\|_{L^2(\Omega)}$ and $\|.\|_{H_0^1(\Omega)}$ will denote the usual normes on $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively. Recalling that the operator A, given by

$$Au = -\Delta u,$$

$$D(A) = \left\{u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\right\},$$

defines an inverse compact on $L^2(\Omega)$ and his spectrum is formed by the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ such that $|\lambda_k| \to +\infty$ and λ_1 the first eigenvalue is positive. Throughout this paper, we denote by λ a simple eigenvalue of A, φ is an eigenfunction associated to λ normalized in $L^2(\Omega)$, Pr designates the orthogonal projection of V on $\left(\varphi^\perp\right)^2$ ($\varphi^\perp$ is the orthogonal of φ in $L^2(\Omega)$). We recall the following proposition proved by T.Gallouet and O.Kavian (see [5]).

We give now a definition of weak solution.

Definition 2.1. We say $(u,v)\in\mathcal{U}$ is a weak solution for the system (1.1) if for any $(\varphi,\psi)\in U$ we have

$$\begin{cases} \left[\begin{array}{l}\int_\Omega a_1(v)\nabla_u.\nabla\varphi dx \\ +\int_\Omega G_1\varphi_1(v).\nabla\varphi dx\end{array}\right] = \int_\Omega f(u,v)\varphi dx, \\ \left[\begin{array}{l}\int_\Omega a_2(u)\nabla_v.\nabla\psi dx \\ +\int_\Omega G_2\varphi_2(u).\nabla\psi dx\end{array}\right] = \int_\Omega g(u,v)\psi dx. \end{cases} \tag{2.1}$$

We write the problem in the form

$$\begin{cases} (u,v)\in\mathcal{U}, \\ \int_\Omega a_1(v)\nabla_u.\nabla\varphi dx + \int_\Omega a_2(u)\nabla_v.\nabla\psi dx \\ = \left\langle L(u,v),(\varphi,\psi)_{\mathcal{U}^*,\mathcal{U}}\ \forall(\varphi,\psi)\right\rangle \in \mathcal{U}, \end{cases}$$

where $L(u,v)$ is, for $(u,v)\in V$, the element of $\mathcal{U}^*$ defined by

$$\begin{aligned} &\langle L(u,v),(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}} \\ &= -\int_\Omega G_1\varphi_1(v).\nabla\varphi dx - \int_\Omega G_2\varphi_2(u).\nabla\psi dx \\ &+\int_\Omega f(u,v)\varphi dx - \int_\Omega g(u,v)\psi dx. \end{aligned}$$

From (1.3), (1.4) and (1.5), it is clear that the application $L(u,v)$ is continuous to V in $\mathcal{U}^*$. For $S\in\mathcal{U}^*$, the linear problem

$$\begin{cases} (w_1,w_2)\in\mathcal{U}, \\ \int_\Omega a_1(v)\nabla_{w_1}.\nabla\varphi dx + \int_\Omega a_2(u)\nabla_{w_2}.\nabla\psi dx \\ = \langle S,(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}}, \end{cases} \tag{2.2}$$

has a unique solution $(w_1,w_2)\in\left(H_0^1(\Omega)\right)^2$. We note the operator $B_{u,v}$ who at S in $\mathcal{U}^*$ associates (w_1,w_2) solution of (2.2). To $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact, we deduced that the operator $B_{u,v}$ is compact to $\mathcal{U}^*$ in V.

The problem (1.1) is equivalent to solving the fixed point problem $(u,v)=B_{u,v}(L(u,v))$. So we show through topological degree, the following problem has a solution

$$\begin{cases} (u,v)\in V, \\ (u,v)=B_{u,v}(L(u,v)). \end{cases}$$

For $t\in[0,1]$, we put $H(t,u,v)=B_{u,v}(tL(u,v))\in V$. The mapping H is defined to $[0,1]\times V$ in V. For $R>0$, let us put

$$B_R = \left\{(u,v)\in V \text{ such that } \|(u,v)\|_V \le R\right\}.$$

Let us show now that

$\{H(t,u,v), t\in[0,1], (u,v)\in B_R\}$ is relatively compact $\in V$.

Let $R>0$, we assume that $\|(u,v)\|_V \le R$. We have $L(u,v)\in\mathcal{U}^*$, and

$$\begin{aligned} &\langle L(u,v),(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}} \\ &= -\int_\Omega G_1\varphi_1(v).\nabla\varphi dx - \int_\Omega G_2\varphi_2(u).\nabla\psi dx \\ &+\int_\Omega f(u,v)\varphi dx + \int_\Omega g(u,v)\psi dx. \end{aligned}$$

We want to estimate $\langle L(u,v),(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}}$.

$$
\begin{aligned}
&\langle L(u,v),(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}} \\
&\le \|G_1\|_\infty \|\varphi_1(v)\|_{L^2(\Omega)} \|\varphi\|_{H_0^1(\Omega)} \\
&\quad + \|G_2\|_\infty \|\varphi_2(u)\|_{L^2(\Omega)} \|\psi\|_{H_0^1(\Omega)} \\
&\quad + \|f(u,v)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + \|g(u,v)\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\
&\le \|G_1\|_\infty C_1 \|v\|_{L^2(\Omega)} \|\varphi\|_{H_0^1(\Omega)} + \|G_2\|_\infty C_2 \|u\|_{L^2(\Omega)} \|\psi\|_{H_0^1(\Omega)} \\
&\quad + C_\Omega C_1'\left(1+\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)\|\varphi\|_{H_0^1(\Omega)} \\
&\quad + C_\Omega C_2'\left(1+\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)\|\psi\|_{H_0^1(\Omega)} \\
&\le \|G_1\|_\infty C_1 \|(u,v)\|_V \|(\varphi,\psi)\|_{\mathcal{U}} + \|G_2\|_\infty C_2 \|(u,v)\|_V \|(\varphi,\psi)\|_{\mathcal{U}} \\
&\quad + C_\Omega C_1'\left(1+\|(u,v)\|_V + \|(u,v)\|_V\right)\|(\varphi,\psi)\|_{\mathcal{U}} \\
&\quad + C_\Omega C_2'\left(1+\|(u,v)\|_V + \|(u,v)\|_V\right)\|(\varphi,\psi)\|_{\mathcal{U}} \\
&\le \begin{pmatrix} \|G_1\|_\infty C_1 R + \|G_2\|_\infty C_2 R \\ + C_1' C_\Omega (1+2R) + C_2' C_\Omega (1+2R) \end{pmatrix} \|(\varphi,\psi)\|_{\mathcal{U}},
\end{aligned}
$$

where C_Ω depends only on Ω (and is given by the Poincar inequality). So $\forall t \in [0,1]$

$$
\begin{aligned}
&t\|L(u,v)\|_{\mathcal{U}^*} \\
&\le \|G_1\|_\infty C_1 R + \|G_2\|_\infty C_2 R \\
&+ C_1' C_\Omega (1+2R) + C_2' C_\Omega (1+2R).
\end{aligned}
$$

Let $H(t,u,v) = B_{u,v}(tL(u,v)) = (w_1,w_2)$ and show that there $\bar{R}$ depending only on R, G_i, C_Ω, C_i, C_i', α_i, for $i=1,2$ such that

$$\|H(t,u,v)\|_{\mathcal{U}} \le \bar{R}.$$

By definition, (w_1,w_2) is a solution of

$$
\begin{cases}
(w_1,w_2) \in \mathcal{U}, \\
\int_\Omega a_1(v) \nabla w_1 . \nabla \varphi dx + \int_\Omega a_2(u) \nabla w_2 . \nabla \psi dx \\
= \langle tL(u,v),(\varphi,\psi)\rangle_{\mathcal{U}^*,\mathcal{U}}.
\end{cases} \tag{2.3}
$$

Taking $\varphi = w_1$, $\psi = w_2$ in (2.3) we obtain

$$
\begin{aligned}
&\min(\alpha_1,\alpha_2)\|(w_1,w_2)\|_{\mathcal{U}}^2 \\
&\le \|tL(u,v)\|_{\mathcal{U}^*,\mathcal{U}} \|(w_1,w_2)\|_{\mathcal{U}} \le \tilde{R}\|(w_1,w_2)\|_{\mathcal{U}},
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{R} = {} & \|G_1\|_\infty C_1 R + \|G_2\|_\infty C_2 R \\
& + C_1' C_\Omega (1+2R) + C_2' C_\Omega (1+2R).
\end{aligned}
$$

We have

$$\|H(t,u,v)\|_{\mathcal{U}} = \|(w_1,w_2)\|_{\mathcal{U}} \le \frac{\tilde{R}}{\min(\alpha_1,\alpha_2)} = \tilde{R}.$$

By Rellich theorem, we deduce that the set $\{H(t,u,v), t\in[0,1], (u,v)\in \bar{B}_R\}$ is relatively compact in V, therefore $H(t,u,v):[0,1]\times V \to V$ is compact.

We now show that H is continous.

Proposition 2.1. The mapping H is continuous to $[0.1]\times \bar{B}_R$ in $\bar{B}_R$.

Proof. Let (t_n,u_n,v_n) converge to (t,u,v) in $[0,1]\times V$.

We want to show that

$$H(t_n,u_n,v_n) \to H(t,u,v) \, in \, V.$$

Let

$$(w_{n,1},w_{n,2}) = H(t_n,u_n,v_n),$$

and

$$(w_1,w_2) = H(t,u,v).$$

To show that

$$(w_{n,1},w_{n,2}) \to (w_1,w_2) \, in \, V,$$

seeking to pass to the limit on the following equation:

$$
\begin{cases}
(w_{n,1},w_{n,2}) \in \mathcal{U}, \\
\int_\Omega a_1(v_n) \nabla w_{n,1} . \nabla \varphi dx + \int_\Omega a_2(u_n) \nabla w_{n,2} . \nabla \psi dx \\
= -t_n \int_\Omega G_1 \varphi_1(v_n) . \nabla \varphi dx - t_n \int_\Omega G_2 \varphi_2(u_n) . \nabla \psi dx \\
+ t_n \int_\Omega f(u_n,v_n)\varphi dx - t_n \int_\Omega g(u_n,v_n)\psi dx.
\end{cases} \tag{2.4}
$$

We know that $(w_{n,1},w_{n,2})$ is bounded in $\mathcal{U}$, because (u_n,v_n) is bounded in V (this is a shown in the previous step: $\|(u_n,v_n)\|_V \le R$ then $\|(w_{n,1},w_{n,2})\|_{\mathcal{U}} \le \bar{R}$.)

The sequence $(w_{n,1},w_{n,2})_{n\in\mathbb{N}}$ is bounded in $\mathcal{U}$, therefore

$$
\begin{aligned}
&(w_{n,1},w_{n,2}) \to (\tilde{w}_1,\tilde{w}_2) \text{ in } \mathcal{U} \text{ weak and} \\
&(w_{n,1},w_{n,2}) \to (\tilde{w}_1,\tilde{w}_2) \text{ in } V, (u_n,v_n) \to (u,v) \text{ a.e.} \\
&\text{and } \exists\, H,G \in L^2(\Omega); |u_n| \le H \text{ a.e and } |v_n| \le G \text{ a.e.}
\end{aligned} \tag{2.5}
$$

Let $(\varphi,\psi)\in\mathcal{U}$; as $a_1(v_n) \to a_1(v)$ a.e therefore

$$a_1(v_n)\nabla\varphi \to a_1(v)\nabla\varphi \text{ a.e},$$

and

$$|a_1(v_n)\nabla\varphi| \le \beta_1 \nabla\varphi,$$

we have

$$a_1(v_n)\nabla\varphi \to a_1(v)\nabla\varphi \text{ in } L^2(\Omega).$$

But $\nabla w_{n,1} \to \nabla \tilde{w}_1$ in $(L^2(\Omega))^N$ weak. We have

$$\int_\Omega a_1(v_n)\nabla w_{n,1} \nabla\varphi dx \to \int_\Omega a_1(v)\nabla\tilde{w}_1 \nabla\varphi dx.$$

Similarly we have

$$\int_\Omega a_2(u_n)\nabla w_{n,2}\nabla\psi dx \to \int_\Omega a_2(u)\nabla\tilde{w}_2\nabla\psi dx.$$

Then we notice that

$$\varphi_1(v_n)\to\varphi_1(v)\,\text{a.e},$$

and

$$|\varphi_1(v_n)|\le C_1|v_n|\le C_1G,$$

thanks to Lebesgue's convergence theorem, we deduce that

$$\varphi_1(v_n)\to\varphi_1(v)\,\text{in}\, L^2(\Omega),$$

and

$$\int_\Omega G_1\varphi_1(v_n).\nabla\varphi dx\to\int_\Omega G_1\varphi_1(v).\nabla\varphi dx.$$

Similarly we have

$$\int_\Omega G_2\varphi_2(u_n).\nabla\psi dx\to\int_\Omega G_2\varphi_2(u).\nabla\psi dx.$$

Finally for the last term,

$$f(u_n,v_n)\to f(u,v)\,\text{a.e.}$$

By dominated convergence (from (1.5) and (2.5)) we have

$$f(u_n,v_n)\to f(u,v)\,\text{in}\, L^2(\Omega),$$

and consequently

$$\int_\Omega f(u_n,v_n)\varphi dx\to\int_\Omega f(u,v)\varphi dx.$$

Similarly we have

$$\int_\Omega g(u_n,v_n)\psi dx\to\int_\Omega g(u,v)\psi dx.$$

Passing to the limit in (2.4), we obtain

$$\begin{aligned}&\int_\Omega a_1(v)\nabla\tilde{w}_1\nabla\varphi dx+\int_\Omega a_2(u)\nabla\tilde{w}_2\nabla\psi dx\\&=-t\int_\Omega G_1\varphi_1(v).\nabla\varphi dx-t\int_\Omega G_2\varphi_2(u).\nabla\psi dx\\&\quad+t\int_\Omega f(u,v)\varphi dx+t\int_\Omega g(u,v)\psi dx,\end{aligned}$$

and therefore $(\tilde{w}_1,\tilde{w}_2)=H(t,u,v)=(w_1,w_2)$.

As $(w_{n,1},w_{n,2})\to(w_1,w_2)$ in V where $(w_{n,1},w_{n,2})=H(t_n,u_n,v_n)$ and $(w_1,w_2)=H(t,u,v)$, then the mapping H is continuous.

3. A Priori Bounds for Solutions of (1.1)

Let $t\in[0,1]$, and $(u,v)=H(t,u,v)=tB_{u,v}(L(u,v))$, that is to say

$$\begin{cases}(u,v)\in\mathcal{U},\\ \int_\Omega a_1(v)\nabla u\nabla\varphi dx+\int_\Omega a_2(u)\nabla v\nabla\psi dx\\ =-t\int_\Omega G_1\varphi_1(v).\nabla\varphi dx-t\int_\Omega G_2\varphi_2(u).\nabla\psi dx\\ \quad+t\int_\Omega f(u,v)\varphi dx+t\int_\Omega g(u,v)\psi dx,\ \forall(\varphi,\psi)\in\mathcal{U}.\end{cases}\tag{3.1}$$

For $s\in\mathbb{R}$, we put $\Phi_i(s)=\int_0^s\varphi_i(\zeta)d\zeta$ (Φ_i is a primitive of φ_i, for $i=1,2$). As $(u,v)\in\left(H_0^1(\Omega)\right)^2$, It is not difficult to show that $\Phi_i(u)\in W_0^{1,1}(\Omega)$ (resp $\Phi_2(v)\in W_0^{1,1}(\Omega)$) and

$$\begin{cases}\int_\Omega G_1\varphi_1(u).\nabla udx=\int_\Omega G_1.\nabla\Phi_1(u)dx,\\ \int_\Omega G_2\varphi_2(v).\nabla vdx=\int_\Omega G_2.\nabla\Phi_2(v)dx.\end{cases}$$

For (1.3), we have

$$\begin{cases}\int_\Omega G_1\varphi_1(u).\nabla udx=\int_\Omega G_1.\nabla\Phi_1(u)dx\\ =-\int_\Omega \text{div}G_1.\Phi_1(u)dx=0,\\ \int_\Omega G_2\varphi_2(v).\nabla vdx=\int_\Omega G_2.\nabla\Phi_2(v)dx\\ =-\int_\Omega\text{div}G_2.\Phi_2(v)dx=0.\end{cases}$$

Taking $u=\varphi$, $v=\psi$ in (3.1). By assumptions (1.2), (1.3), (1.4), (1.5) and (1.6), we have

$$\underbrace{\min(\alpha_1,\alpha_2)}_{\alpha}\|(u,v)\|_{\mathcal{U}}^2\le\int_\Omega|f(u,v)u|dx+\int_\Omega|g(u,v)v|dx.$$

Lemma 3.1. *There exist $R>0$ such that for all $t\in[0,1]$ and all $(u,v)\in V$,*

$$(u,v)-H(t,u,v)=0\Rightarrow\|(u,v)\|_V<R.$$

Proof. To prove this lemma we assume by contradiction, that for all $R>0$ there exists $(t,u,v)\in[0,1]\times V$ such that

$$(u,v)-H(t,u,v)=0 \text{ and } \|(u,v)\|_V<R.$$

In other words, we can find a sequence $(t_n,u_n,v_n)\in[0,1]\times V$ such that

$$(u_n,v_n)-H(t_n,u_n,v_n)=0 \text{ and } \|(u_n,v_n)\|_V<n.\tag{3.2}$$

Taking

$$(\tilde{u}_n,\tilde{v}_n)=\left(\frac{u_n}{\|(u_n,v_n)\|_V},\frac{v_n}{\|(u_n,v_n)\|_V}\right),$$

we have

$$\|\tilde{u}_n,\tilde{v}_n\|_V=1,$$

and

$$\alpha\|\tilde{u}_n,\tilde{v}_n\|_{\mathcal{U}}^2\le\int_\Omega\left|\frac{f(u_n,v_n)}{\|(u_n,v_n)\|_V}\tilde{u}_n\right|dx+\int_\Omega\left|\frac{g(u_n,v_n)}{\|(u_n,v_n)\|_V}\tilde{v}_n\right|dx.$$

For (1.5), we have

$$\begin{aligned}&\alpha\|\tilde{u}_n,\tilde{v}_n\|_{\mathcal{U}}^2\\&\le\int_\Omega\frac{C_1'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}|\tilde{u}_n|dx+\int_\Omega\frac{C_2'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}|\tilde{v}_n|dx\end{aligned}$$

$$\le \int_\Omega \frac{C_1'|\tilde{u}_n|}{\|(u_n,v_n)\|_V}dx + C_1'\int_\Omega |\tilde{u}_n||\tilde{u}_n|dx + C_1'\int_\Omega |\tilde{v}_n||\tilde{u}_n|dx$$
$$+\int_\Omega \frac{C_2'|\tilde{v}_n|}{\|(u_n,v_n)\|_V}dx + C_2'\int_\Omega |\tilde{u}_n||\tilde{v}_n|dx + C_1'\int_\Omega |\tilde{v}_n||\tilde{v}_n|dx$$
$$\le \frac{1}{\|(u_n,v_n)\|_V}\int_\Omega \left(C_1'|\tilde{u}_n| + C_2'|\tilde{v}_n|\right)dx$$
$$+2\max\left(C_1',C_2'\right)\int_\Omega |\tilde{u}_n,\tilde{v}_n|^2 dx$$
$$\le \frac{1}{\|(u_n,v_n)\|_V}\left(C_1'|\tilde{u}_n|_{L^2(\Omega)} + C_2'|\tilde{v}_n|_{L^2(\Omega)}\right)$$
$$+2\max\left(C_1',C_2'\right)\|\tilde{u}_n,\tilde{v}_n\|_V^2$$
$$\le \frac{1}{\|(u_n,v_n)\|_V}\left(C_1'\|\tilde{u}_n,\tilde{v}_n\|_V + C_2'\|\tilde{v}_n,\tilde{v}_n\|_V\right)$$
$$+2\max\left(C_1',C_2'\right)$$
$$\le \frac{\max\left(C_1',C_2'\right)}{\|(u_n,v_n)\|_V} + 2\max\left(C_1',C_2'\right),$$

Moreover, by (3.2) we have

$$\frac{\max\left(C_1',C_2'\right)}{\|(u_n,v_n)\|_V} \le \frac{\max\left(C_1',C_2'\right)}{n} \le \max\left(C_1',C_2'\right).$$

Then

$$\|\tilde{u}_n,\tilde{v}_n\|_{\mathcal{U}}^2 \le k, \text{ such that } k = \frac{3\max\left(C_1',C_2'\right)}{\alpha},$$

that is, $\|\tilde{u}_n,\tilde{v}_n\|_{\mathcal{U}}^2$ is bounded in $\mathcal{U}$.

Since $(\tilde{u}_n,\tilde{v}_n) \in \left(H_0^1(\Omega)\right)^2$ and the embedding $(H_0^1(\Omega) \hookrightarrow L^2(\Omega))$ is compact, we can extract a subsequence $(\tilde{u}_n,\tilde{v}_n)$, still denoted by $(\tilde{u}_n,\tilde{v}_n)$, which converges in V. Let $(\tilde{u},\tilde{v})$ be the limit of $(\tilde{u}_n,\tilde{v}_n)$ in V. We have threrfore $\|(u_n,v_n)\|_V = 1$ (which give $\tilde{u} \ne 0$, and $\tilde{v} \ne 0$). We also have

$$\begin{gathered}(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v}) \text{ a.e,}\\ |\tilde{u}_n| \le H \text{ and } |\tilde{v}_n| \le G, \text{ with } H,G \in L^2(\Omega).\end{gathered} \tag{3.3}$$

Finally, using the Poincare inequality, there is C_Ω, depending only on Ω such that

$$\frac{\alpha}{C_\Omega^2} = \frac{\alpha}{C_\Omega^2}\|(u_n,v_n)\|_V^2 \le \alpha\|\tilde{u}_n,\tilde{v}_n\|_{\mathcal{U}}^2$$
$$\le \int_\Omega \frac{|f(u_n,v_n)|}{\|(u_n,v_n)\|_V}|\tilde{u}_n|dx + \int_\Omega \frac{|g(u_n,v_n)|}{\|(u_n,v_n)\|_V}|\tilde{v}_n|dx.$$

Let us put

$$Z_n = \int_\Omega \frac{|f(u_n,v_n)||\tilde{u}_n|}{\|(u_n,v_n)\|_V}dx + \int_\Omega \frac{|g(u_n,v_n)||\tilde{v}_n|}{\|(u_n,v_n)\|_V}dx.$$

and now we show that $Z_n \to 0$ when $n \to \infty$, which is impossible since Z_n is reduced by constant α / C_Ω^2 which is strictly positive.

Show that

$$\begin{cases} \dfrac{f(u_n,v_n)|\tilde{u}_n|}{\|(u_n,v_n)\|_V} \to 0 \text{ a.e in } \Omega,\\ \dfrac{g(u_n,v_n)|\tilde{v}_n|}{\|(u_n,v_n)\|_V} \to 0 \text{ a.e in } \Omega,\end{cases}$$

with domination (in $L^1(\Omega)$), we have by the dominated convergence theorem that $Z_n \to 0$ when $n \to \infty$.

We first show dominance.

From (1.5) and (3.3), we have

$$\begin{cases} \dfrac{|f(u_n,v_n)|}{\|(u_n,v_n)\|_V} \le \dfrac{C_1'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}\\ \le C_1' + C_1'|\tilde{u}_n| + C_1'|\tilde{v}_n| \le C_1' + C_1'H + C_1'G,\\ \dfrac{|G(u_n,v_n)|}{\|(u_n,v_n)\|_V} \le \dfrac{C_2'(1+|u_n|+|v_n|)}{\|(u_n,v_n)\|_V}\\ \le C_2' + C_2'|\tilde{u}_n| + C_2'|\tilde{v}_n| \le C_2' + C_2'H + C_2'G.\end{cases}$$

Then

$$\begin{cases} \left|\dfrac{f(u_n,v_n)}{\|(u_n,v_n)\|_V}\tilde{u}_n\right| \le \left(C_1' + C_1'H + C_1'G\right)H \in L^1(\Omega),\\ \left|\dfrac{g(u_n,v_n)}{\|(u_n,v_n)\|_V}\tilde{v}_n\right| \le \left(C_2' + C_2'H + C_2'G\right)G \in L^1(\Omega).\end{cases}$$

We now show the convergence a.e.

We have

$$(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v}) \text{ a.e.}$$

Let $x \in A$. From the hypothesis (1.6) and (1.7) it follows that

Case I

If $\tilde{u}(x) > 0$ and $\tilde{v}(x) > 0$, therefore $\tilde{u}(x).\tilde{v}(x) > 0$ (resp if $\tilde{u}(x) < 0$ and $\tilde{v}(x) < 0$), $\tilde{u}_n(x) \to \tilde{u}(x)$ and $\tilde{v}_n(x) \to \tilde{v}(x)$ but

$$\lim_{n\to\pm\infty}\|u_n,v_n\|_V = \pm\infty$$

therefore $u_n(x) = \tilde{u}_n(x)\|u_n,v_n\|_V \to \pm\infty$ and

$$v_n(x) = \tilde{v}_n(x)\|u_n,v_n\|_V \to \pm\infty.$$

$$\begin{cases} \dfrac{f(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{u}_n(x) = \dfrac{f(u_n(x),v_n(x))v_n(x)}{v_n(x)\|u_n,v_n\|_V}\tilde{u}_n(x)\\ = \dfrac{f(u_n(x),v_n(x))}{v_n(x)}\tilde{u}_n(x).\tilde{v}_n(x) \underset{n\to\infty}{\to} 0,\\ \dfrac{g(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{v}_n(x) = \dfrac{g(u_n(x),v_n(x))u_n(x)}{u_n(x)\|u_n,v_n\|_V}\tilde{v}_n(x)\\ = \dfrac{g(u_n(x),v_n(x))}{u_n(x)}\tilde{u}_n(x).\tilde{v}_n(x) \underset{n\to\infty}{\to} 0.\end{cases}$$

Case II

If $\tilde{u}(x)>0$ and $\tilde{v}(x)>0$, therefore $\tilde{u}(x).\tilde{v}(x)<0$ (resp if $\tilde{u}(x)<0$ and $\tilde{v}(x)>0$), $\tilde{u}_n(x)\to\tilde{u}(x)$ and $\tilde{v}_n(x)\to\tilde{v}(x)$ but

$$\lim_{n\to\pm\infty}\|u_n,v_n\|_V=\pm\infty$$

therefore $u_n(x)=\tilde{u}_n(x)\|u_n,v_n\|_V\to\pm\infty$ and

$$v_n(x)=\tilde{v}_n(x)\|u_n,v_n\|_V\to\pm\infty.$$

$$\begin{cases}\dfrac{f(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{u}_n(x)=\dfrac{f(u_n(x),v_n(x))\tilde{v}_n(x)}{\tilde{v}_n(x)\|u_n,v_n\|_V}\tilde{u}_n(x)\\ =\dfrac{f(u_n(x),v_n(x))}{v_n(x)}\tilde{u}_n(x).\tilde{v}_n(x)\underset{n\to\infty}{\to}0,\\ \dfrac{g(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{v}_n(x)=\dfrac{g(u_n(x),v_n(x))\tilde{u}_n(x)}{\tilde{u}_n(x)\|u_n,v_n\|_V}\tilde{v}_n(x)\\ =\dfrac{g(u_n(x),v_n(x))}{u_n(x)}\tilde{u}_n(x).\tilde{v}_n(x)\underset{n\to\infty}{\to}0.\end{cases}$$

Case III

If $\tilde{u}(x)=\tilde{v}(x)=0$

$$\begin{aligned}&\left|\frac{f(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{u}_n(x)\right|\\ &\le\frac{C_1'(1+|u_n(x)|+|v_n(x)|)}{\|u_n,v_n\|_V}|\tilde{u}_n(x)|\\ &\le\left(C_1'+C_1'|\tilde{u}_n(x)|+C_1'|\tilde{v}_n(x)|\right)|\tilde{u}_n(x)|\\ &\to0\ \text{ because } \tilde{u}(x)=0,\end{aligned}$$

$$\begin{aligned}&\left|\frac{g(u_n(x),v_n(x))}{\|u_n,v_n\|_V}\tilde{v}_n(x)\right|\\ &\le\frac{C_2'(1+|u_n(x)|+|v_n(x)|)}{\|u_n,v_n\|_V}|\tilde{v}_n(x)|\\ &\le\left(C_2'+C_2'|\tilde{u}_n(x)|+C_2'|\tilde{v}_n(x)|\right)|\tilde{v}_n(x)|\\ &\to0\ \text{ because } \tilde{v}(x)=0.\end{aligned}$$

In summary we have

$$\begin{cases}\dfrac{f(u_n,v_n)}{\|u_n,v_n\|_V}\tilde{u}_n\to0\ \text{ a.e,}\\ \dfrac{g(u_n,v_n)}{\|u_n,v_n\|_V}\tilde{v}_n\to0\ \text{ a.e.}\end{cases}$$

It was also shown that $\lim_{n\to+\infty}Z_n=0$, this is contradiction with $Z_n\ge\alpha/C_\Omega^2$ for all $n\in\mathbb{N}^*$. We have shown that there exists $R>0$ suth that

$$(u,v)=H(t,u,v)\Rightarrow\|u_n,v_n\|_V<R.$$

Now, we give the proof of our main result.

Proof. of Theorem (1.1). We have no solution to the equation $(u,v)-H(t,u,v)=0$ on the edge of the ball B_R such that $B(0,R)=(u,v)\in V,\ \|u_n,v_n\|_V<R.$

By invariance of the topological degree we have

$$t\in[0,1],\ \deg(H(t,.,.),B(0,R),0),$$

is constant.

In particular $t=0$. By the homotopy invariance property, we have

$$\deg(H(t,.,.),B(0,R),0)=\deg(H(1,.,.),B(0,R),0)=1$$

We infer the existence of $(u,v)\in B(0,R)$ as $(u,v)-H(1,u,v)=0.$

Therefore (u,v) is a solution of (1.1) (and theorem(1.1) is shown).

4. Conclusion

We have discussed in this article the conditions of existence for a nonlinear elliptic system. This problem has been treated by Leray-Schauder degree theory, the latter is a more tool powerful, more general and often even easier to use. The problem proposed in this paper is a generalization of working of T. Gallouët (see [1,5,6]).

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