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Fractional Black Scholes Option Pricing with Stochastic Arbitrage Return

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Abstract Option price and random arbitrage returns change on different time scales allow the development of an asymptotic pricing theory involving the options rather than exact prices. The role that random arbitrage opportunities play in pricing financial derivatives can be determined. In this paper, we construct Green's functions for terminal boundary value problems of the fractional Black-Scholes equation. We follow further an approach suggested in literature and focus on the pricing bands for options that account for random arbitrage opportunities and got similar result for the fractional Black- Scholes option pricing.

Keywords: *arbitrage returns, option pricing, green function, FBS equation*

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1. Introduction

Fractional calculus has become of increasing use for analyzing not only stochastic processes driven by fractional Brownian processes [16], but also non -random fractional phenomena in physics [8], like the study of porous systems, for instance, and quantum mechanics [14]. Whichever the framework is, we believe that the very reason for introducing and using fractional derivative is to deal with non-differentiable functions. In financial literature for example, stochastic volatility models the Merton jump-diffusion model [9], non-Gaussian option pricing models [4,5], amongst others have been proposed. Each of these is based on the assumption of the absence of arbitrage. However, it is well-known that arbitrage opportunities always exist in the real world (see Refs. [6,15]). Of course, arbitragers ensure that the prices of securities do not get out of line with their equilibrium values, and therefore virtual arbitrage is always short-lived. An arbitrage possibility is essentially equivalent to the possibility of making a positive amount of money out of nothing without taking any risk. It is thus essentially a riskless money making machine. An arbitrage possibility is a serious case of mispricing in the market. It is well-known that arbitrage opportunities always exist in the real world [10]. Of course, arbitragers ensure that the prices of securities do not get out of line with their equilibrium values, and therefore virtual arbitrage is always short-lived. The first attempt to take into account virtual arbitrage in option pricing was made by Physicists Refs [1,7,13]. The authors assume that arbitrage returns exist, appearing and disappearing over a short time scale. Asma et al [2] applied the homotopy perturbation method for fractional Black-Scholes equation by using He's polynomials and Sumudu transform to obtain the solution of fractional Black-Scholes equation. At this point, Belgacem et al. [3,9] had applied the Laplace transform and extended the theory and the applications of the Sumudu transform to the solution of fractional differential equations.

In this work, a technique is proposed for the construction of Green's functions for terminal boundary value problems of the fractional Black-Scholes equation. The technique is based on the method of integral Laplace transform and the method of variation of parameters. It provides closed form analytic representations for the constructed Green's functions [12]. We follow further an approach suggested in Ref. [15] and focus on the pricing bands for options that account for random arbitrage opportunities and got similar result for the fractional Black- Scholes option pricing.

2. Derivation of the Black-Scholes Equation

We base our derivation on replicating portfolio that ensures that no arbitrage opportunities are allowed. As in the discrete case, consider a portfolio $\Lambda = \{\Lambda_t\}_{t>0}$, which is F_t- measurable (we can choose as we go, but at any point in time the choice is deterministic), Λ_t denotes the proportion of shares invested at time t, the rest of the money is invested in the money market account, giving risk-free rate of return, r, say. In what follows, we state:

Theorem 2.1

Let a generic payoff function $G(t) = V(s,t)$, the PDE associated with the price of derivative on the stock price is

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} - rV = 0. \qquad (2.1)$$

Proof:

The stock price S_t follows the fractional Brownian motion process [9]

$$\frac{dS}{S}=\mu dt+\sigma dB_H(t),S(0)=s, \qquad (2.2)$$

and the wealth of an investor X_t, follows a diffusion driven by (with time suppressed)

$$dX=\Delta dS+r(X-\wedge S)dt. \qquad (2.3)$$

Putting equation (2.2) into equation (2.3) yields;

$$dX=\{rX+\wedge S(\mu-r)\}dt+\wedge S\sigma dB_H(t)\}, \qquad (2.4)$$

where $\mu-r$ is the risk premium. Suppose that the value of this claim at time t is given by

$$G(t)=V(S,t),S=S_t. \qquad (2.5)$$

Applying the fractional Ito's formula on equation 2.5, we have

$$dG=\begin{bmatrix}\frac{\partial V}{\partial t}+\mu S\frac{\partial V}{\partial S}\\ +H\sigma^2S^2t^{2H-1}\end{bmatrix}dt+\sigma S\frac{\partial V}{\partial S}d\mathrm{B}_H. \qquad (2.6)$$

To track $G(t)$ at all times, we have under the assumption of complete market that

$$X(t)\ =G(t)=V(S,t)\forall t\in[0,T]. \qquad (2.7)$$

Thus

$$\frac{\partial V}{\partial t}+\mu s\frac{\partial V}{\partial S}+H\sigma^2s^2t^{2H-1}\frac{\partial^2V}{\partial S^2}=rV+\wedge_tS(\mu-r) \qquad (2.8)$$

and

$$\sigma S\frac{\partial V}{\partial S}=\wedge_tS\sigma \qquad (2.9)$$

$$\wedge_t=\frac{\partial v}{\partial s}(s,t). \qquad (2.10)$$

While equation (2.6) with $\wedge_t=\frac{\partial v}{\partial s}$ gives

$$\frac{\partial V}{\partial t}+\mu S\frac{\partial V}{\partial S}+H\sigma^2S^2t^{2H-1}\frac{\partial^2V}{\partial S^2}=rV+S\mu\frac{\partial V}{\partial S}-Sr\frac{\partial V}{\partial S} \qquad (2.11)$$

$$\frac{\partial V}{\partial t}+H\sigma^2S^2t^{2H-1}\frac{\partial^2V}{\partial S^2}+Sr\frac{\partial V}{\partial S}-rV=0. \qquad (2.12)$$

Thus, the fractional Black-Scholes equation for valuing an option with value V is obtained.

3. Construction of Green's Function for the Fractional Black-Scholes Equation

For a call option with maturity date T, strike price K, and payoff function G, the value price $V=V(S,t)$ satisfies the following fBm, [9],

$$\frac{\partial V}{\partial t}+Ht^{2H-1}S^2\sigma^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial v}{\partial s}-rV=0, \qquad (3.1)$$

with homogeneous boundary value problem corresponding to

$$V(s,0)=h(s), \qquad (3.2)$$

and

$$|V(-\infty,t)|<\infty \ and\ |V(\infty,t)|<\infty. \qquad (3.3)$$

We set $S=e^x\Rightarrow x=\ln s$, $u(x,t)=V(e^x,t)$ and $h(e^x)=g(x)$, to get

$$\frac{\partial u}{\partial t}+Ht^{2H-1}\sigma^2\left(\frac{\partial^2u}{\partial x^2}-\frac{\partial u}{\partial x}\right)+r\frac{\partial u}{\partial x}-ru=0$$

or

$$\frac{\partial u}{\partial t}+Ht^{2H-1}\sigma^2\frac{\partial^2u}{\partial x^2}-\left(Ht^{2H-1}\sigma^2-r\right)\frac{\partial u}{\partial x}-ru=0. \qquad (3.4)$$

Let $\lambda=Ht^{2H-1}\sigma^2$ and $\alpha=Ht^{2H-1}\sigma^2-r$, then we have

$$\frac{\partial u}{\partial t}=-\lambda\sigma^2\frac{\partial^2u}{\partial x^2}+\alpha\frac{\partial u}{\partial x}+ru$$

$$u(x,0)=h\left(e^x\right),|V(-\infty,t)|<\infty\ and\ |V(\infty,t)|<\infty. \qquad (3.5)$$

Applying Laplace transform to (3.5) gives

$$-\lambda\sigma^2\frac{d^2u}{dx^2}+\alpha\frac{du}{dx}+ru=-u(x,0)+su,$$

or for $c=\frac{1}{\lambda}$

$$\frac{d^2u}{dx^2}+c\frac{du}{dx}+(r-s)u=ch\left(e^x\right). \qquad (3.6)$$

To find a fundamental set of solutions to the homogeneous equation corresponding to (3.6), consider its characteristic equation:

$$k^2+c\alpha k+c(r-s)=0.$$

Solving, we have

$$k=\frac{-c\alpha\pm\sqrt{(c\alpha)^2-4c(r-s)}}{2}=\frac{-c\alpha}{2}\pm\sqrt{c\left(\frac{c\alpha^2}{4}-r+s\right)}$$

For simplicity, set $c=1$, then $k=\frac{-\alpha}{2}\pm\sqrt{\left(\frac{\alpha^2}{4}-r+s\right)}$.

Thus the roots are $k_1=\gamma+\omega$ and $k_2=\gamma-\omega$ where $\omega=(s+\beta)^{1/2}$, while the parameters $\gamma=\frac{-\alpha}{2}$ and $\beta=\frac{\alpha^2}{4}-r$. This yields two linearly independent solutions to the homogeneous equation corresponding to (3.6) as

$$U_1(x,s)=\exp(\gamma+\omega)x$$

and

$$U_2(x,s)=\exp(\gamma-\omega)x$$

with their linear combination given as

$$U(x,s) = A(x,s)\exp(\gamma+\omega)x + B(x,s)\exp(\gamma-\omega)x \quad (3.7)$$

Now representing according to the method of variation of parameters, the general solution to (3.6). Following the procedure of this method, one arrives at the well-posed system

$$\begin{bmatrix} \exp(\gamma+\omega)x & \exp(\gamma-\omega)x \\ [(\gamma+\omega)\exp(\gamma+\omega)x] & [(\gamma-\omega)\exp(\gamma-\omega)x] \end{bmatrix} \begin{bmatrix} A'(x,s) \\ B'(x,s) \end{bmatrix} = \begin{bmatrix} 0 \\ -h(e^x) \end{bmatrix}$$

of linear algebraic equations in the derivatives with respect to x of the coefficients A(*x*,s) and B(*x*,s) of the linear combination in (3.7). The solution of the above system is obtained as

$$A'(x,s) = \frac{-\exp[-(\gamma+\omega)x]}{2\omega} h(e^x)$$

and

$$B'(x,s) = \frac{\exp[-(\gamma-\omega)x]}{2\omega} h(e^x).$$

Upon integration, the coefficients $A(x,s)$ and $B(x,s)$ are found in the form

$$A(x,s) = \frac{-1}{2\omega}\int_{-\infty}^{x} \exp[-(\gamma+\omega)\xi]h(e^{\xi})d\xi + M(s)$$

and

$$B(x,s) = \frac{1}{2\omega}\int_{-\infty}^{x} \exp[-(\gamma-\omega)\xi]h(e^{\xi})d\xi + N(s).$$

Substitution of these in (3.7) yields the general solution of (3.5) in the form:

$$U(x,s) = \frac{1}{2\omega}\int_{-\infty}^{x} exp\gamma(\xi - x)[exp\omega(\xi - x) - exp\omega(x-\xi)]d\xi \quad (3.8)$$
$$+ M(s)\exp(\gamma+\omega)x + N(s)\exp(\gamma-\omega)x.$$

The constants of integration M(s) and N(s) can be obtained upon satisfying the boundary conditions of (3'6) as

$$N(s) = 0,\ M(s) = \frac{1}{2w}\int_{-\infty}^{\infty} \exp[-(\gamma+w)\xi]h(e^{\xi})d\xi.$$

Substituting in (3.8), we obtain the solution to the boundary value problem in (3.5) and (3.6) in the form.

$$U(x,s) = \int_{-\infty}^{x} \frac{\exp\gamma(x-\xi)}{2w}[\exp w(\xi - x) - \exp w(x-\xi)]h(e^{\xi})d\xi \quad (3.9)$$
$$+ \int_{-\infty}^{\infty} \frac{\exp\gamma(x-\xi)}{2w}[\exp w(x-\xi)h(e^{\xi})d\xi.$$

Equation (3.9) can be re-written in a compact single-integral form as

$$U(x,s) = \int_{-\infty}^{\infty} \frac{\exp\gamma(x-\xi)}{2w}\exp(-w|x-\xi|)h(e^{\xi})d\xi. \quad (3.10)$$

The solution $U(x,t)$ to the initial boundary value problem stated by (3.1) can be obtained from $U(x,s)$ with the aid of the inverse Laplace transform. In doing so, we keep in mind that the parameter w has earlier been introduced in terms of the parameter s of the Laplace transform as

$$w = (s+\beta)^{\frac{1}{2}}.$$

This yields

$$u(x,t) = L^{-1}[U(x,s)]$$
$$= \int_{-\infty}^{\infty} \exp\gamma(x-\xi)L^{-1}\{\exp[-(s+\beta)^{\frac{1}{2}}|x-\xi|]\}h(e^{\xi})d\xi \quad (3.11)$$
$$= \int_{-\infty}^{\infty} \frac{\exp\gamma(x-\xi)\exp(-\beta t)}{2(\pi t)^{\frac{1}{2}}}\exp[-\frac{(x-\xi)^2}{4t}]h(e^{\xi})d\xi$$

To obtain the solution $V(s,t)$, we make the backward substitutions,

$$x = In\, s, \xi = In\,\tilde{s}$$
$$\Rightarrow d\xi = \frac{1}{\tilde{s}}d\tilde{s},$$

While the interval of integration $(-\infty,\infty)$ in (3.11) transforms according to the relation $\xi = In\tilde{s}$ to the interval $[0,\infty)$ with respect to $\tilde{s}$. With all these in mind, one arrives at the solution to the terminal-boundary value problem as

$$V(s,t)$$
$$= \int_0^{\infty} \frac{\begin{bmatrix}\exp\gamma(Ins - In\tilde{s}) \\ \exp(-\beta t)\end{bmatrix}}{2\tilde{s}(\pi t)^{\frac{1}{2}}}\exp\left[\frac{-(Ins - In\tilde{s})^2}{4t}\right]h\left(e^{In\tilde{s}}\right)d\tilde{s}$$
$$= \int_0^{\infty} \frac{\exp\gamma(In\frac{s}{\tilde{s}})\exp(-\beta t)}{2\tilde{s}(\pi t)^{1/2}}\exp\left[\frac{-\left(In\frac{s}{\tilde{s}}\right)^2}{4t}\right]h(\tilde{s})d\tilde{s}.$$

Revealing the Green's function to the problem as

$$G(s,t,\tilde{s}) = \frac{1}{2\tilde{s}(\pi t)^{\frac{1}{2}}}\exp\gamma(In\, s/\tilde{s}) - \beta t - \frac{In\left(s/\tilde{s}\right)^2}{4t}. \quad (3.12)$$

4. Fractional B-S Equation with Random Arbitrage Return

The associated option price $V^{\varepsilon}(\tau,s)$, obeys the following Stochastic P.D.E

$$\frac{\partial V^{\varepsilon}}{\partial \tau}=H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2 v^{\varepsilon}}{\partial s^2}+rs\frac{\partial v^{\varepsilon}}{\partial s} -rv^{\varepsilon}+\xi(\frac{\tau}{\varepsilon})\left(s\frac{\partial v^{\varepsilon}}{\partial s}-v^{\varepsilon}\right). \quad (4.1)$$

Subject to the initial condition

$$V^{\varepsilon}(0,s)=\max(s-k,0).$$

Here $\xi(t)$ is the random arbitrage return that describes the fluctuations of return around rdt, [7]. According to the law of large numbers V^{ε} converges in probability to the Black-Scholes price V_{BS} as $\varepsilon\to 0$. One can split $V^{\varepsilon}(\tau,s)$ into the sum of the Black-Scholes price V_{BS} and the random field Z^{ε} with the scaling factor $\sqrt{\varepsilon}$, giving

$$V^{\varepsilon}(\tau,s)=V_{BS}(\tau,s)+\sqrt{\varepsilon}Z^{\varepsilon}(\tau,s). \quad (4.2)$$

Substituting (4.2) into (4.1), we have;

$$\frac{\partial}{\partial \tau}\left(V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon}\right)=H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2}{\partial s^2}(V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon})$$
$$+rs\frac{\partial}{\partial s}\left(V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon}\right)-r[V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon}]+$$
$$\xi\left(\frac{\tau}{\xi}\right)\left\{[S\frac{\partial}{\partial s}(V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon}]-(V_{BS}+\sqrt{\varepsilon}Z^{\varepsilon})\right\}$$

$$\frac{\partial z^{\varepsilon}}{\partial \tau}=\frac{1}{\sqrt{\varepsilon}}\left\{\begin{array}{l}\frac{-\partial}{\partial \tau}V_{BS}+H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2}{\partial s^2}V_{BS}\\ +rs\frac{\partial}{\partial s}V_{BS}-rV_{BS}\\ +\xi\left(\frac{\tau}{\xi}\right)\left[S\frac{\partial V_{BS}}{\partial s}-V_{BS}\right]\end{array}\right\} \quad (4.3)$$
$$+H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2 z^{\varepsilon}}{\partial s^2}+rs\frac{\partial z^{\varepsilon}}{\partial s}$$
$$-rZ^{\varepsilon}+\xi\left(\frac{\tau}{\xi}\right)\left[S\frac{\partial}{\partial s}Z^{\varepsilon}-Z^{\varepsilon}\right]$$

Substituting (4.1) into (4.3), we have

$$\frac{\partial z^{\varepsilon}}{\partial \tau}=\frac{\xi\left(\frac{\tau}{\varepsilon}\right)}{\sqrt{\varepsilon}}\left[S\frac{\partial V_{BS}}{\partial s}-V_{BS}\right]$$
$$+H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2 Z^{\varepsilon}}{\partial s^2}+\left[r+\xi\left(\frac{\tau}{\varepsilon}\right)\right]\left[S\frac{\partial Z^{\varepsilon}}{\partial s}-Z^{\varepsilon}\right].$$

Ergodic theory implies that $\xi\left(\frac{\tau}{\varepsilon}\right)$ in its integral form converges to zero as $\varepsilon\to 0$, while $\frac{\xi\left(\frac{\tau}{\varepsilon}\right)}{\sqrt{\varepsilon}}$ converges weakly to a white Gaussian noise $\eta(t)$. Furthermore, as $\varepsilon\to 0$, the random field $Z^{\varepsilon}(\tau,s)$ converges weakly to the field $Z(\tau,s)$ that obeys the asymptotic stochastic P.D.E given by

$$\frac{\partial z}{\partial \tau}=H\sigma^2 s^2\tau^{2H-1}\frac{\partial^2 z}{\partial s^2}+r\left(S\frac{\partial z}{\partial s}-Z\right)+\left[S\frac{\partial V_{BS}}{\partial s}-V_{BS}\right]\eta(\tau). \quad (4.4)$$

With initial condition

$$Z(0,s)=0$$

Where $\eta(\tau)=\frac{dB(\tau)}{d\tau}$.

Equation (4.4) can be solved in terms of the fractional Black-Scholes Green function; $G(s,s_{1,}\tau,\tau_1)$ to give

$$Z(\tau,s)=\int_0^{\tau}\int_0^{\infty}G(s,s_{1,}\tau,\tau_1)\left(\begin{array}{c}S_1\frac{\partial V_{BS}}{\partial s_1}\\ -V_{BS}(\tau_{1,}s_1)\end{array}\right)\eta(\tau_1)ds_1 d\tau_1 \quad (4.5)$$

Where

$$G(s,s_{1,}\tau,\tau_1)=\frac{1}{2s_1(\pi t)^{1/2}}\exp\gamma\left(In\ {}^{s}\!/_{s_1}\right)-\beta t-In\frac{\left({}^{s}\!/_{s_1}\right)^2}{4t}.$$

It follows from (4.5) that since $\eta(t)$ is the Gaussian noise with zero mean, $Z(\tau,s)$ is also the Gaussian field with zero mean.

The pricing bands for the options for the case of arbitrage opportunities can be given by

$$V_{BS}(\tau,s)\pm 2\sqrt{\varepsilon u(\tau,s)} \quad (4.6)$$

Where

$$U(\tau,s)=R(\tau,s,s) \quad (4.7)$$

and $R(\tau,s,s)$ is the covariance

$$R(\tau,s,s)=\langle Z(\tau,s)Z(\tau,s)\rangle.$$

The variance $U(\tau,s)$ quantifies the fluctuations around the fractional Black-Scholes price. It should be noted that it is independent of the detailed statistical characteristics of the arbitrage return.

It is note-worthy that an investor who employs the arbitrage opportunities band hedging sells the option for

$$V_{BS}(\tau,s)+2\sqrt{\varepsilon u(\tau,s)}. \quad (4.8)$$

5. Conclusion

In this paper, we have introduced a technique for constructing the Green function for the fractional Black-Scholes equation. We also investigated the implications of random arbitrage return for option pricing. We derived the asymptotic equation for the random field that quantifies the fluctuations around the fractional Black-Scholes price and showed that it is independent of the detailed statistical characteristics of the arbitrage return.

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