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Modeling of Stress Distribution in a Semi-infinite Piecewise-homogeneous Body

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Abstract In this paper the Fourier vector integral transforms method with discontinuous coefficients developed by authors is used for elasticity theory problems solving. The analytical solving dynamic problems for theory of elasticity in piecewise homogeneous half-space is found. The explicit construction of direct and inverse Fourier vector transforms with discontinuous coefficients is presented. Unknown tension in the boundary conditions and in the internal conjugation conditions don't commit splitting in a considered dynamic problem, so the application of the scalar Fourier integral transforms with piece-wise constant coefficients does not lead to success. Conformable theoretical bases of a method are presented in this paper. The technique of applying Fourier vector transforms for solving problems of the dynamic problems the elasticity theory.

Keywords: *piecewise homogeneous medium, theory of elasticity, Fourier vector transform*

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1. Introduction

The purpose of the mathematical theory of elasticity is to define the tension and deformations on border and inside the elastic body any form under all load conditions. In dynamic problems of the theory of elasticity required values are functions of coordinates and time. The problems about oscillations of constructions and buildings are dynamic problems. Forms of oscillations and their possible changes, amplitudes of oscillations and their increase or decrease in the course of time, resonance modes, dynamic tension, methods of excitation and extinguish of oscillation and others, and also problems about distribution of elastic waves; seismic waves, and their influence on constructions and buildings, waves arising at explosions and blows, thermoelastic waves etc. are defined in the given problems. Different representations of the equilibrium equation solutions through functions of tension are used when solving problems by the variable separating method. The required problem is taken for solutions of differential equations of a more simple structure with the help of such representations. Each functions of tension in these equations "is not fastened" with others, but it enters into boundary conditions together with the others. A.F.Ulitko [7] has offered rather effective method to research the mathematical physics problems - a method Eigen vectorvalued functions. This method is the vector analogue of Fourier integral transforms method. This method is an analytical method for elasticity theory problems solution. In this article we consider and develop the Eigen vector valued method. We come to the most simple problem in space of images with the help of the integral transforms (Fourier, Laplace, Hankel, etc.). The finding of direct transforms formula is the main difficulty in solving problems of this approach. Extensive enough bibliography of works on use of this method is resulted in J.S.Ufljand's monography [2]. Elasticity theory problems for heterogeneous bodies are of great practical interest. Lame coefficients are not constant in these problems. They are the functions of coordinates defining the field of elastic properties of bodies. Application of analytical methods is connected with considerable mathematical difficulties because there is no corresponding mathematical apparatus, when the tension-strain state of bodies of the complex configuration is researched. Fourier vector integral transforms method is equivalent the method Eigen vector-valued functions, however, it can be successfully applied to solve elasticity theory problems in a piece-wise homogeneous medium. The theory of Fourier integral transforms with piece-wise constant coefficients in a scalar case was studied by Ufljand J.S. [16,17], Najda L.S. [11], Protsenko V. S [12,13], Lenjuk M. P [8,9,10]. The vector method is developed by the author in [2,19]. is adapted for the solution of problems in piece-wise homogeneous medium. Unknown tension in the boundary conditions and in the internal conjugation conditions don't commit splitting in a considered dynamic problem, so the application of the scalar Fourier integral transforms with piece-wise constant coefficients does not lead to success. In this paper the Fourier vector integral transforms method with discontinuous coefficients developed by authors is used for elasticity theory problems solving. Conformable theoretical bases of a method are presented in item 4. The necessary proofs are developed in [2] and [19]. The closed form of the dynamic problem solution is found in the use of this method in item 4.

2. Problem Statement

Let's consider a problem about distribution of tension in an n+1-layer elastic semi-infinite solid

$$I_n^+ \times R = \left\{(x, y) : x \in I_n^+, y \in R\right\},$$

where $I_n^+ = \bigcup_{i=1}^{n+1}(l_{i-1}, l_i)$. The vector of displacement $\bar{u}_i$ has components $u_i, v_i, 0$ in the case of plane strain. If introduced two functions tension $\phi_i(x, y, t)$ and $\psi_i(x, y, t)$, under the condition [14], functions are defined by the relations

$$u_i = \frac{\partial \phi_i}{\partial x} + \frac{\partial \psi_i}{\partial y}, v_i = \frac{\partial \phi_i}{\partial y} - \frac{\partial \psi_i}{\partial x}, \tag{1}$$

than expressions for the component of pressure become [14]

$$\sigma_{ix} = \lambda_i \Delta\phi_i + 2\mu_i\left(\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial x \partial y}\right),$$

$$\sigma_{iy} = \lambda_i \Delta\phi_i + 2\mu_i\left(\frac{\partial^2 \phi_i}{\partial y^2} - \frac{\partial^2 \psi_i}{\partial x \partial y}\right)$$

$$\tau_{ixy} = \mu_i\left(2\frac{\partial^2 \phi_i}{\partial x \partial y} - \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2}\right), \tag{2}$$

where λ_i, μ_i -elastic Lame constants. If to choose functions of tension ϕ_i and ψ_i in the form of solutions of a system of wave equations

$$\frac{\partial^2 \phi_i}{\partial t^2} = c_{1i}^2 \Delta\phi_i, \frac{\partial^2 \psi_i}{\partial t^2} = c_{2i}^2 \Delta\psi_i \tag{3}$$

$$t > 0, -\infty < y < \infty, l_{i-1} < x < l_i$$

with zero initial conditions

$$\phi_i(x, y, 0) = 0, \quad \psi_i(x, y, 0) = 0,$$
$$\frac{\partial \phi_i(x, y, 0)}{\partial t} = 0, \quad \frac{\partial \psi_i(x, y, 0)}{\partial t} = 0 \tag{4}$$

than the movement equations will be satisfied. The tension $p(y, t)$, changing with time, is applied on the border of the body. If tangent tension is equal to zero, than the boundary conditions become

$$\sigma_{1x} = -p(y, t), \quad \tau_{1xy} = 0 \;\; as \;\; x = 0. \tag{5}$$

Let the components of the vector of displacement $\bar{u}_i$ and the components of the tension tensor σ_{ix}, τ_{ixy} be continuous, we get internal boundary conditions, so-called conjugation conditions [5]:

$$u_i = u_{i+1}, \quad v_i = v_{i+1}, \quad \sigma_{ix} = \sigma_{i+1x}, \quad \tau_{ixy} = \tau_{i+1xy}, \tag{6}$$

$$x = l_i, \quad i = 1, ..., n.$$

3. Vector Fourier Transform with Discontinuous Coefficients

Let's develop the method of vector Fourier transform for the solution this problem. Let's consider Sturm–Liouville vector theory [1] about a design bounded on the set of non-trivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients

$$\left(A_m^2 \frac{d^2}{dx^2} + \lambda^2 \mathrm{E} + \Gamma_m^2\right) y_m = 0,$$
$$q_m^2 = \lambda^2 \mathrm{E} + \Gamma_m^2, \; m = \overline{1, n+1} \tag{7}$$

on the boundary conditions.

$$\left(\left(\alpha_{11}^0 + \lambda^2 \delta_{11}^0\right)\frac{d}{dx} + \left(\beta_{11}^0 + \lambda^2 \gamma_{11}^0\right)\right) y_1\Big|_{x=l_0} = 0,$$
$$\left\|y_{n+1}\right\|\Big|_{x=\infty} < \infty \tag{8}$$

and conditions of the contact in the points of conjugation of intervals

$$\left(\left(\alpha_{j1}^k + \lambda^2 \delta_{j1}^k\right)\frac{d}{dx} + \left(\beta_{j1}^k + \lambda^2 \gamma_{j1}^k\right)\right) y_k =$$
$$= \left(\left(\alpha_{j2}^k + \lambda^2 \delta_{j2}^k\right)\frac{d}{dx} + \left(\beta_{j2}^k + \lambda^2 \gamma_{j2}^k\right)\right) y_{k+1}, \tag{9}$$

$$x = l_k, \; k = \overline{1, n}, \; j = 1, 2.,$$

where

$$y_m(x, \lambda) = \begin{pmatrix} y_{1m}(x, \lambda) \\ \vdots \\ y_{rm}(x, \lambda) \end{pmatrix},$$

$$\left\|y_m\right\| = \sqrt{y_{1m}^2 + ... + y_{rm}^2}, m = \overline{1, n+1}.$$

Let for some λ the considered the boundary problem has a non-trivial solution

$$y(x, \lambda) = \sum_{k=1}^{n} \theta(x - l_{k-1})\theta(l_k - x) y_k(x, \lambda) +$$
$$+ \theta(x - l_n) y_{n+1}(x, \lambda).$$

The number λ is called an Eigen value in this case, and the corresponding decision $y(x, \lambda)$ is called Eigen vector-valued function.

$$\alpha_{11}^0, \beta_{11}^0, \gamma_{11}^0, \delta_{11}^0, \alpha_{j1}^k, \beta_{j1}^k, \gamma_{j1}^k,$$
$$\delta_{j1}^k, \alpha_{j2}^k, \beta_{j2}^k, \gamma_{j2}^k, \delta_{j2}^k, A_j -$$

are matrixes of the size $r \times r$. We will required invertible

$$\det \; M_{mk} \neq 0, \;\; \lambda \in [0, \infty) \tag{10}$$

for matrixes

$$M_{mk} \equiv \begin{pmatrix} \beta_{1m}^{k} + \lambda^2 \gamma_{1m}^{k} & \alpha_{1m}^{k} + \lambda^2 \delta_{1m}^{k} \\ \beta_{2m}^{k} + \lambda^2 \gamma_{2m}^{k} & \alpha_{2m}^{k} + \lambda^2 \delta_{2m}^{k} \end{pmatrix},$$
$$m = 1,2; \; k = \overline{1,n}.$$

Matrixes A_m^2 and Γ_m^2, are is $m = \overline{1,n+1}$ -positive-defined [6]. We denote

$$\Phi_{n+1}(x) = e^{q_{n+1}xi}; \; \Psi_{n+1}(x) = e^{-q_{n+1}xi};$$
$$q_{n+1}^2 = A_{n+1}^{-2}\left(\lambda^2 \mathrm{E} + \Gamma^2\right).$$

Define the induction relations the others n-pairs a matrix-importance functions (Φ_k, Ψ_k), $k = 1,n$:

$$\left[\left(\alpha_{j1}^{k} + \lambda^2 \delta_{j1}^{k}\right)\frac{d}{dx} + \left(\beta_{j1}^{k} + \lambda^2 \gamma_{j1}^{k}\right)\right](\Phi_k, \Psi_k) =$$
$$= \left[\left(\alpha_{j2}^{k} + \lambda^2 \delta_{j2}^{k}\right)\frac{d}{dx} + \left(\beta_{j2}^{k} + \lambda^2 \gamma_{j2}^{k}\right)\right](\Phi_{k+1}, \Psi_{k+1}), \quad (11)$$
$$k = \overline{1,n}, \quad j = \overline{1,2}.$$

Let us introduce the following notation

$$\overset{0}{\Phi}_1(\lambda) = \left.\begin{bmatrix} \left(\alpha_{11}^{0} + \lambda^2 \delta_{11}^{0}\right)\frac{d}{dx} \\ + \left(\beta_{11}^{0} + \lambda^2 \gamma_{11}^{0}\right) \end{bmatrix} \Phi_1(x,\lambda)\right|_{x=l_0},$$

$$\overset{0}{\Psi}_1(\lambda) = \left.\begin{bmatrix} \left(\alpha_{11}^{0} + \lambda^2 \delta_{11}^{0}\right)\frac{d}{dx} \\ + \left(\beta_{11}^{0} + \lambda^2 \gamma_{11}^{0}\right) \end{bmatrix} \Psi_1(x,\lambda)\right|_{x=l_0},$$

$$\Omega_k = \begin{pmatrix} \Phi_k & \Psi_k \\ \Phi_k' & \Psi_k' \end{pmatrix}, \quad i = \overline{1,n+1}.$$

Theorem 1. The spectrum of the problem (7),(8),(9) is a continuous and fills all semi axis $(0,\infty)$. Sturm–Liouville theory r time is degenerate. To each Eigen value λ corresponds to exactly r linearly independent vector-valued functions. As the last it is possible to take r columns matrix-importance functions.

$$u(x,\lambda) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\theta(l_k - x) u_k(x,\lambda),$$

$$u_j(x,\lambda) = \Phi_j(x,\lambda)\overset{0}{\Phi}{}_1^{-1}(\lambda) - \Psi_j(x,\lambda)\overset{0}{\Psi}{}_1^{-1}(\lambda). \quad (12)$$

That is

$$y^m(x,\lambda) = \begin{pmatrix} u_{1m}(x,\lambda) \\ \vdots \\ u_{rm}(x,\lambda) \end{pmatrix}.$$

Dual Sturm–Liouville theory consists in a finding of the non-trivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients.

$$\left(A_m^2 \frac{d^2}{dx^2} + \lambda^2 \mathrm{E} + \Gamma_m^2\right) y_m = 0, \quad (13)$$
$$q_m^2 = \lambda^2 \mathrm{E} + \Gamma_m^2, \; m = \overline{1,n+1}$$

on the boundary conditions

$$\left.\begin{pmatrix} \frac{d}{dx} y_1^* \left(\beta_{11}^{0} + \lambda^2 \gamma_{11}^{0}\right)^{-1} \\ + y_1^* \left(\alpha_{11}^{0} + \lambda^2 \delta_{11}^{0}\right)^{-1} \end{pmatrix}\right|_{x=l_0} = 0, \quad (14)$$
$$\left\| y_{n+1}^* \right\| < \infty,$$

and conditions of the contact in the points of conjugation of intervals

$$\left(-\frac{d}{dx} y_k^*, y_k^*\right)\begin{pmatrix} \beta_{11}^{k} + \lambda^2 \gamma_{11}^{k} & \alpha_{11}^{k} + \lambda^2 \delta_{11}^{k} \\ \beta_{21}^{k} + \lambda^2 \gamma_{21}^{k} & \alpha_{21}^{k} + \lambda^2 \delta_{21}^{k} \end{pmatrix}^{-1} =$$
$$= \left(-\frac{d}{dx} y_{k+1}^*, y_{k+1}^*\right)\begin{pmatrix} \beta_{12}^{k} + \lambda^2 \gamma_{12}^{k} & \alpha_{12}^{k} + \lambda^2 \delta_{12}^{k} \\ \beta_{22}^{k} + \lambda^2 \gamma_{22}^{k} & \alpha_{22}^{k} + \lambda^2 \delta_{22}^{k} \end{pmatrix}^{-1}, \quad (15)$$
$$x = l_k, \quad k = \overline{1,n}.$$

The solution of the boundary value problem we write in the form of

$$y^*(\xi,\lambda) = \sum_{k=1}^{n+1} \theta(\xi - l_{k-1})\theta(l_k - \xi) y_k^*(\xi,\lambda),$$
$$y_m^*(\xi,\lambda) = \left(y_{m1}^*(\xi,\lambda) \; \cdots \; y_{mr}^*(\xi,\lambda)\right),$$
$$\left\| y_m^* \right\| = \sqrt{\left(y_{1m}^*\right)^2 + \ldots + \left(y_{rm}^*\right)^2}, m = \overline{1,n+1}.$$

Theorem 2. The spectrum of the problem (7),(8),(9) is a continuous and fills semi axis $(0,\infty)$. Sturm–Liouville theory r time is degenerate. To each Eigen value λ corresponds to exactly r linearly independent vector-valued functions. As the last it is possible to take r rows matrix-importance functions.

$$u^*(x,\lambda) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\theta(l_k - x) u_k^*(x,\lambda),$$
$$u_j^*(x,\beta) = \left(\overset{0}{\Phi}_1(\beta), \overset{0}{\Psi}_1(\beta)\right)\Omega_j^{-1}(x,\beta)\begin{pmatrix} 0 \\ \mathrm{E} \end{pmatrix} A_j^{-2},$$

That is

$$y^{*j}(\xi,\lambda) = \left(u_{j1}^*(\xi,\lambda) \; \cdots \; u_{jr}^*(\xi,\lambda)\right), j = \overline{1,r}. \quad (16)$$

The existence of spectral functions $u(x,\lambda)$ and the conjugate spectral function $u^*(x,\lambda)$ allows to write the a vector decomposition theorem on the set of I_n^+.

Theorem 3. Let the vector-valued function *f (x)* is defined on I_n^+ continuous, absolutely integrated and has the bounded total variation. Then for any $x \in I_n^+$ true formula of decomposition

$$f(x) = -\frac{1}{\pi j}\int_0^\infty u(x,\lambda)\Bigg(\int_{l_0}^\infty u^*(\xi,\lambda) f(\xi) d\xi +$$
$$+ \left(\gamma_{11}^{0} f_1(l_0) + \delta_{11}^{0} f_1'(l_0)\right) +$$

$$+\sum_{k=1}^{n}\left(\overset{0}{\phi}_1(\lambda),\overset{0}{\psi}_1(\lambda)\right)\Omega_k^{-1}(l_k,\lambda)M_{k1}^{-1}(\lambda)\cdot$$
$$\cdot\left\{\begin{matrix}\begin{pmatrix}\gamma_{21}^k & \delta_{21}^k\\ \gamma_{22}^k & \delta_{22}^k\end{pmatrix}\begin{pmatrix}f_{k+1}(l_k)\\ f'_{k+1}(l_k)\end{pmatrix}\\ -\begin{pmatrix}\gamma_{11}^k & \delta_{11}^k\\ \gamma_{12}^k & \delta_{12}^k\end{pmatrix}\begin{pmatrix}f_k(l_k)\\ f'_k(l_k)\end{pmatrix}\end{matrix}\right\}\Bigg)\lambda d\lambda. \quad (17)$$

The decomposition theorem allows to enter the direct and inverse matrix integral Fourier transform on the real semi axis with conjugation points:

$$F_{n+}[f](\lambda)=\int_{l_0}^{\infty}u^*(\xi,\lambda)f(\xi)d\xi+$$
$$+\left(\gamma_{11}^0 f_1(l_0)+\delta_{11}^0 f_1'(l_0)\right)+$$
$$+\sum_{k=1}^{n}\left(\overset{0}{\phi}_1(\lambda),\overset{0}{\psi}_1(\lambda)\right)\Omega_k^{-1}(l_k,\lambda)M_{k1}^{-1}(\lambda)\cdot \quad (18)$$
$$\cdot\left\{\begin{matrix}\begin{pmatrix}\gamma_{21}^k & \delta_{21}^k\\ \gamma_{22}^k & \delta_{22}^k\end{pmatrix}\begin{pmatrix}f_{k+1}(l_k)\\ f'_{k+1}(l_k)\end{pmatrix}\\ -\begin{pmatrix}\gamma_{11}^k & \delta_{11}^k\\ \gamma_{12}^k & \delta_{12}^k\end{pmatrix}\begin{pmatrix}f_k(l_k)\\ f'_k(l_k)\end{pmatrix}\end{matrix}\right\}\equiv\tilde{f}(\lambda),$$

$$F_{n+}^{-1}\left[\tilde{f}\right](x)=-\frac{1}{\pi j}\int_0^{\infty}\lambda u(x,\lambda)\tilde{f}(\lambda)d\lambda\equiv f(x), \quad (19)$$

when

$$f(x)=\sum_{k=1}^{n+1}\theta(l_k-x)\theta(x-l_{k-1})f_k(x).$$

Let's apply the obtained integral formulas for the solution of the problem of elasticity theory (1),(2),(3),(4). Let's result the basic identity of integral transform of the differential operator

$$B=\sum_{j=1}^{n+1}\theta(x-l_{j-1})\theta(l_j-x)\left(A_j^2\frac{d^2}{dx^2}+\Gamma_j^2\right).$$

Theorem 3. If vector-valued function

$$f(x)=\sum_{k=1}^{n+1}\theta(x-l_{k-1})\theta(l_k-x)f_k(x),$$

is continuously differentiated on set three times, has the limit values together with its derivatives up to the third order inclusive

$$f_k^{(m)}(l_{k-1})=\lim_{x\to l_{k-1}+0}f_k^{(m)}(x),$$
$$m=0,1,2,3;\quad k=\overline{1,n+1}$$

Satisfies to the boundary condition on infinity

$$\lim_{x\to\infty}\left(u^*(x,\lambda)\frac{d}{dx}f(x)-\frac{d}{dx}u^*(x,\lambda)f(x)\right)=0$$

satisfies to homogeneous conditions of conjugation (9), that basic identity of integral transform of the differential operator B hold

$$F_{n+}[B(f)](\lambda)$$
$$=-\lambda^2\tilde{f}(\lambda)-\left\{\left(\beta_{11}^0 f_1(l_0)+\alpha_{11}^0 f_1'(l_0)\right)-\right. \quad (20)$$
$$\left.-\left(\gamma_{11}^0 A_1^2 f_1''(l_0)+\delta_{11}^0 A_1^2 f_1'''(l_0)\right)\right\}.$$

The proof of theorems 1,2,3,4 is spent by a method of the method of contour integration. Similarly presented to work of the author [19].

4. The Solution of Dynamic Problems of the Theory of Elasticity

Let's apply on the variable y Fourier transformation [4], and let's apply on the variable x the vector integral transforms of Fourier (18). In the images of Fourier series in the variable y the problem (1), (2), (3), (4) takes the form of the simultaneous equations

$$\frac{\partial^2\bar{\phi}_i}{\partial t^2}=c_{1i}^2\frac{\partial^2\bar{\phi}_i}{\partial x^2}-c_{1i}^2\xi^2\bar{\phi}_i,\ \frac{\partial^2\bar{\psi}_i}{\partial t^2}=c_{2i}^2\frac{\partial^2\bar{\psi}_i}{\partial x^2}-c_{2i}^2\xi^2\bar{\psi}_i,\ (21)$$
$$t>0,\quad l_{i-1}<x<l_i$$

with initial conditions

$$\bar{\phi}_i(x,y,0)=0,\quad \bar{\psi}_i(x,y,0)=0,$$
$$\frac{\partial\bar{\phi}_i(x,y,0)}{\partial t}=0,\quad \frac{\partial\bar{\psi}_i(x,y,0)}{\partial t}=0 \quad (22)$$

where $\bar{\phi}_i,\bar{\psi}_i$ - images of Fourier series in the variable y functions of tension

$$\bar{\phi}_i=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_i(x,y,t)e^{-j\xi y}dy,$$
$$\bar{\psi}_i=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi_i(x,y,t)e^{-j\xi y}dy$$

with boundary conditions

$$\sigma_{1x}=\lambda_1\frac{\partial^2\bar{\phi}_1}{\partial x^2}-\lambda_1\xi^2\bar{\phi}_1+$$
$$+2\mu_1\left(\frac{\partial^2\bar{\phi}_1}{\partial x^2}+j\xi\frac{\partial\bar{\psi}_1}{\partial x}\right)=-\bar{p}(\xi,t),x=0$$

$$\bar{\tau}_{1xy}=\mu_1\left(2j\xi\frac{\partial\bar{\phi}_1}{\partial x}-\frac{\partial^2\bar{\psi}_1}{\partial x^2}-\xi^2\bar{\psi}_1\right)=0,x=0, \quad (23)$$

with the internal conditions of conjugation

$$\frac{\partial\bar{\phi}_i}{\partial x}+j\xi\bar{\psi}_i=\frac{\partial\bar{\phi}_{i+1}}{\partial x}+j\xi\bar{\psi}_{i+1},$$
$$j\xi\bar{\phi}_i-\frac{\partial\bar{\psi}_i}{\partial x}=j\xi\bar{\phi}_{i+1}-\frac{\partial\bar{\psi}_{i+1}}{\partial x},\text{as } x=l_i$$

$$\lambda_i \frac{\partial^2 \overline{\phi}_i}{\partial x^2} - \lambda_i \xi^2 \overline{\phi}_i + 2\mu_i \left(\frac{\partial^2 \overline{\phi}_i}{\partial x^2} + j\xi \frac{\partial \overline{\psi}_i}{\partial x} \right)$$
$$= \lambda_{i+1} \frac{\partial^2 \overline{\phi}_{i+1}}{\partial x^2} - \lambda_{i+1} \xi^2 \overline{\phi}_{i+1} + 2\mu_{i+1} \left(\frac{\partial^2 \overline{\phi}_{i+1}}{\partial x^2} + j\xi \frac{\partial \overline{\psi}_{i+1}}{\partial x} \right)$$
$$as \quad x = l_i$$

$$\mu_i \left(2j\xi \frac{\partial \overline{\phi}_i}{\partial x} - \frac{\partial^2 \overline{\psi}_i}{\partial x^2} - \xi^2 \overline{\psi}_i \right)$$
$$= \mu_{i+1} \left(2j\xi \frac{\partial \overline{\phi}_{i+1}}{\partial x} - \frac{\partial^2 \overline{\psi}_{i+1}}{\partial x^2} - \xi^2 \overline{\psi}_{i+1} \right) \quad as \quad x = l_i \tag{24}$$

Denote $c = \max_i \{c_{1i}, c_{2i}\}$. Let's apply to a problem (21), (22), (23), (24) vector integral Fourier transform with discontinuous coefficients, defined by formulas (18) - (19). Let's put in simultaneous equations (7)

$$r = 2, A_i^2 = \begin{pmatrix} c_{i1}^2 & 0 \\ 0 & c_{i2}^2 \end{pmatrix}, \Gamma_i^2 = \begin{pmatrix} \left(c^2 - c_{i1}^2\right)\xi^2 & 0 \\ 0 & \left(c^2 - c_{i2}^2\right)\xi^2 \end{pmatrix},$$

in boundary conditions (8) let's consider

$$\alpha_{11}^0 = \begin{pmatrix} 0 & 2j\mu_1\xi \\ 2j\mu_1\xi & 0 \end{pmatrix},$$
$$\beta_{11}^0 = -\begin{pmatrix} \lambda_1 + 2\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix} A_1^{-2}\Gamma_1^2 - \begin{pmatrix} \lambda_1\xi^2 & 0 \\ 0 & \mu_1\xi^2 \end{pmatrix},$$
$$\gamma_{11}^0 = -\begin{pmatrix} \lambda_1 + 2\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix} A_1^{-2}, \delta_{11}^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

in the conditions of conjugation (9) we will put

$$\alpha_{11}^k = \begin{pmatrix} 0 & 2j\mu_k\xi \\ 2j\mu_k\xi & 0 \end{pmatrix},$$
$$\beta_{11}^k = -\begin{pmatrix} \lambda_k + 2\mu_k & 0 \\ 0 & -\mu_k \end{pmatrix} A_k^{-2}\Gamma_k^2 - \begin{pmatrix} \lambda_k\xi^2 & 0 \\ 0 & \mu_k\xi^2 \end{pmatrix},$$
$$\gamma_{11}^k = -\begin{pmatrix} \lambda_k + 2\mu_k & 0 \\ 0 & -\mu_k \end{pmatrix} A_k^{-2}, \delta_{11}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\alpha_{12}^k = \begin{pmatrix} 0 & 2j\mu_{k+1}\xi \\ 2j\mu_{k+1}\xi & 0 \end{pmatrix},$$
$$\beta_{12}^k = -\begin{pmatrix} \lambda_{k+1} + 2\mu_{k+1} & 0 \\ 0 & -\mu_{k+1} \end{pmatrix} A_{k+1}^{-2}\Gamma_{k+1}^2$$
$$-\begin{pmatrix} \lambda_{k+1}\xi^2 & 0 \\ 0 & \mu_{k+1}\xi^2 \end{pmatrix},$$
$$\gamma_{12}^k = -\begin{pmatrix} \lambda_{k+1} + 2\mu_{k+1} & 0 \\ 0 & -\mu_{k+1} \end{pmatrix} A_{k+1}^{-2}, \delta_{12}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\alpha_{2i}^k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta_{2i}^k = \begin{pmatrix} 0 & j\xi \\ j\xi & 0 \end{pmatrix},$$
$$\gamma_{2i}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \delta_{2i}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2.$$

Let's apply to a problem (21), (22), (23), (24) transforms of Fourier F_{n+} on the variable x. Using identity (20), we get Cauchy problem

$$\frac{d^2}{dt^2}\begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix} = -c^2\xi^2 \begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix} - \eta^2 \begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix} + \begin{pmatrix} \overline{p}(\xi, t) \\ 0 \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix}(\xi, \eta, 0) = 0, \quad \frac{d}{dt}\begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix}(\xi, \eta, 0) = 0, \tag{26}$$

Here denote

$$\begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix}(\eta, \xi) = F_{n+}\begin{pmatrix} \overline{\phi} \\ \overline{\psi} \end{pmatrix}(\eta),$$

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=1}^{n+1} \theta(l_k - x)\theta(x - l_{k-1}) \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix}.$$

Let's result the solution of the problem (25)-(26)

$$\begin{pmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{pmatrix}(\eta, \xi, t) = \int_0^t \frac{\sin\left(\sqrt{c^2\xi^2 + \eta^2}\,(t-\tau)\right)}{\sqrt{c^2\xi^2 + \eta^2}} \begin{pmatrix} \overline{p}(\xi, \tau) \\ 0 \end{pmatrix} d\tau.$$

Let's apply the inverse Fourier transform on y and inverse integral transform of Fourier series F_{n+}^{-1} on the variable x. Using (19), we get functions of tension ϕ_i, ψ_i:

$$\begin{pmatrix} \phi_i(x, y, t) \\ \psi_i(x, y, t) \end{pmatrix}$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} H_i(x, y - s, t - \tau) \begin{pmatrix} p(s, \tau) \\ 0 \end{pmatrix} ds d\tau, \tag{27}$$

when

$$H_i(x, y - s, t - \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-\frac{1}{j\pi} \int_0^{\infty} e^{j\xi} u_j(x, \eta, \xi) \cdot \right.$$
$$\left. \cdot \frac{\sin\left(\sqrt{c^2\xi^2 + \eta^2}\,(t-\tau)\right)}{\sqrt{c^2\xi^2 + \eta^2}} d\eta d\xi \right).$$

The formula (27) takes the form

$$\begin{pmatrix} \phi(x, y, t) \\ \psi(x, y, t) \end{pmatrix} = -\frac{1}{j\sqrt{2\pi}} \int_0^t \int_{y - c(t-\tau)}^{y + c(t-\tau)}$$
$$H\left(x, \sqrt{(t-\tau)^2 - \frac{(y-s)^2}{c^2}}\right) \begin{pmatrix} p(s, \tau) \\ 0 \end{pmatrix} ds d\tau.$$

In the case of a homogeneous environment, that is not the dependence of the λ_i, μ_i -elastic Lama constants,

$$H(x, z) =$$
$$= \int_0^{\infty} \operatorname{Im}\left(\frac{e^{jx\eta}}{j\eta\left(\alpha_{11}^0 + \eta^2\delta_{11}^0\right) + \left(\beta_{11}^0 + \eta^2\gamma_{11}^0\right)} \right) J_0(\eta z) d\eta.$$

when J_0 is Bessel function [3]. The expressions (27) for the functions of tension allow to find components of the vector of displacements $u_i, v_i, 0$ and the components of the tension tensor $\sigma_{ix}, \sigma_{iy}, \tau_{ixy}$ according to the formulae (1), (2).

Remark. The dynamic problem of the theory of elasticity for semi space was considered in the known monograph [15]. However, this problem was solved without initial conditions. The authors apply the Fourier transform of the time variable. It leads them to imprecision in the received formulas for the functions of tension. In our opinion the solution by the method of integral transforms of Fourier (18),(19) on a spatial variable also is more natural.

5. Conclusion

In the work the dynamic problem of elasticity theory are considered: the problem of oscillations of constructions and buildings, the problem of the propagation of elastic waves; thermo elastic waves. The method of integral transforms developed in solving problems. Using the integral transformation (Fourier, Laplace, Hankel) we came to a more simple task in the pattern space. Problem of elasticity theory for inhomogeneous bodies studied. These tasks are of great use in practice. The method of the vector integral transforms of Fourier with discontinuous coefficients used for the decision of problems of the theory of elasticity in a piecewise-homogeneous media. The solution of the dynamic problem in the analytical form found.

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