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hApproximate Solution of Stochastic Partial Differential Equation with Random Neumann Boundary Condition

A. H. EL_Bassiouny*, W. W. Mohammed, F. Eskander

Department of Mathematics, Faculty of Science, Mansoura University, Egypt
*Corresponding author: el_bassiouny@mans.edu.eg

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Abstract In this paper we approximate the solution of a parabolic nonlinear stochastic partial differential equation (SPDE) with cubic nonlinearity and with random Neumann boundary condition via a stochastic ordinary differential equation (SODE) which is a stochastic amplitude equation near a change of stability.

Keywords: *amplitude equations, SPDEs, random boundary conditions, multiscale analysis, Ginzburg-Landau equation.*

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1. Introduction

Stochastic partial differential equations (SPDEs) appear naturally as models for dynamical systems abided by random influences. The SPDEs have a wide range of applications outside mathematics. For instance, biology, chemistry, epidemiology, economics, microelectronics, mechanics, and finance.

For some applications the noise not affects only inside the medium, but on its physical boundary, too. This happens for heat transfer in a solid in contact with afield [6], the air-sea interactions on the ocean surface [8] and chemical reactor theory [7]. Thus, this topic has a rapidly developing as a fascinating research field with many interesting unanswered questions.

To approximate the SPDEs near a change of stability, we use a rigorous technique, so it is important to make the reduction of the dynamics of SPDEs to obtain simpler equations that are the amplitude or the modulation equations.

In this paper we deal with a parabolic equation (typically, the heat equation) perturbed by a Neumann boundary noise involve additive degenerate noise. More specifically, consider for $t \geq 0$;

$$\begin{aligned} &\partial_t u(x,t) = A_u(x,t) + \varepsilon^2 \nu u(x,t) \\ &+F(u(x,t)) + \varepsilon^2 \partial_t W(t),\ for\ x \in D \\ &\partial_t u(0,t) = 0,\ \partial_t u(1,t) = \varepsilon^2 \sigma \partial_t B(t), \\ &u(x,0) = u_0,\ or\ x \in D \end{aligned} \qquad (1)$$

where A is a non-positive self-adjoint operator with finite dimensional kernel, $\varepsilon^2 \nu u(t)$ is a small deterministic perturbation, the constant $\nu \in [-1,1]$, F is a cubic nonlinearity, W is a Wiener process, B is a real valued Brownian motion and σ is the positive noise intensity parameter.

In the case of no homogenous boundary conditions (i.e., $\sigma \neq 0$). Sowers [9] investigated general reaction diffusion equation with Neumann boundary conditions. Da Prato and Zabczyk [4,5] explained the difference between the problems with Dirichlet and Neumann boundary noises. Recently, Cerrai and Freidlin [2] have considered a nonlinear stochastic parabolic equation with Neumann boundary noise. The Ginzburg-Landau equation with random Neumann boundary conditions is solved numerically by Xu and Duan [10].

The paper is organized as follows. In the next section we state some definitions, notation and assumptions that we need for our result. In Section 3 we give a formal derivation for the amplitude equation, also we state and prove the main result of this paper. Finally, we give applications to the nonlinear heat equation.

2. Preliminaries

Let $H = L^2(D)$ be a Hilbert space with L^2-norm denoted by ||.|| and inner product by <.,.>, where D is a bounded domain with smooth boundary ∂D.

The linear operator $A = \partial_x^2$ generates an analytic semigroup $\{e^{tA}\}_{T\geq 0}$ on H. Moreover, denote by $\{e_k\}_{k=0}^{\infty}$, which forms a complete orthonormal basis in H; a family of eigenfunctions of A and $Ae_k = -\lambda_k e_k$ for the eigenvalues $\{e_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \lambda_k = \infty$. If we take $\{e_k\}_{k=0}^{\infty}$ in the form

$$e_k(x)=\begin{cases}1 & if\ k=0,\\ \sqrt{2}\cos(k\pi x) & if\ k\geq 1,\end{cases}$$

then $\lambda_k=\pi^2k^2$. Define N: = ker A = {1}, and $S=N^{\perp}$ the orthogonal component of N in H. Also, define the projection $p_c:H\to N$ and $P_s:=I-P_c$. Let the projections P_c and P_s are commute with A.

Definition 1. For $\alpha\in\mathbb{R}$; we define the space H^{α} as

$$H^{\alpha}=\left\{\sum_{k=0}^{\infty}\gamma_k e_k:\sum_{k=0}^{\infty}\gamma_k^2k^{2\alpha}<\infty\right\}\ with\ norm$$

$$\left\|\sum_{k=0}^{\infty}\gamma_k e_k\right\|_{\alpha}^2=\gamma_0^2+\sum_{k=0}^{\infty}\gamma_k^2k^{2\alpha},$$

where $\{e_k\}_{k\in\mathbb{N}_0}$ is an orthonormal basis of H and $\{\gamma_k\}_{k\in\mathbb{N}_0}$ are real numbers.

Lemma 2. For all t > 0 and $\beta\leq\alpha$, there are constants M > 0 and $\omega\geq 0$ such that for all $u\in H^{\beta}$

$$\left\|e^{tA}u\right\|_{\alpha}\leq Mt^{-\frac{\alpha-\beta}{2}}e^{-\omega t}\|u\|_{\beta}. \tag{2}$$

Definition 3. (Stopping time) For the N × S- valued stochastic process (a,ψ) defined in the next section. We define, for some T_0 > 0 and $\kappa\in\left(0,\frac{1}{5}\right)$, the stopping time τ^* as

$$\tau^*:=T_0\wedge\inf\left\{T>0:\|u(T)\|_{\alpha}>\varepsilon^{-\kappa}\right\}. \tag{3}$$

Also we have the following hypotheses.

H₁: Assume that the nonlinearity $F:\left(H^{\alpha}\right)^3\to H^{\alpha-\beta}$ with β is trilinear, symmetric and satisfies the following conditions, for some C > 0,

$$\|F(u,v,w)\|_{\alpha-\beta}\leq C\|u\|_{\alpha}\|v\|_{\alpha}\|w\|_{\alpha}$$
$$for\ all\ u,v,w\in H^{\alpha},$$
$$\langle F_c(u),u\rangle\leq 0\ for\ all\ u\in N.$$

and

$$\langle F_c(u,u,w),w\rangle\leq 0\ \forall u,w\in N.$$

We use F(u) = F(u,u,u) and $F_c=P_cF$ for short.

H₂: Let W be a cylindrical Wiener process on H. Suppose for t ≥ 0;

$$W(t)=\sum_{k=0}^{\infty}\alpha_k\beta_k(t)e_k,$$

where the $(\beta_k)_{k\in\mathbb{N}_0}$ are independent, standard Brownian motions in $\mathbb{R}$ and the $(\alpha_k)_{k\in\mathbb{N}_0}$ are real numbers for all k. Also, we assume that

$$\sum_{k=1}^{\infty}k^{2\alpha}\lambda_k^{2\delta-1}\alpha_k^2<\infty\ for\ \delta\in\left(0,\frac{1}{2}\right).$$

3. Amplitude Equation and Main Result

In this section we state and prove the main theorem after we derive the amplitude equation of the Equation (1). First, let us derive the Amplitude equation with error term. According to [3] the mild solution of Equation (1) is

$$\begin{aligned}u(t)&=e^{tA}u_0+\int_0^t e^{(t-s)A}\left[\varepsilon^2\nu u+F(u)\right]ds\\&+\varepsilon^2\int_0^t e^{(t-s)A}dW(s)+\varepsilon^2\sigma(1-A)\int_0^t e^{(t-s)A}DdB,\end{aligned} \tag{4}$$

where D_{γ} is the Neumann map and it is defined for any $\gamma\in\mathbb{R}$ by the solution of

$$(1-A)D_{\gamma}=0,\partial_x D_{\gamma}(0)=0\ and\ \partial_x D_{\gamma}(1)=\gamma.$$

Fortunately, we have an explicit expression for the Neumann map D_{γ} as follows:

$$D_{\gamma}(x)=\frac{e^x+e^{-x}}{e-e^{-1}}\gamma.$$

Define Z(t) as

$$Z(t)=(1-A)\int_0^t e^{(t-s)A}DdB.$$

In the following, we write Z as explicit formula in terms of Fourier series

$$Z(t)=\sum_{k=0}^{\infty}\langle Z(t),e_k\rangle e_k. \tag{5}$$

Hence,

$$\begin{aligned}\langle Z(t),e_k\rangle&=\left\langle(1-A)\int_0^t e^{(t-s)A}DdB,e_k\right\rangle\\&=\int_0^t e^{-(t-s)\lambda_k}\langle DdB,(1-A)e_k\rangle\\&=e_k(1)\int_0^t e^{-(t-s)\lambda_k}dB\end{aligned} \tag{6}$$

By substituting from (6) into (5) we have

$$Z(t)=\sum_{k=0}^{\infty}e_k(1)\int_0^t e^{-(t-s)\lambda_k}dB(s)e_k \tag{7}$$

Now, we can rewrite the mild solution (4) in the following form

$$\begin{aligned}u(t)&=e^{tA}u(0)+\int_0^t e^{(t-s)A}\left[\varepsilon^2\nu u+F(u)\right]ds\\&+\varepsilon^2W(t)+\varepsilon^2\sigma Z(t),\end{aligned} \tag{8}$$

where $W(t)=\int_0^t e^{(t-s)A}dW(s)$ and Z(t) is defined in (7). Thus

$$\begin{aligned}\partial_t u(t)&=Au(t)+\varepsilon^2\nu u(t)+F(u(t))\\&+\varepsilon^2\partial_t W(t)+\varepsilon^2\sigma Z(t)\end{aligned} \tag{9}$$

In order to rescale (9) to the slow time-scale, we consider the following ansatz

$$T=\varepsilon^2 t \tag{10}$$

to obtain

$$\partial_T \vartheta = \varepsilon^{-2} A\vartheta + \nu\vartheta + F(\vartheta) + \partial_T \tilde{W}(T) + \partial_T \tilde{Z}(T), \quad (11)$$

where

$$\tilde{W}(T) := \varepsilon W(\varepsilon^{-2}T) \text{ and } \tilde{Z}(T) := \sigma \sum_{k=0}^{\infty} \tilde{B}(T) e_k(1) e_k,$$

with $\tilde{B}(T) = \varepsilon B(\varepsilon^{-2}T)$. To get the amplitude equation with error term, let

$$\vartheta(T) = a(T) + \varepsilon\psi(T), \quad (12)$$

where $a \in N$ and $\psi \in S$ Substituting from (12) into (11) to have

$$\begin{aligned} \partial_T a + \varepsilon\partial_T \psi &= \varepsilon^{-2} A(a+\varepsilon\psi) + \nu(a+\varepsilon\psi) \\ &+ F(a+\varepsilon\psi) + \partial_T \tilde{W}(T) + \partial_T \tilde{Z}(T), \end{aligned} \quad (13)$$

Taking projection onto Pc for (13) we obtain

$$\partial_T a = \nu a + F_c(a+\varepsilon\psi) + \partial_T \tilde{W}_c(T) + \partial_T \tilde{Z}_c(T). \quad (14)$$

Taking projection onto Ps for (13) we obtain

$$\begin{aligned} \partial_T \psi &= \varepsilon^{-2} A_s \psi + \nu\psi + \varepsilon^{-1} F_s(a+\varepsilon\psi) \\ &+ \varepsilon^{-1}\partial_T \tilde{W}_s(T) + \varepsilon^{-1}\partial_T \tilde{Z}_s(T). \end{aligned} \quad (15)$$

In the next lemma, we can easy to show that the non-dominant modes ψ are not too large as asserted in Definition 3 for $T \le \tau^*$.

Lemma 4. Assume the hypothesis H_1 and H_2 hold. Then for all p $\ge$ 1 there is a constant C > 0 such that

$$E \sup_{T\in[0,T_0]} \|\psi(T)\|_a^p \le C\varepsilon^{-\kappa_0}.$$

for $\kappa_0 < \kappa < \frac{1}{4}$.

Proof. See the proof of the Corollary 4.3 in [1].

Lemma 5. Under the hypothesis H_1 and $E\|\Psi(0)\| \le C$ and for $\left(0, \frac{1}{3}\right)$ from the definition of τ^*, then

$$\begin{aligned} a(T) &= a(0) + \int_0^T \nu a(\tau)d\tau + \int_0^T F_c(a)d\tau \\ &+ Q_c(T) + R(T). \end{aligned} \quad (16)$$

with

$$R(T) = O(\varepsilon^{1-3\kappa}) \text{ and } Q_c(T) = \tilde{W}_c(T) + \tilde{Z}_c(T).$$

Proof. We have from the previous lemma that

$$\psi(T) = O(\varepsilon^{-\kappa_0}), \quad (17)$$

Substituting into (14) and integrating the resulted equation from 0 to T, we obtain

$$\begin{aligned} a(T) &= a(0) + \int_0^T \nu a(\tau)d\tau + \int_0^T F_c(\tau)d\tau \\ &+ Q_c(T) + R(T), \end{aligned}$$

where

$$\begin{aligned} R(T) &= 3\varepsilon\int_0^T F_c(a,a,\psi)d\tau + 3\varepsilon^2\int_0^T F_c(a,\psi,\psi)d\tau \\ &+ \varepsilon^2\int_0^T F_c(\psi)d\tau. \end{aligned}$$

We can find that the bound of R is $O(\varepsilon^{1-3\kappa})$ when we use equation (17).

Lemma 6. Let the hypotheses H1 and H2 hold. De.ne the stochastic process b(T) in N with $E|b(0)| \le C$ as the solution of

$$b(T) = b(0) + \int_0^T \nu b(\tau)d\tau + \int_0^T F_c(b(\tau)) + Q_c(T). \quad (18)$$

Then for T0 > 0 there exists a constant C > 0 such that $E \sup_{T\in[0,T_0]} |b(T)|^p \le C.$

Proof. We define X as

$$X(T) = b(T) - Q_c(T). \quad (19)$$

Substituting into (18), we obtain

$$\partial_T X = \nu(X+Q_c) + F_c(X+Q_c). \quad (20)$$

Taking the scalar product h:;XiR on both sides of (20)

$$\begin{aligned} \langle \partial_T X, X\rangle_{\mathbb{R}} &= \frac{1}{2}\partial_T |X|^2 = \langle \nu(X+Q_c), X\rangle_{\mathbb{R}} \\ &+ \langle F_c(X+Q_c), X\rangle_{\mathbb{R}} \end{aligned}$$

Using Cauchy-Schwartz and Young inequalities and the hypothesis H_1 we have

$$\frac{1}{2}\partial_T |X|^2 \le C\left(|X|^2 + |Q_c|^4\right) - \delta$$

By integrating the above equation from 0 to T .we obtain

$$\begin{aligned} |X(T)|^2 &\le \left(|X(0)|^2 + CT + CT\sup_{[0,T]}|Q_c|^4\right) \\ &\le C + CT_0 \sup_{[0,T]}|Q_c|^4. \end{aligned}$$

Taking $\frac{p}{2}$ -th power and using Gronwall's lemma, then the supremum and expectation, we obtain

$$E\sup_{[0,T_0]}|X|^p \le C^{\frac{p}{2}} + CT_0^{\frac{p}{2}} E\sup_{[0,T_0]}|Q_c|^{2p} \le C. \quad (21)$$

Using (21) and (19), we have

$$E\sup_{[0,T_0]}|b|^p \le CE\sup_{[0,T_0]}|X|^p + CE\sup_{[0,T_0]}|Q_c|^p \le C.$$

Definition 3: Define the set $\Omega^* \subset \Omega$ such that all these estimates

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < C\varepsilon^{-\frac{3}{2}\kappa}, \quad (22)$$

$$\sup_{[0,\tau^*]} \|R\|_\alpha < C\varepsilon^{1-4\kappa}, \quad (23)$$

$$\sup_{[0,\tau^*]} |b| < C\varepsilon^{-\frac{1}{2}\kappa}, \tag{24}$$

hold on Ω^*.

Theorem 1: Assume that the hypotheses H_1 and H_2 hold. Let a be the solution of (16) and b be the solution of (18). If the initial condition satisfies $a(0) = b(0)$, then

$$\sup_{T\in\left[0,\tau^*\right]} \|a(T) - b(T)\|_\alpha \le C\varepsilon^{1-5\kappa},$$

and

$$\sup_{T\in\left[0,\tau^*\right]} \|a(T)\|_\alpha \le C\varepsilon^{-\frac{1}{2}\kappa},$$

for $\kappa < \frac{1}{5}$ on Ω^*.

Proof: Define $\varphi(T)$ as

$$\varphi(T) := a(T) - R(T).$$

From (16) we obtain

$$\begin{aligned}\varphi(T) &= a(0) + \int_0^T \nu\left(\varphi(\tau) + R(\tau)\right) d\tau \\ &+ \int_0^T F_c(\varphi(\tau) + R(\tau))\, d\tau + Q_c(T).\end{aligned} \tag{25}$$

Subtracting (25) from (18) and defining $h(T) := b(T) - \varphi(T)$, we obtain

$$h(T) = \int_0^T \nu\left(h - R\right) d\tau + \int_0^T F_c(b) d\tau - \int_0^T F_c(b - h + R) d\tau.$$

Thus,

$$\partial_T h = \nu\left(h - R\right) + F_c(b) - F_c(b - h + R). \tag{26}$$

Taking the scalar product $\langle ., h\rangle$ on both sides of (26), we have

$$\begin{aligned}\frac{1}{2}\partial_T |h|^2 &= \nu\langle h, h\rangle - \nu\langle R, h\rangle + \langle F_c(b), h\rangle \\ &- \langle F_c(b - h + R), h\rangle.\end{aligned}$$

Using Cauchy-Schwartz and Young inequalities, we obtain the following linear ordinary differential inequality

$$\partial_T |h|^2 \le C|h|^2 [1 + |h|^2] + C|R|^2 [1 + |R|^4 + |b|^4], \tag{27}$$

holds on Ω^* By substituting from (23) and (24) into (27). As long as $|h| \le 1$,

Integrating from 0 to T and using Gronwall's lemma, we obtain

$$|h|^2 \le C\varepsilon^{2-10\kappa}, \text{ on } \Omega^*.$$

Hence,

$$\sup_{[0,\tau^*]} |h| \le C\varepsilon^{1-5\kappa}, \text{ on } \Omega^*.$$

Then,

$$\begin{aligned}\sup_{[0,\tau^*]} |a - b|_\alpha &= \sup_{[0,\tau^*]} |h - R| \le \sup_{[0,\tau^*]} |h| + \sup_{[0,\tau^*]} |R| \\ &\le C\varepsilon^{1-5\kappa},\end{aligned}$$

for $\kappa < \frac{1}{5}$. For the second part of the theorem, using the triangle inequality, we have

$$\begin{aligned}\sup_{[0,\tau^*]} |a|_\alpha &\le \sup_{[0,\tau^*]} |a - b| + \sup_{[0,\tau^*]} |b| \\ &\le C\varepsilon^{1-5\kappa} + C\varepsilon^{-\frac{1}{2}\kappa} \\ &\le C\varepsilon^{-\frac{1}{2}\kappa}.\end{aligned}$$

Theorem 2. ***(Approximation theorem):*** Under hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, let u be the solution of (1) defined in (10) and (12) with the initial conditions $u(0) = \varepsilon\, a(0) + \varepsilon^2\, \psi(0)$ where $a(0) \in N$ and $\psi(0) \in S$, b is the solution of (18) with $b(0) = a(0)$. Then for $T_0 > 0$ and for $k \in (0, \frac{1}{5})$, there exists $C > 0$ such that

Proof: First we note that by using triangle inequality, we obtain

$$\begin{aligned}&\sup_{T\in\left[0,\tau^*\right]} \left\|u(\varepsilon^{-2}T) - \varepsilon\, b(T)\right\|_\alpha \\ &\le \varepsilon \sup_{\left[0,\varepsilon^{-2}\tau^*\right]} \|a - b\|_\alpha + \varepsilon^2 \sup_{\left[0,\varepsilon^{-2}\tau^*\right]} \|\psi\|_\alpha \\ &\le \varepsilon\, C\varepsilon^{1-4\kappa} + \varepsilon^2\, C\varepsilon^{-\frac{3}{2}\kappa} \le C\varepsilon^{2-4\kappa}, \text{ on } \Omega^*,\end{aligned}$$

and

$$\sup_{T\in[0,T_0]} \left\|u(\varepsilon^{-2}T) - \varepsilon b(T)\right\|_\alpha = \sup_{T\in\left[0,\tau^*\right]} \left\|u(\varepsilon^{-2}T) - \varepsilon b(T)\right\|_\alpha,$$

on Ω^*.

For the probability of Ω^* we have,

$$\begin{aligned}P\left(\Omega^*\right) &\ge 1 - P\left\{\sup_{[0,\tau^*]} \|\psi\|_a \ge \varepsilon^{-\frac{3}{2}\kappa}\right\} \\ &- P\left\{\sup_{[0,\tau^*]} \|R\|_a \ge \varepsilon^{1-4\kappa}\right\} - P\left\{\sup_{[0,\tau^*]} \|b\|_a \ge \varepsilon^{-\frac{1}{2}\kappa}\right\}\end{aligned}$$

Hence,

$$\begin{aligned}&1 - P\left(\Omega^*\right) \\ &\le P\left\{\sup_{\left[0,\tau^*\right]} \|\psi\|_\alpha \ge \varepsilon^{-\frac{3}{2}\kappa}\right\} + P\left\{\sup_{\left[0,\tau^*\right]} \|R\|_\alpha \ge \varepsilon^{1-4\kappa}\right\} \\ &+ P\left\{\sup_{\left[0,\tau^*\right]} \|b\|_\alpha \ge \varepsilon^{-\frac{1}{2}\kappa}\right\} \le C\varepsilon^{\frac{1}{2}q\kappa} + C\varepsilon^{q\kappa} + C\varepsilon^{\frac{1}{2}q\kappa} \\ &\le C\varepsilon^{\frac{1}{2}q\kappa},\end{aligned} \tag{28}$$

we used Chebychev's inequality. Thus

$$P[\sup_{t\in\left[0,\varepsilon^{-2}T_0\right]}\left\|u(t)-\varepsilon b(\varepsilon^2 t)\right\| > C\varepsilon^{2-5\kappa}] \le C\varepsilon^{p\kappa},$$

where $p=\frac{1}{2}q\kappa$.

4. Application

We apply our results to heat equation. The heat equation is a partial differential equation that describe the distribution of heat in a given area in a given time interval. Generally, given a certain area in space, because of heat movement from warmer are ask to colder ones, the warm spots will cool down and the colder spots will begin to warm up. Solutions for which there is no heat moving are called "equilibrium solutions".

Also, we can set boundary conditions for this PDE. For instance, if we have a rod with one end on a block of ice and the other end attached to a heater. Here we find that the interior point on the rod will not excede the temperature of the heater and will not drop below the temperature of the ice. Therefore we can apply our work on this kind (heat equation) with Neumman boundary condition which has the form

$$\begin{gathered}\partial_t u=\Delta u+\varepsilon^2\nu u-u^3+\varepsilon^2 dW \\ \partial_x u(0,t)=0,\ \text{and}\ \partial_x u(1,t)=\varepsilon^2\sigma\,\partial_t B(t).\end{gathered} \tag{29}$$

Now, we can satisfy the conditions of stability:

For $u=\gamma_1$, and $w=\gamma_2\ \in N$,

$$\left\langle F_c(u),u\right\rangle \le -C\gamma_1^4 \le 0,$$

and

$$\left\langle F_c(u,u,w),w\right\rangle \le -C\gamma_1^2\gamma_2^2 \le 0.$$

The main theorem states that the solution of the heat Equation (29) is approximated by

$$u(t)=\varepsilon\vartheta(\varepsilon^2 t),$$

and

$$\vartheta(T)=b(T)+O(\varepsilon^{1-}),$$

where b is the solution of the amplitude equation that takes the form

$$\partial_T b=\nu b-b^3+\alpha\ \ \partial_T\tilde{\beta}+\sigma\,\partial_T\tilde{B}.$$

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