*International Journal of Partial Differential Equations and Applications*, 2015, Vol. 3, No. 1, 12-19
Available online at http://pubs.sciepub.com/ijpdea/3/1/3

DOI:10.12691/ijpdea-3-1-3

# The Approximate Solutions of the stochastic Generalized Swift-Hohenberg Equation with Neumann Boundary Conditions

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*Received January 20, 2015, Revised February 05, 2015, Accepted February 10, 2015*

**Abstract** We consider the stochastic Generalized Swift-Hohenberg (GSSH) equation with respect to Neumann boundary conditions on the interval [0, π] in this form

$$\partial_t u = -\left(q_c^2 + \partial_x^2\right)^2 u + v\varepsilon^4 u + bu^2 - u^3 + \varepsilon^{\frac{5}{2}}\partial_t W(t).$$

Our aim of this paper is to approximate the solutions of (GSSH) via the amplitude equation with quintic term.

***Keywords:*** *Generalized Swift-Hohenberg equation, Neumann boundary conditions, amplitude equations, time-scales*

**Cite This Article:** E. M Elabbasy, Wael W. Mohammed, and Mahmoud A. Nagy, "The Approximate Solutions of the stochastic Generalized Swift-Hohenberg Equation with Neumann Boundary Conditions." *International Journal of Partial Differential Equations and Applications*, vol. 3, no. 1 (2015): 12-19. doi: 10.12691/ijpdea-3-1-3.

## 1. Introduction

The stochastic Swift-Hohenberg equation (SH, for short) is one of important equations for description localized structures in the modern physics, which was. rst used as a toy model for the convective instability in Rayleigh-Bénard problem [1] or [2]. Blömker et al [3,5] studied the stochastic Swift-Hohenberg equation in this form

$$\partial_t u = -\left(1 + \partial_x^2\right)^2 u + \mu_\varepsilon u - u^3 + \sigma_\varepsilon \partial_t W(t), \qquad (1)$$

near its change of stability. While in [4] they studied the stochastic Swift-Hohenberg Equation (1) on large domains near a change of stability. They approximated the solutions by amplitude equation with cubic and noise terms.

In this paper we consider the generalized stochastic Swift-Hohenberg equation (GSSH, for short) in this form

$$\partial_t u = -\left(q_c^2 + \partial_x^2\right)^2 u + v_\varepsilon u + bu^2 - u^3 + \sigma_\varepsilon \partial_t W(t), \qquad (2)$$

where b is constant, $v_\varepsilon u(t)$ is a small deterministic (linear) perturbation and W(t) be a finite Wiener process. Mohammed et al [6] studied (GSSH) wtih $v_\varepsilon = v\varepsilon^2$ and $\sigma_\varepsilon = \varepsilon^{\frac{3}{2}}$, they obtained the amplitude equation with cubic term.

For simplicity of the presentation here we study the Equation (2) with respect to Neumann boundary conditions on [0, π] in the case of

$$q_c \neq 1, v_\varepsilon = v\varepsilon^4 \text{ and } \sigma_\varepsilon = \varepsilon^{\frac{5}{2}}$$

to obtain the amplitude equation with quintic term as follows

$$\partial_t \gamma = v\gamma - c_0\gamma^5 + \sum_{k \neq q_c}^{N} \frac{b\alpha_k^2}{4\lambda_k} B_c\left(e_k, e_k\right)$$

where $c_0$ constant. Also, we show that near a change of stability on a time-scale of order $\varepsilon^{-4}$ the solution u(t) of (2) is well approximated by

$$u(t) \simeq \varepsilon\gamma\left(\varepsilon^4 t\right) + \mathrm{O}\left(\varepsilon^{2-}\right). \qquad (3)$$

The organization of this paper as follows. In Section 2 we state the assumptions and definition that we need, while in Section 3 we give the formal derivation of the amplitude equation of (GSSH). In Section 4 we give bounds for high modes.

In Section 5 we state and proof the main results. Finally, In Section 6 we give application to (GSSH) in one dimension with respect to Neumann boundary conditions.

## 2. Preliminaries

Before we state our assumptions and Definitions that we need, let us rewrite the equation (2) in abstract form as follows

$$\partial_t u(t) = Au(t) + \varepsilon^4 \nu u(t) + bB(u(t)) + F(u(t)) + \varepsilon^{\frac{5}{2}} \partial_t W(t), \tag{4}$$

where

$$A = -\left(q_c^2 + \partial_x^2\right)^2$$

is a linear operator with finite dimensional kernel $\{\cos(q_c x)\}$, $\varepsilon^4 \nu u$ is a small deterministic perturbation, B(u) is a quadratic nonlinearity and F (u) is a cubic nonlinearity. Let $[e_k]_{k=0}^{\infty}$ be an eigenfunctions of A in H with the corresponding eigenvalues $\lambda_k$, which satisfies $Ae_k = -\lambda_k e_k$.

Define N := kerA, S = $N^{\perp}$ the orthogonal complement of N in H, and Pc for the projection Pc : H→N and Define Ps := I-Pc where I is the identity operator on H. As the dimension of N is finite, it is well known that both Pc and Ps are bounded linear operators on H (cf. Weidmann[7]).

**Definition 1** *For* $\alpha \in \mathbb{R}$, *we define the space* $H^{\alpha}$ *as*

$$H^{\alpha} = \left\{ \sum_{k=0}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \right\}$$

*with norm*

$$\left\| \sum_{k=0}^{\infty} \gamma_k e_k \right\|_{\alpha}^2 = \gamma_0^2 + \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha},$$

*where* $[e_k]_{k=0}^{\infty}$ *be an orthonormal basis of H and* $[\gamma_k]_{k=0}^{\infty}$ *are real numbers.*

For the quadratic nonlinearity B we assume that:

**Assumption 2** *Assume that* $B : \left(H^{\alpha}\right)^2 \to H^{\alpha-\beta}$ *be a bounded bilinear, symmetric (i.e. B(u, v) = B(v, u)) and satisfies* $P_c B(e_k, e_k) \neq 0$ *for k∈(0,1). We use B(u) = B(u, u),* $B_s = P_s B$ *and* $B_c = P_c B$ *for short.*

For the cubic nonlinearity F we assume that:

**Assumption 3** *Assume that* $F : \left(H^{\alpha}\right)^3 \to H^{\alpha-\beta}$ *is trilinear, symmetric and satisfies the following conditions, for some C > 0,*

$$\|F(u,v,w)\|_{\alpha-\beta} \le C \|u\|_{\alpha} \|v\|_{\alpha} \|w\|_{\alpha} \ \forall u, v, w \in H^{\alpha} \tag{5}$$

$$\langle F_c(u), v \rangle \le 0 \ \forall u \in N, \tag{6}$$

*and*

$$\langle F_c(u,u,w), w \rangle \le 0 \ \forall u, w \in N. \tag{7}$$

We use F(u) = F(u, u, u), $F_s = P_s F$ and $F_c = P_c F$ for short.

**Assumption 4** *Assume that*

$$-2b^2 B_c\left(a, A_s^{-1} B_s(a)\right) + F_c(a) = 0,$$

$$B_c\left(a, A_s^{-1} B_s\left(a, A_s^{-1} B_s(a)\right)\right) = 0,$$

$$B_c\left(a, A_s^{-1} F_s(a)\right) = 0,$$

*and*

$$B_c\left(B_k(a) e_k, B_{\ell}(a) e_{\ell}\right) = 0,$$

*where*

$$B_k(a) = \langle B_k(a) e_k \rangle$$

For the noise we suppose:

**Assumption 5** *Let W be a finite Wiener process on H. Suppose for t ≥ 0,*

$$W(t) = \sum_{k \neq q_c}^{N} \alpha_k \beta_k e_k,$$

*where* $(\beta_k)_k$ *are independent, standard Brownian motions in* $\mathbb{R}$ *and* $(\alpha_k)_k$ *are real numbers and* $P_c W = 0$.

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large, as given by the next definition.

**Definition 6** *For the N X S-valued stochastic process (a, ψ) defined later in (10) we define, for some* $T_0 > 0$ *and* $\kappa \in \left(0, \frac{1}{15}\right)$, *the stopping time* $\tau^*$ *as*

$$\tau^* := T_0 \wedge \inf \left\{ \begin{array}{l} T > 0 : \|a(T)\|_{\alpha} > \varepsilon^{-\kappa} \\ or \\ \|\psi(T)\|_{\alpha} > \varepsilon^{-3\kappa} \end{array} \right\}. \tag{8}$$

**Definition 7** *For a real-valued family of processes* $\{X_{\varepsilon}(t)\}_{t \ge 0}$ *we say* $X_{\varepsilon} = O(f_{\varepsilon})$, *if for every p ≥ 1 there exists a constant Cp such that*

$$E \sup_{t \in [0, \tau^*]} |X_{\varepsilon}(t)|^p \le C_p f_{\varepsilon}^p. \tag{9}$$

We use also the analogous notation for time-independent random variables.

## 3. The amplitude Equation

In this section we present a short formal derivation of the amplitude equation.

We interest here the studying behavior of solution of (4) on time-scales of order $\varepsilon^{-4}$. So, we split the solution u(t) into

$$u(t) = \varepsilon a\left(\varepsilon^4 t\right) + \varepsilon^2 \psi\left(\varepsilon^4 t\right), \tag{10}$$

where a ∈ N and ψ ∈ S. After rescaling to the slow time-scale $T = \varepsilon^4 t$, we obtain the following system of equations:

$$da(T) = \left[ \begin{array}{l} \nu a(T) + 2b\varepsilon^{-2} B_c(a, \psi) \\ + b\varepsilon^{-1} B_c(\psi) - \varepsilon^{-2} F_c(a) \\ -3\varepsilon^{-1} F_c(a, a, \psi) \\ -3F_c(a, \psi, \psi) - \varepsilon F_c(\psi) \end{array} \right] dT, \tag{11}$$

and

$$d\psi(T)=\begin{bmatrix}\varepsilon^{-4}A_s\psi(T)+\nu\psi(T)\\ +b\varepsilon^{-6}B_s(a+\psi)\\ -\varepsilon^{-6}F_s(a+\psi)\end{bmatrix} +\varepsilon^{\frac{-3}{2}}\sum_{k\neq q_c}^{N}a_k d\tilde{\beta}_k(T)e_k, \tag{12}$$

where $\tilde{\beta}_k(T):=\varepsilon^2\beta_k\left(\varepsilon^{-4}T\right)$ is a rescaled version of the Brownian motion. Equation (11) reads in integrated form as

$$\begin{aligned}&a(T)=\\ &a(0)+\nu\int_0^T a ds+2b\varepsilon^{-2}\int_0^T B_c(a,\psi)ds\\ &+b\varepsilon^{-1}\int_0^T B_c(\psi)ds-\varepsilon^{-2}\int_0^T F_c(a)ds\\ &-3\varepsilon^{-1}\int_0^T F_c(a,a,\psi)ds-3\int_0^T F_c(a,\psi,\psi)\\ &-\varepsilon^{-1}\int_0^T F_c(\psi)ds\end{aligned} \tag{13}$$

First step, we want to remove the terms depending on $\varepsilon^{-2}$. Therefore, we apply Itô formula to $B_c\left(a,A_s^{-1},\psi\right)$ to obtain

$$\begin{aligned}&2b\varepsilon^{-2}\int_0^T B_c(a,\psi)dT\\ &=-\sum_{\ell,k}\frac{8b^4}{\lambda_\ell(\lambda_\ell+\lambda_k)}\int_0^T B_c\begin{pmatrix}B_c(a,B_k(a)e_k),\\ A_s^{-1}B_\ell(a)e_\ell\end{pmatrix}ds\\ &-2b^2\int_0^T B_c\left(F_c(a),A_s^{-1}A_s^{-1}B_s(a)\right)ds\\ &-4b^2\int_0^T B_c\left(a,A_s^{-1}B_s\left(a,A_s^{-1},F_s(a)\right)\right)ds\\ &-8b^4\int_0^T B_c\left(a,A_s^{-1}B_s\begin{pmatrix}a,A_s^{-1},\\ B_s\left(a,A_s^{-1},B_s(a)\right)\end{pmatrix}\right)ds\\ &-6b^2\int_0^T B_c\left(a,A_s^{-1}F_s\left(a,a,A_s^{-1},B_s(a)\right)\right)ds\\ &-\sum_{\ell,k}\frac{4b^4}{\lambda_k(\lambda_k+\lambda_\ell)}\int_0^T B_c\left(B_c\begin{pmatrix}a,A_s^{-1},\\ B_s\begin{pmatrix}B_k(a)e_k,\\ B_\ell(a)e_\ell\end{pmatrix}\end{pmatrix}\right),ds\\ &-2b^2\varepsilon^{-2}\int_0^T B_c\left(a,A_s^{-1}B_s(a)\right)ds\\ &+O\left(\varepsilon^{1-8\kappa}\right).\end{aligned} \tag{14}$$

Second step, we want to remove the term depending on $\varepsilon^{-1}$. Therefore, we apply Itô formula to $B_c(\psi_k e_k,\psi_\ell e_\ell)$ and $F_c(a,a,\psi)$ to obtain

$$\begin{aligned}&b\varepsilon^{-1}\sum_{\ell,k}B_c(\psi_k e_k,\psi_\ell e_\ell)ds\\ &=-\sum_{\ell,k}\frac{4b^4}{\lambda_\ell(\lambda_\ell+\lambda_k)}\int_0^T B_c\begin{pmatrix}B_k(a)e_k,\\ B_\ell\left(a,A_s^{-1}B_s(a)\right)\end{pmatrix}e_\ell ds\\ &-\sum_{\ell,k}\frac{2b^2}{\lambda_\ell(\lambda_\ell+\lambda_k)}\int_0^T B_c\left(B_k(a)e_k,F_\ell(a)e_\ell\right)ds\\ &-\sum_{\ell,k}\frac{2b^2}{\lambda_\ell(\lambda_\ell+\lambda_k)}\int_0^T B_c\left(F_k(a)e_k,B_\ell(a)e_\ell\right)ds\\ &+\sum_{\ell,k,j}\frac{4b^2}{\lambda_\ell(\lambda_\ell+\lambda_j)(\lambda_k+\lambda_\ell)}\int_0^T B_c\begin{pmatrix}B_k\begin{pmatrix}a,\\ B_j(a)e_j\end{pmatrix}e_k,\\ B_\ell(a)e_\ell\end{pmatrix}ds\\ &+\sum_{\ell,k,j}\frac{4b^2}{\lambda_\ell(\lambda_\ell+\lambda_j)(\lambda_k+\lambda_\ell)}\int_0^T B_c\begin{pmatrix}B_k\begin{pmatrix}a,\\ B_j(a)e_j\end{pmatrix}e_k,\\ B_\ell(a)e_\ell\end{pmatrix}ds\\ &+\sum_{k\neq q_c}^{N}\frac{b\alpha_k^2}{4\lambda_k}\int_0^T B_c(e_k,e_k)ds+O\left(\varepsilon^{1-8\kappa}\right).\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\varepsilon^{-1}\int_0^T F_c(a,a,\psi)ds=\varepsilon^{-1}\int_0^T F_c\left(a,a,A_s^{-1}B_s(a)\right)ds\\ &+2b^2\int_0^T F_c\left(a,a,A_s^{-1}B_s\left(a,A_s^{-1}B_s(a)\right)\right)ds\\ &+\int_0^T F_c\left(a,a,A_s^{-1}F_s(a)\right)ds\\ &+O\left(\varepsilon^{1-8\kappa}\right).\end{aligned} \tag{16}$$

We use Assumption 4 to obtain

$$\int_0^T F_c\left(a,a,A_s^{-1}F_s(a)\right)ds=0.$$

Third step, we apply Itô formula to $F_c(a,\psi,\psi)$ to get

$$\begin{aligned}&\int_0^T F_c(a,\psi,\psi)ds=\\ &\sum_{\ell,k}\frac{2b^2}{\lambda_k(\lambda_k+\lambda_\ell)}\int_0^T F_c\begin{pmatrix}a,B_k(a)e_k,\\ B_\ell(a)e_\ell\end{pmatrix}ds\\ &+O\left(\varepsilon^{1-15\kappa}\right)\end{aligned} \tag{17}$$

Substituting from Equations (14), (15), (17) and (16) into Equation (13) to obtain

$$\begin{aligned}a(T)&=a(0)+\int_0^T\left[\nu a+\tilde{G}(a)\right]ds\\ &+\sum_{k\neq q_c}^{N}\frac{b\alpha_k^2}{4\lambda_k}\int_0^T B_c(e_k,e_k)ds+R(T).\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
\tilde{G}(a) = &\sum_{\ell,k} \frac{8b^4 B_k(a) B_\ell(a)}{\lambda_l^2(\lambda_\ell+\lambda_k)} B_c\left(B_c(a,e_k),e_l\right) \\
&-2b^2 B_c\left(F_c(a), A_s^{-1}A_s^{-1}B_s(a)\right) \\
&-4b^2 B_c\left(a, A_s^{-1}B_s\left(a, A_s^{-1}F_c(a)\right)\right) \\
&-8b^4 B_c\left(a, A_s^{-1}B_s\left(a, A_s^{-1}B_s(a)\left(a, A_s^{-1}B_s(a)\right)\right)\right) \\
&-6b^2 B_c\left(a, A_s^{-1}F_s\left(a,a,A_s^{-1}B_s(a)\right)\right) \\
&-\sum_{\ell,k} \frac{4b^4 B_k(a) B_\ell(a)}{\lambda_k(\lambda_k+\lambda_\ell)} B_c\left(a, A_s^{-1}B_s(e_k,e_\ell)\right) \\
&-\sum_{\ell,k} \frac{4b^4 B_k(a)}{\lambda_\ell(\lambda_\ell+\lambda_k)} B_c\left(e_k, B_\ell\left(a, A_s^{-1}B_s(a)\right)e_\ell\right) \\
&-\sum_{\ell,k} \frac{2b^4 B_k(a) F_\ell(a)}{\lambda_\ell \lambda_k} B_c(e_k,e_\ell) \\
&+\sum_{\ell,k,j} \frac{8b^4 B_j(a) B_\ell(a)}{\lambda_\ell(\lambda_\ell+\lambda_j)(\lambda_k+\lambda_\ell)} \\
&B_c\left(B_k(a,e_j)e_k,e_l\right) - 3F_c\left(a,a,A_s^{-1}F_s(a)\right) \\
&-\sum_{\ell,k} \frac{6b^4 B_k(a) B_\ell(a)}{\lambda_k(\lambda_k+\lambda_\ell)} F_c(a,e_k,e_\ell) \\
&-6b^2 F_c\left(a,a,A_s^{-1}B_s\left(a,A_s^{-1}B_s(a)\right)\right)ds
\end{aligned}
$$

and

$$
R(T) = -\varepsilon\int_0^T F_c(\psi)\,ds + O\left(\varepsilon^{1-15\kappa}\right). \tag{19}
$$

# 4. Bounds for the high modes

Define Ornstein-Uhlenbeck process Z (OU, for short) as follows

$$
Z(T) := \sum_k Z_k(T) e_k, \tag{20}
$$

where

$$
Z_k(T) := \varepsilon^{-2}\alpha_k \int_0^T e^{-\varepsilon^{-4}\lambda_k(T-s)}\, d\tilde{\beta}_k(s). \tag{21}
$$

Equation (12) reads in integrated form as

$$
\begin{aligned}
\psi(T) = &e^{\varepsilon^{-4}TA_s\psi(0)} \\
&+\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}\left[\nu\psi(T) + 2b\varepsilon^{-3}B_s(a,\psi)\right] \\
&+b\varepsilon^{-2}B_s(\psi,\psi) - \varepsilon^{-3}F_s(a+\varepsilon\psi)(s)\,ds \\
&+b\varepsilon^{-4}\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}\left[B_s(a,a)\right](\tau)\,d\tau + \varepsilon Z(T)
\end{aligned} \tag{22}
$$

where Z(T) is Defined in (20) and $e^{\varepsilon^{-4}(T-\tau)A_s}$ is the semi group created by the operator $-\left(q_c^2+\partial_x^2\right)^2$ (cf. [8] ).

In next lemma, we will approximate $\psi$ by the fast Ornstein-Uhlenbeck process Z as follows.

**Lemma 8** *There is a constant C > 0 such that, for* $\kappa > 0$ *from the definition of* $\tau^*$ *and* $p \ge 1$,

$$
E\sup_{t\in[0,\tau^*]} \left\|\psi(T) - Q(T)\right\|_\alpha^p \le C\varepsilon^{p-9p\kappa}, \tag{23}
$$

*where* $Q(T)$

$$
\begin{aligned}
Q(T) = &e^{\varepsilon^{-4}TA_s}\psi(0) + \\
&b\varepsilon^{-4}\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}\left[B_s(a,a)\right](\tau)\,d\tau + \varepsilon Z(T)
\end{aligned} \tag{24}
$$

*and Z(T) is defined in (20).*

**Proof.** From Equations (22) and (24) we obtain

$$
\psi(T) = \int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}\begin{bmatrix} \nu\psi(T) \\ +2b\varepsilon^{-3}B_s(a,\psi) \\ +b\varepsilon^{-2}B_s(\psi,\psi) \\ -\varepsilon^{-3}F_s(a+\varepsilon\psi) \end{bmatrix}(\tau)\,d\tau + Q(T) \tag{25}
$$

Taking the norm of both sides and using triangle inequality to obtain

$$
\begin{aligned}
&\left\|\psi(T) - Q(T)\right\|_\alpha \\
&\le C\left\|\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}\psi\,d\tau\right\|_\alpha \\
&+C\varepsilon^{-3}\left\|\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}B_s(a,\psi)\,d\tau\right\|_\alpha \\
&+C\varepsilon^{-2}\left\|\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}B_s(\psi,\psi)\,d\tau\right\|_\alpha \\
&+C\varepsilon^{-3}\left\|\int_0^T e^{\varepsilon^{-4}(T-\tau)A_s}F_s(a+\varepsilon\psi)\,d\tau\right\|_\alpha \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Where now bound these terms separately. For the. rst term, we obtain

$$
\begin{aligned}
I_1 &\le C\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|\psi(\tau)\right\|_\alpha d\tau \\
&\le C\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|\psi(\tau)\right\|_\alpha d\tau \\
&\le C\varepsilon^4 \sup_{t\in[0,\tau^*]}\left\|\psi(\tau)\right\|_\alpha \int_0^{\varepsilon^{-4}\omega T} e^{-\eta}\,d\eta \le C\varepsilon^{4-3\kappa},
\end{aligned}
$$

where we used the definition of $\tau^*$. For the second term and the third term, we obtain

$$
\begin{aligned}
I_2 &\le C\varepsilon^{-3}\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|B_s(a(\tau)),\psi(\tau)\right\|_\alpha d\tau \\
&\le C\varepsilon^{-3}\sup_{t\in[0,\tau^*]}\left\|B_s(a(\tau)),\psi(\tau)\right\|_\alpha \int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}d\tau \\
&\le C\varepsilon \sup_{t\in[0,\tau^*]}\left\{\left\|(a(\tau))\right\|_\alpha \left\|\psi(\tau)\right\|_\alpha\right\}\int_0^{\varepsilon^{-4}\omega T} e^{-\eta}\,d\eta \\
&\le C\varepsilon^{1-4\kappa},
\end{aligned}
$$

and

$$I_3 \le C\varepsilon^{-2}\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|B_s\left(a(\tau)\right),\psi(\tau)\right\|_\alpha d\tau$$
$$\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]}\left\|B_s\left(a(\tau)\right),\psi(\tau)\right\|_\alpha \int_0^{\varepsilon^{-4}\omega T} e^{-\eta}d\eta$$
$$\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]}\left\|\psi(\tau)\right\|_\alpha^2 \int_0^{\varepsilon^{-4}\omega T} e^{-\eta}d\eta$$
$$\le C\varepsilon^{2-6\kappa},$$

Analogously, we derive for the fourth term

$$I_4 \le C\varepsilon^{-3}\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|F_s\left(a(\tau)\right)+\varepsilon\psi(\tau)\right\|_\alpha d\tau$$
$$\le C\varepsilon^{-3}\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|a(\tau)+\varepsilon\psi(\tau)\right\|_\alpha^3 d\tau$$
$$\le C\varepsilon \sup_{\tau\in[0,\tau^*]}\left\|a(\tau)+\varepsilon\psi(\tau)\right\|_\alpha^3 \int_0^{\varepsilon^{-4}\omega T} e^{-\eta}d\eta$$
$$\le C\varepsilon\left(\sup_{[0,\tau^*]}\left\|a(\tau)\right\|_\alpha^3+\varepsilon^3\sup_{[0,\tau^*]}\left\|a(\psi)\right\|_\alpha^3\right)$$
$$\le C\varepsilon^{1-9\kappa},$$

where we used again the definition of $\tau^*$. Combining all results, yields (23).

**Lemma 9** *There is a constant C > 0 such that, for $\kappa$ > 0 from the definition of $\tau^*$ and p≥1,*

$$E\sup_{T\in[0,\tau^*]}\left\|\int_0^T e^{-\varepsilon^{-4}(T-\tau)A_s}\left[B_s(a,a)\right](\tau)d\tau\right\|_\alpha^p \le C\varepsilon^{4p-2p\kappa}, \tag{26}$$

**Proof.** We obtain for T $\le\tau^*$.

$$\left\|\int_0^T e^{-\varepsilon^{-4}(T-\tau)A_s}\left[B_s(a,a)\right](\tau)d\tau\right\|_\alpha$$
$$\le C\int_0^T e^{-\varepsilon^{-4}(T-\tau)A_s}\left\|B_s(a,a)\right\|_\alpha dd\tau$$
$$\le C\sup_{\tau\in[0,\tau^*]}\left\|B_s(a,a)\right\|_\alpha \int_0^T e^{-\varepsilon^{-4}(T-\tau)A_s}d\tau$$
$$\le C\varepsilon^4\sup_{\tau\in[0,\tau^*]}\left\|a\right\|_\alpha^2 \int_0^{\varepsilon^{-4}\omega T} e^{-\eta}d\eta$$
$$\le C\varepsilon^{4-2\kappa}.$$

**Lemma 10** *Under Assumation 5, for every $\kappa$ > 0 and p ≥ 1, there is a constant C, depending on p, $\alpha_k$, $\lambda_k$, $\kappa$ and $T_0$, such that*

$$E\sup_{T\in[0,T_0]}\left|Z_k(T)\right|^p \le C\varepsilon^{-\kappa_0},$$

*where $Z_k(T)$ is defined in (21).*

**Proof.** See Lemma 4.2 in [6].

**Lemma 11** *Let $\psi$ (0) = O(1). Then for p ≥ 0, $\kappa$ > 0 and from the definition of $\tau^*$, there is a constant C > 0 such that*

$$E\left(\sup_{T\in[0,\tau^*]}\left\|\psi(T)\right\|_\alpha^p\right)\le C\varepsilon^{-2p\kappa_0}. \tag{27}$$

**Proof.** From Equations (25), by triangle inequality, Lemma 8, Lemma 10 and Lemma 9 we obtain

$$E\left(\sup_{T\in[0,\tau^*]}\left\|\psi(T)\right\|_\alpha^p\right)\le\left\|\psi(T)-Q(T)\right\|_\alpha^p+\left\|e^{\tau\varepsilon^{-4}A_s}\psi(0)\right\|_\alpha^p$$
$$+\varepsilon^{-4p}\left\|\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}B_s\left(a(\tau),a(\tau)\right)d\tau\right\|_\alpha^p+\varepsilon\left\|Z(T)\right\|_\alpha^p$$
$$\le C\varepsilon^{4p}+C\varepsilon^{4p-2\kappa p}+C\varepsilon^{-2\kappa_0 p}+C\varepsilon^{p-5p\kappa}$$
$$\le C\varepsilon^{-2p\kappa_0}.$$

## 5. The main Result

Before we state and proof the main Theorem for the approximation of the solution of Equation (18). We define $\omega$ (T) in N as solution of

$$\omega(T)=\omega(0)+\int_0^T\left[\nu\omega+\tilde{G}(\omega)\right]ds+\frac{b}{4}\sum_{k\ne q_c}^{N}\frac{\alpha_k^2}{\lambda_k}\int_0^T B_c\left(e_k,e_k\right)ds \tag{28}$$

**Lemma 12** *Let $\omega$ (T) be a the solution of Equation (28). Assume that the initial condition satisfies $\left|\omega(0)\right|^p\le C$ for some p > 1, then for all $T_0$ > 0 there exists another constant C such that*

$$E\sup_{T\in[0,T_0]}\left|\omega(T)\right|^p\le C. \tag{29}$$

**Proof.** Taking the scalar product $\langle\cdot,\omega\rangle$ on both sides of Equation (28) yields

$$\frac{1}{2}\partial_T\left|\omega\right|^2=\nu(\omega,\omega)+\left\langle\tilde{G}(\omega),\omega\right\rangle+\sum_{k\ne q_c}^{N}\frac{b\alpha_k^2}{4\lambda_k}\left\langle B_c\left(e_k,e_k\right),\omega\right\rangle.$$

Using Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2}\partial_T\left|\omega\right|^2\le C\left|\omega\right|^2+C_1.$$

Using Gronwalls lemma, we obtain for 0≤ T that

$$\left|\omega(T)\right|^2\le e^{2CT}\left|\omega(0)\right|^2+C_1e^{2CT}.$$

Taking supremum on both sides on [0, $T_0$] yields (29).

**Lemma 13** *Let R(T) is defined Equation (19), then for all $T_0$ > 0 there exists another constant C such that*

$$E\sup_{T\in[0,T_0]}\left|R(T)\right|\le C\varepsilon^{1-15\kappa}, \tag{30}$$

**Proof.** We follow the same steps of Lemma 8.

In the following we are not able to calculate moments of error terms. Thus we restrict ourselves to a sufficiently large subset of $\Omega$, where our estimate go through.

**Definition 14** *Define the set $\Omega^* \subset \Omega$ such that all these estimates*

$$\sup_{[0,\tau^*]} \|\psi - Q\|_\alpha \le C\varepsilon^{1-10\kappa}, \tag{31}$$

$$\sup_{[0,\tau^*]} \|R\|_\alpha \le C\varepsilon^{1-15\kappa}, \tag{32}$$

*and*

$$\sup_{[0,\tau^*]} \|\omega\|_\alpha \le C\varepsilon^{-\frac{1}{2}\kappa}, \tag{33}$$

*hold on $\Omega^*$.*

**Proposition 15** *The set $\Omega^*$ has approximately probability 1*

**Proof.**

$$\begin{aligned} &P(\Omega^*) \\ &\ge 1 - P\left(\sup_{[0,\tau^*]} \|R\|_\alpha \le C\varepsilon^{1-15\kappa}\right) \\ &-P\left(\sup_{[0,\tau^*]} \|\psi - Q\|_\alpha \le C\varepsilon^{1-10\kappa}\right) \\ &-P\left(\sup_{[0,\tau^*]} \|\omega\|_\alpha \le C\varepsilon^{-\frac{1}{2}\kappa}\right), \end{aligned}$$

Using Chebychev inequality and Lemmas 8 and 12, we obtain for sufficiently large q

$$\begin{aligned} &P(\Omega^*) \\ &\ge 1 - C\left[\varepsilon^{q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa}\right] \\ &\ge 1 - C\varepsilon^{\frac{1}{2}q\kappa} \ge 1 - C\varepsilon^{p}. \end{aligned} \tag{34}$$

**Theorem 16** *Assume that Assumption (5) hold and suppose a(0) = O(1) and $\psi$ (0) = O(1). Let $\omega(T)$ be a solution of (28) and a(T) as defined in (18). If the initial condition satisfies a(0) = $\omega$ (0)and for all $\kappa \in \left(0, \frac{1}{15}\right)$, then*

$$E \sup_{T\in[0,\tau^*]} |a(T) - \omega(T)| \le C\varepsilon^{p-15\kappa}. \tag{35}$$

**Proof.** Define $\varphi$ (T) as

$$\varphi(T) := a(T) - R(T).$$

From (18) we get

$$\begin{aligned} \varphi(T) &= \varphi(0) + \int_0^T \begin{bmatrix} \nu(\varphi(s) + R(s)) \\ +\tilde{G}(\varphi(s) + R(s)) \end{bmatrix} ds \\ &+ \sum_{k\ne q_c}^{N} \frac{b\alpha_k^2}{4\lambda_k} \int_0^T B_c(e_k, e_k)\, ds. \end{aligned} \tag{36}$$

Define now

$$\varsigma(T) := \omega(T) - \varphi(T). \tag{37}$$

Subtracting Equation (36), Equations (28) and using (37) to obtain

$$\begin{aligned} \varsigma(T) &= \nu\int_0^T \varsigma(T)\,ds - \nu\int_0^T R(T)\,ds \\ &+\int_0^T \left[\tilde{G}(\omega) - \tilde{G}(\omega - \varsigma + R)\right] ds. \end{aligned} \tag{38}$$

From Equation (38) we obtain

$$\partial_T \varsigma = \nu\varsigma - \nu R + \left[\tilde{G}(\omega) - \tilde{G}(\omega - \varsigma + R)\right]. \tag{39}$$

Taking the scalar product $\langle \cdot, \varsigma \rangle$ on both sides (39), yields

$$\begin{aligned} \frac{1}{2}\partial_T |\varsigma|^2 &= \langle \partial_T \varsigma, \varsigma \rangle = \langle \nu\varsigma, \varsigma \rangle - \langle \nu R, \varsigma \rangle \\ &+ \left\langle \tilde{G}(\omega) - \tilde{G}(\omega - \varsigma + R), \varsigma \right\rangle. \end{aligned}$$

Using Young and Cauchy-Schwartz inequalities, we obtain the following linear ordinary differential inequality

$$\begin{aligned} &\partial_T |\varsigma|^2 \le C\left[|\varsigma|^2 + |\varsigma|^6\right] \\ &+C|R|^2 \begin{bmatrix} 1 + |R|^2 + |R|^4 + |\omega|^4 + |\omega|^6 \\ +|\omega|^2|R|^4 + |\omega|^2|R|^2 + |\omega|^4|R|^2 \end{bmatrix} \\ &\le C\left[|\varsigma|^2 + |\varsigma|^6\right] + C|R|^2\left[c + c|R|^6 + c|\omega|^6\right]. \end{aligned}$$

Using Gronwalls lemma we obtain (as $\varsigma$ (0) = 0) for T $\le \tau^* \le T_0$

$$\begin{aligned} |\varsigma|^2 &\le C\int_0^T |R(s)|^2\left[1 + |\omega|^6\right] e^{2C(T-s)} ds \\ &\le \sup_{T\in[0,\tau^*]} |R(s)|^2\left[1 + |\omega|^6\right]. \end{aligned}$$

Taking the supremum on both sides and using Equations (32) and (33). The expectation yields

$$E \sup_{T\in[0,\tau^*]} |\varsigma|^p \le C\varepsilon^{p-15p\kappa}.$$

Using Equations (29) and (36). We. nish the proof by using Equations (37), (39) and

$$\begin{aligned} &E \sup_{T\in[0,\tau^*]} |a(T) - \omega(T)|^p \le E \sup_{T\in[0,\tau^*]} |\varsigma - R|^p \\ &E \sup_{T\in[0,\tau^*]} |\varsigma|^p + E \sup_{T\in[0,\tau^*]} |R|^p. \end{aligned}$$

Now, we state and prove the main result of this paper as follow:

**Theorem 17** *(Approximation) Under Assumptions 5 let h be a solution of (4) defined in (10) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon^2 \psi(0)$ with $a(0) \in N$ and $\psi(0) \in S$ where a(0) and $\psi$ (0) are of order one, and $\omega$ (T) is a solution of Equation (28) with $\omega$ (0) = a(0). Then for all p > 1 and $T_0$ > 0 and all $\kappa \in \left(0, \frac{2}{15}\right)$, there exists C > 0 such that*

$$P\left(\sup_{t\in\left[0,\varepsilon^{-4}T_0\right]}\left\|u(t)-\varepsilon\omega\left(\varepsilon^4 t\right)-\varepsilon Q\left(\varepsilon^4 t\right)\right\|_\alpha > \varepsilon^{2-15\kappa}\right) \quad (40)$$
$$\le C\varepsilon^p.$$

*where Q(T)Defined in (24) with $Z_k(T)$ Defined in (21).*

**Proof.** For the stopping time, we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \left\{\begin{array}{l}\sup_{T\in[0,T_0]}\|a(T)\|_\alpha < \varepsilon^{-\kappa},\\ \sup_{T\in[0,T_0]}\|\psi(T)\|_\alpha < \varepsilon^{-3\kappa}\end{array}\right\} \supseteq \Omega^*.$$

Now let us turn to the approximation result. Using (10) and triangle inequality, yields

$$\sup_{T\in[0,T_0]}\left\|u\left(\varepsilon^{-4}T\right)-\varepsilon\omega(T)-\varepsilon^2 Q(T)\right\|_\alpha$$
$$< \varepsilon \sup_{[0,\tau^*]}\|a-\omega\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]}\|\psi-Q\|_\alpha .$$

From Equations (31) and (35), we obtain

$$\sup_{t\in\left[0,\varepsilon^{-2}T_0\right]}\left\|u(t)-\varepsilon\omega\left(\varepsilon^4 t\right)-\varepsilon^2 Q\left(\varepsilon^4 t\right)\right\|_\alpha$$
$$= \sup_{t\in\left[0,\varepsilon^{-2}\tau^*\right]}\left\|u(t)-\varepsilon\omega\left(\varepsilon^4 t\right)-\varepsilon^2 Q\left(\varepsilon^4 t\right)\right\|_\alpha$$
$$\le C\varepsilon^{2-15\kappa} \text{ on } \Omega^*.$$

Thus

$$P\left(\sup_{t\in\left[0,\varepsilon^{-2}T_0\right]}\left\|u(t)-\varepsilon\omega\left(\varepsilon^4 t\right)-\varepsilon^2 Q\left(\varepsilon^4 t\right)\right\|_\alpha > \varepsilon^{2-15\kappa}\right)$$
$$\le 1-P(\Omega^*).$$

Using Equation (34), yields (40).

# 6. Applications

In this section, consider the stochastic generalized Swift-Hohenberg equation (2) with respect to Neumann boundary conditions on [0, π]. Let

$$e_k(x) = \begin{cases}\sqrt{\frac{1}{\pi}} & \text{if } k=0,\\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k>0,\end{cases} \quad (41)$$

then N:=span{cos($q_c$x)} and $\lambda_k = \left(q_c^2-k^2\right)^2$. Moreover, all conditions of Assumptions 3 and 2 are satisfied for the operators B (u, u) = $u^2$ and F (u, u, u) =$u^3$ with α = 1 and β = 0, it is easy to check that

$$\|F\ (u,\upsilon,\omega)\|_{H1} \le C\|u\|_{H1}\|\upsilon\|_{H1}\|\omega\|_{H1},$$

and

$$B_c(e_k,e_k) = \frac{1}{\pi}\cos(2kx) \ne 0 \ \text{for}\ k\in(0,\infty).$$

**First Case:** If we choose $q_c$ = 2, then the stochastic generalized Swift-Hohenberg equation (2) takes this form

$$\partial_t u = -\left(\partial_x^2\right)^2 u + \varepsilon^4 \nu u + bu^2 - u^3 + \varepsilon^{\frac{5}{2}}\partial_t W(t), \quad (42)$$

For this model we note that

$$A = -\left(4+\partial_x^2\right)^2, \lambda_k = \left(4-k^2\right)^2, N := span\{\cos(2x)\}.$$

If $b^2 = \frac{216}{19}$ we see that Assumation 4 is hold where

$$B_s(a) = \frac{\gamma^2}{2}\left(1+\cos(4x)\right), B_k(a) = \left\langle B_s(a), e_k\right\rangle,$$

and

$$F(a) = \frac{\gamma^2}{4}\left(3\cos(2x)+\cos(6x)\right).$$

For the noise we study two cases:

(1) If $W(t) = \sigma\beta_1\cos(x)$, then the amplitude equation takes the form

$$\partial_t\gamma = \nu\gamma - c_0\gamma^5 + \frac{\sigma^2}{\pi\sqrt{6}}, c_0 \simeq 1.36, \quad (43)$$

where $\frac{\sigma^2}{\pi\sqrt{6}}$ comes from the noise term.

(2) If $W(t) = \sum_{k=3}^N \alpha_k\beta_k\cos(kx)$, then the amplitude equation takes the form

$$\partial_t\gamma = \nu\gamma - c_0\gamma^5, c_0 \simeq 1.36. \quad (44)$$

In the two Cases near a change of stability on a time-scale of order $\varepsilon^{-4}$ the solution u(t) of (42) is well approximated by

$$u(t) \simeq \varepsilon\gamma\left(\varepsilon^4 t\right) + O\left(\varepsilon^{2-15\kappa}\right),$$

where is a solution of (43) (or (44)).

**Second Case:** If we choose $q_c$ = 3, then the stochastic generalized Swift-Hohenberg equation (2) takes this form

$$\partial_t u = -\left(9+\partial_x^2\right)^2 u + \varepsilon^4\nu u + bu^2 - u^3 + \varepsilon^{\frac{5}{2}}\partial_t W(t), \quad (45)$$

and we note that

$$N := span\{\cos(3x)\} \text{ and } \lambda_k = \left(9-k^2\right)^2.$$

Assumation 4 is hold if $b^2 \simeq 57.05$ where

$$B_s(a) = \frac{\gamma^2}{2}\left(1+\cos(6x)\right) \text{ and } B_k(a) = \left\langle B_s(a), e_k\right\rangle,$$

and

$$F(a) = \frac{\gamma^2}{4}\left(3\cos(3x)+\cos(9x)\right).$$

In this case if we choose $W(t) = \sum_{k\ne 3}^N \alpha_k\beta_k\cos(kx)$ then the amplitude equation takes the form

$$\partial_t \gamma = \nu\gamma - c_0\gamma^5, c_0 \simeq 0.028. \qquad (46)$$

and the solution u(t) of (45) is well approximated by

$$u(t) \simeq \varepsilon\gamma\left(\varepsilon^4 t\right) + O\left(\varepsilon^{2-15\kappa}\right),$$

where is a solution of (46).

Finally, we note that from First and Second cases if

$$q_c = L \text{ and } W(t) = \sum_{k\neq 3}^{N} \alpha_k \beta_k \cos(kx),$$

then there is influence from the noise only in the case of L is even and k = L/2.

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